SOLUTIONS MANUAL
for elementary mechanics & thermodynamics

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Contents

Chapter 1

MOTION ALONG A STRAIGHT LINE

1. The following functions give the position as a function of time:

   i) $x = A$
   ii) $x = Bt$
   iii) $x = Ct^2$
   iv) $x = D \cos \omega t$
   v) $x = E \sin \omega t$

   where A, B, C, D, E, ω are constants.

A) What are the units for A, B, C, D, E, ω?

B) Write down the velocity and acceleration equations as a function of time. Indicate for what functions the acceleration is *constant.*

C) Sketch graphs of x, v, a as a function of time.

SOLUTION

A) X is always in m.
Thus we must have A in m; B in $m\,\text{sec}^{-1}$, C in $m\,\text{sec}^{-2}$.
ωt is always an angle, θ is radius and $\cos\theta$ and $\sin\theta$ have no units.

Thus ω must be sec^{-1} or radians sec^{-1}.

D and E must be m.

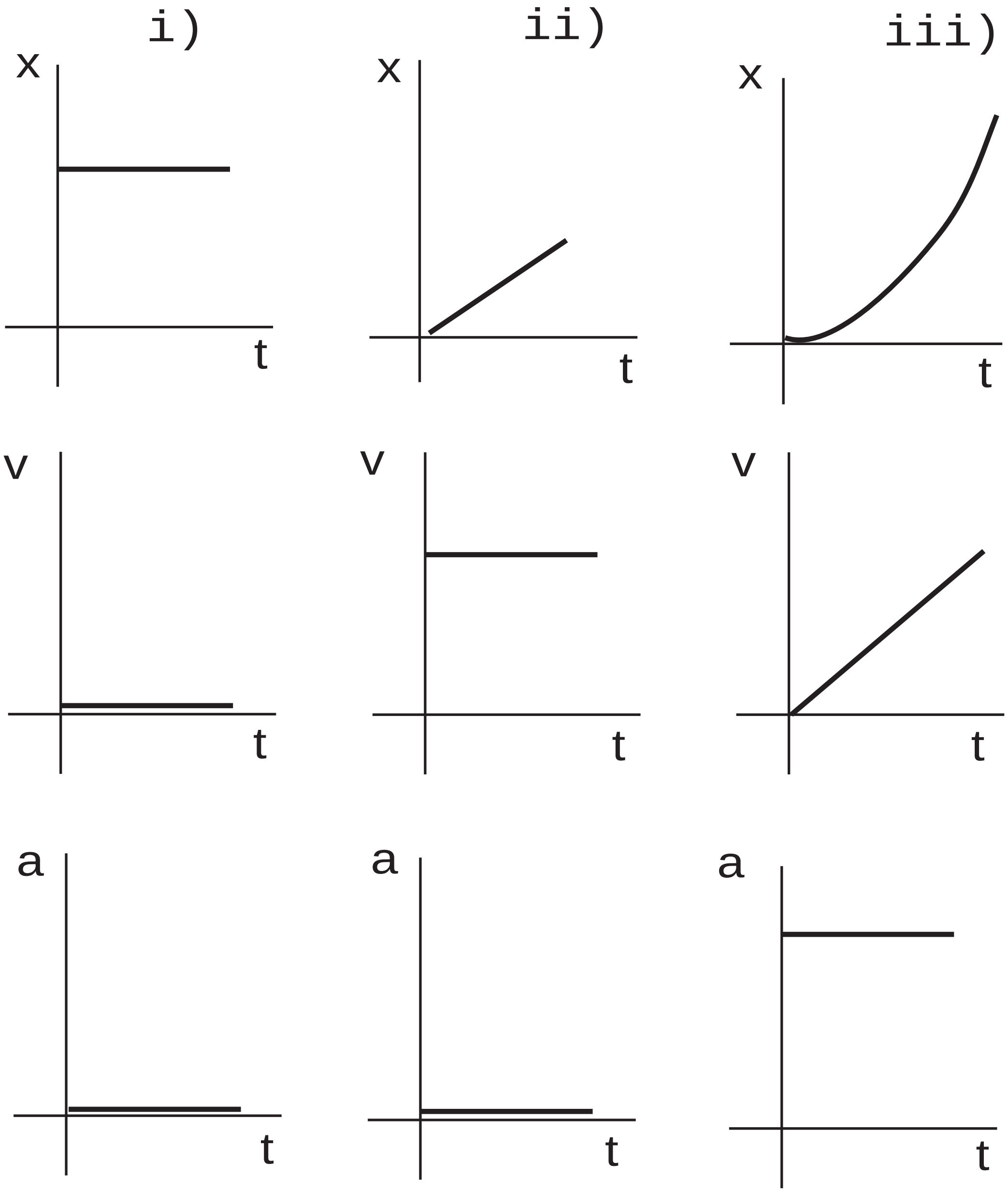
B) $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$. Thus

i) $v = 0$ ii) $v = B$ iii) $v = Ct$
iv) $v = -\omega D \sin \omega t$ v) $v = \omega E \cos \omega t$

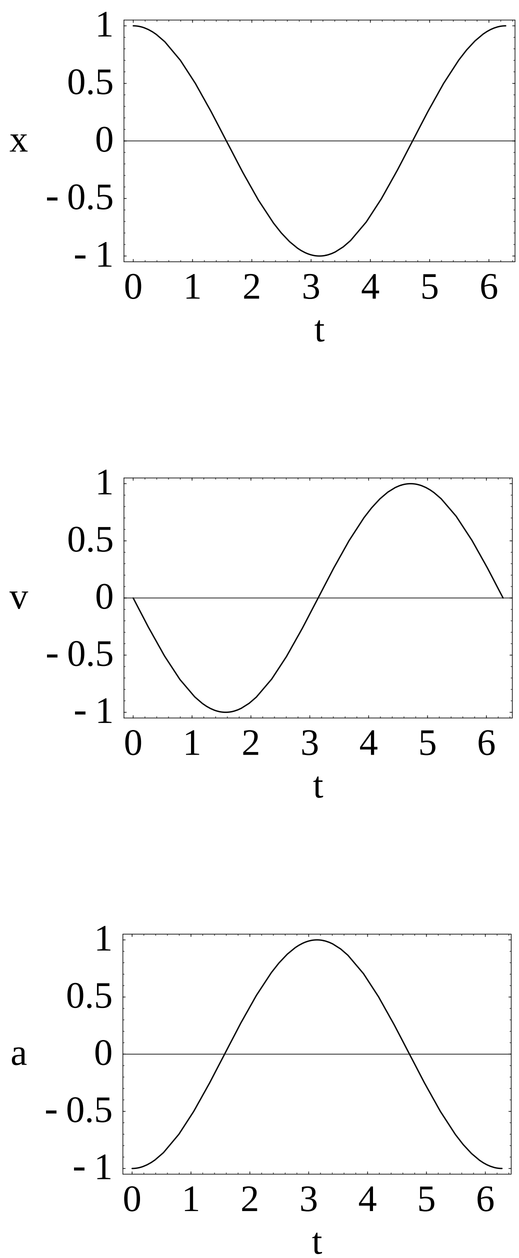
and notice that the units we worked out in part A) are all consistent with v having units of $m\cdot\ \text{sec}^{-1}$. Similarly

i) $a = 0$ ii) $a = 0$ iii) $a = C$
iv) $a = -\omega^2 D \cos \omega t$ v) $a = -\omega^2 E \sin \omega t$

C)



iv)



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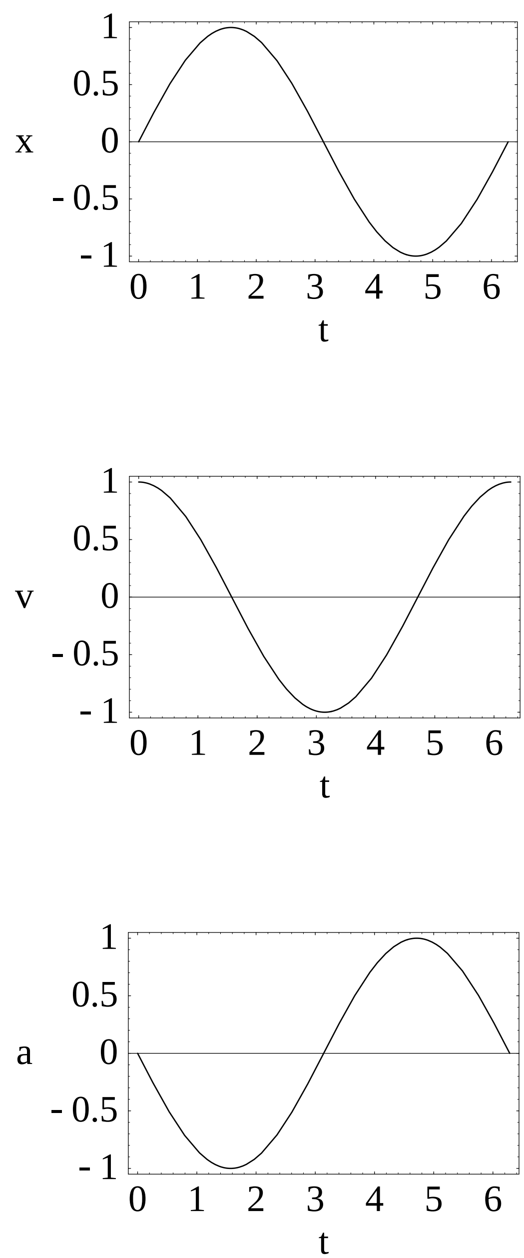
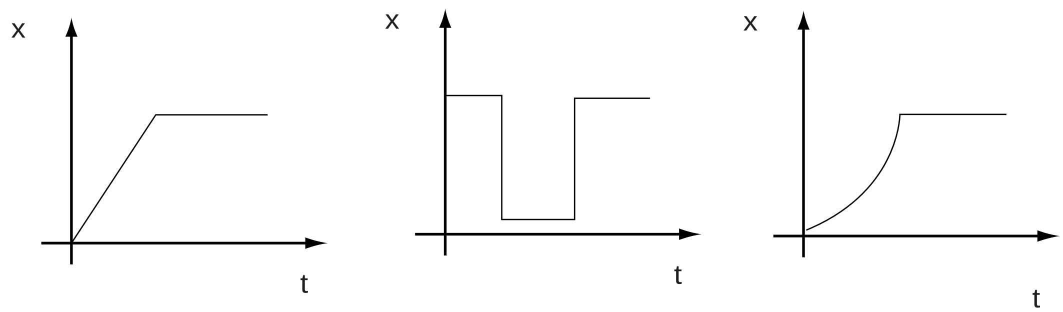
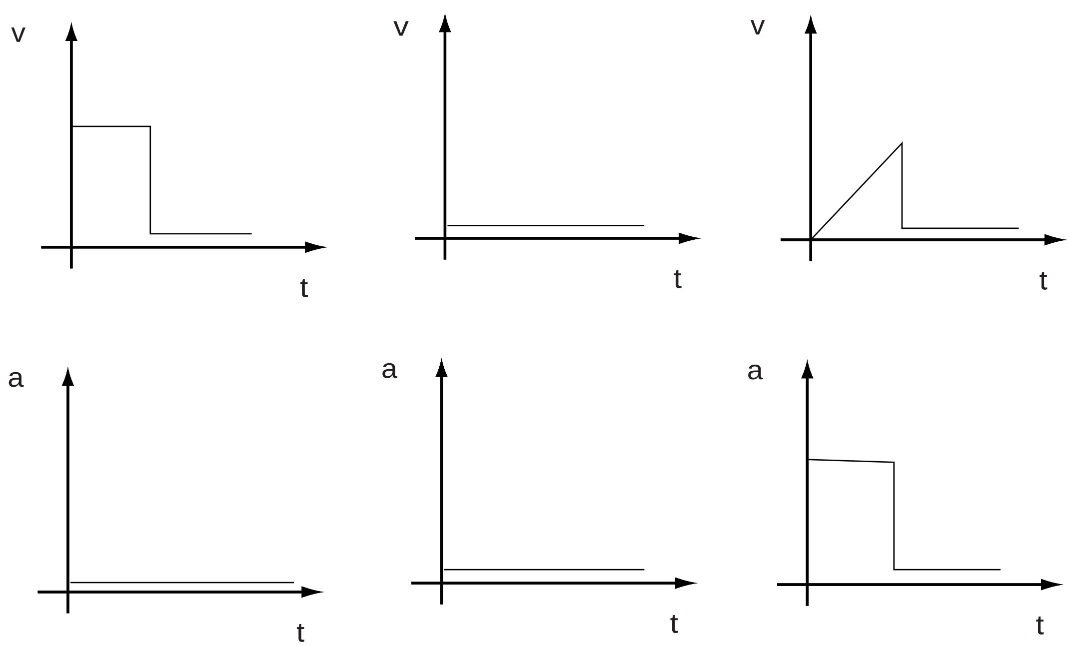


2. The figures below show position-time graphs. Sketch the corresponding velocity-time and acceleration-time graphs.



<u>SOLUTION</u>

The velocity-time and acceleration-time graphs are:



3. If you drop an object from a height H above the ground, work out a formula for the speed with which the object hits the ground.

SOLUTION

$$v^2 = v_0^2 + 2a(y - y_0)$$

In the vertical direction we have:

$v_0 = 0,$ $a = -g,$ $y_0 = H,$ $y = 0.$

Thus

$$\begin{aligned} v^2 &= 0 - 2g(0 - H) \\ &= 2gH \\ \Rightarrow v &= \sqrt{2gH} \end{aligned}$$

4. A car is travelling at constant speed v_1 and passes a second car moving at speed v_2. The instant it passes, the driver of the second car decides to try to catch up to the first car, by stepping on the gas pedal and moving at acceleration a. Derive a formula for how long it takes to catch up. (The first car travels at constant speed v_1 and does not accelerate.)

<u>SOLUTION</u>

Suppose the second car catches up in a time interval t. During that interval, the first car (which is not accelerating) has travelled a distance $d = v_1 t$. The second car also travels this distance d in time t, but the second car is accelerating at a and so it's distance is given by

$$
\begin{aligned}
x - x_0 &= d = v_0 t + \frac{1}{2}at^2 \\
&= v_1 t = v_2 t + \frac{1}{2}at^2 \qquad \textit{because} \quad v_0 = v_2 \\
v_1 &= v_2 + \frac{1}{2}at \\
\Rightarrow t &= \frac{2(v_1 - v_2)}{a}
\end{aligned}
$$

5. If you start your car from rest and accelerate to $30mph$ in 10 *seconds*, what is your acceleration in *mph per sec* and in *miles per hour*^{2} ?

SOLUTION

$$1hour = 60 \times 60sec$$
$$1sec = \frac{1}{60 \times 60}hour$$

$$\begin{aligned}
v &= v_0 + at \\
a &= \frac{v - v_0}{t} \\
&= \frac{30\ mph - 0}{10\ sec} \\
&= 3\ mph\ per\ sec \\
&= 3\ mph\frac{1}{sec} = 3\ mph\ \frac{1}{(\frac{1}{60} \times \frac{1}{60}hour)} \\
&= 3 \times 60 \times 60\ miles\ hour^{-2} \\
&= 10,800\ miles\ per\ hour^2
\end{aligned}$$

6. If you throw a ball up vertically at speed V, with what speed does it return to the ground ? Prove your answer using the constant acceleration equations, and neglect air resistance.

SOLUTION

We would guess that the ball returns to the ground at the same speed V, and we can actually prove this. The equation of motion is

$$\begin{aligned} v^2 &= v_0^2 + 2a(x - x_0) \\ & \quad \textit{and} \qquad x_0 = 0, \qquad x = 0, \qquad v_0 = V \\ \Rightarrow v^2 &= V^2 \\ \textit{or} \qquad v &= V \end{aligned}$$

Chapter 2

VECTORS

1. Calculate the angle between the vectors $\vec{r} = \hat{i} + 2\hat{j}$ and $\vec{t} = \hat{j} - \hat{k}$.

SOLUTION

$$\begin{aligned}
\vec{r}.\vec{t} \equiv |\vec{r}||\vec{t}|\cos\theta &= (\hat{i} + 2\hat{j}).(\hat{j} - \hat{k}) \\
&= \hat{i}.\hat{j} + 2\hat{j}.\hat{j} - \hat{i}.\hat{k} - 2\hat{j}.\hat{k} \\
&= 0 + 2 - 0 - 0 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
|\vec{r}||\vec{t}|\cos\theta &= \sqrt{1^2 + 2^2}\sqrt{1^2 + (-1)^2}\cos\theta \\
&= \sqrt{5}\sqrt{2}\cos\theta \\
&= \sqrt{10}\cos\theta
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \cos\theta &= \frac{2}{\sqrt{10}} = 0.632 \\
\Rightarrow \theta &= 50.8^0
\end{aligned}$$

2. Evaluate $(\vec{r} + 2\vec{t}\,).\vec{f}$ where $\vec{r} = \hat{\imath} + 2\hat{\jmath}$ and $\vec{t} = \hat{\jmath} - \hat{k}$ and $\vec{f} = \hat{\imath} - \hat{\jmath}$.

<u>SOLUTION</u>

$$\begin{aligned}
\vec{r} + 2\vec{t} &= \hat{\imath} + 2\hat{\jmath} + 2(\hat{\jmath} - \hat{k}) \\
&= \hat{\imath} + 2\hat{\jmath} + 2\hat{\jmath} - 2\hat{k} \\
&= \hat{\imath} + 4\hat{\jmath} - 2\hat{k}
\end{aligned}$$

$$\begin{aligned}
(\vec{r} + 2\vec{t}\,).\vec{f} &= (\hat{\imath} + 4\hat{\jmath} - 2\hat{k}).(\hat{\imath} - \hat{\jmath}) \\
&= \hat{\imath}.\hat{\imath} + 4\hat{\jmath}.\hat{\imath} - 2\hat{k}.\hat{\imath} - \hat{\imath}.\hat{\jmath} - 4\hat{\jmath}.\hat{\jmath} + 2\hat{k}.\hat{\jmath} \\
&= 1 + 0 - 0 - 0 - 4 + 0 \\
&= -3
\end{aligned}$$

3. Two vectors are defined as $\vec{u} = \hat{j} + \hat{k}$ and $\vec{v} = \hat{i} + \hat{j}$. Evaluate:

   A) $\vec{u} + \vec{v}$

   B) $\vec{u} - \vec{v}$

   C) $\vec{u}.\vec{v}$

   D) $\vec{u} \times \vec{v}$

SOLUTION

A)

$$\vec{u} + \vec{v} = \hat{j} + \hat{k} + \hat{i} + \hat{j} = \hat{i} + 2\hat{j} + \hat{k}$$

B)

$$\vec{u} - \vec{v} = \hat{j} + \hat{k} - \hat{i} - \hat{j} = -\hat{i} + \hat{k}$$

C)

$$\begin{aligned}
\vec{u}.\vec{v} &= (\hat{j} + \hat{k}).(\hat{i} + \hat{j}) \\
&= \hat{j}.\hat{i} + \hat{k}.\hat{i} + \hat{j}.\hat{j} + \hat{k}.\hat{j} \\
&= 0 + 0 + 1 + 0 \\
&= 1
\end{aligned}$$

D)

$$\begin{aligned}
\vec{u} \times \vec{v} &= (\hat{j} + \hat{k}) \times (\hat{i} + \hat{j}) \\
&= \hat{j} \times \hat{i} + \hat{k} \times \hat{i} + \hat{j} \times \hat{j} + \hat{k} \times \hat{j} \\
&= -\hat{k} + \hat{j} + 0 - \hat{i} \\
&= -\hat{i} + \hat{j} - \hat{k}
\end{aligned}$$

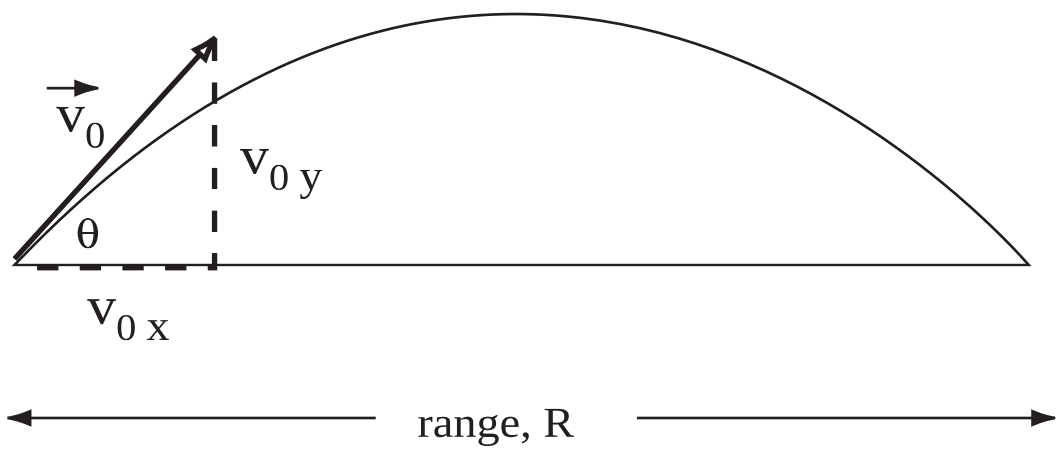
Chapter 3

MOTION IN 2 & 3 DIMENSIONS

1. A) A projectile is fired with an initial speed v_o at an angle θ with respect to the horizontal. Neglect air resistance and derive a formula for the horizontal range R, of the projectile. (Your formula should make no explicit reference to time, t). At what angle is the range a maximum ?

   B) If $v_0 = 30\,km/hour$ and $\theta = 15^o$ calculate the numerical value of R.

SOLUTION



$$v_{0y} = v_0 \sin\theta$$
$$v_{0x} = v_0 \cos\theta$$

In the x direction we have:

$$\begin{aligned} a_x &= 0 \\ x - x_0 &\equiv R \\ v_x &= v_{0x} + a_x t \\ \Rightarrow v_x &= v_{0x} \end{aligned}$$

$$R \;=\; x - x_0 = \frac{v_x + v_{0x}}{2} t = \frac{2v_{0x}}{2} t = v_0 \cos\theta \; t$$

In the y direction we have:

$$\begin{aligned} a_y &= -g \\ y - y_0 &= 0 \end{aligned}$$

$$\begin{aligned} 0 = y - y_0 &= v_{0y}t + \frac{1}{2}a_y t^2 \\ &= v_0 \sin\theta\, t - \frac{1}{2}gt^2 \\ \Rightarrow v_0 \sin\theta = \frac{1}{2}gt & \\ \Rightarrow t &= \frac{2v_0 \sin\theta}{g} \\ \Rightarrow R &= v_0 \cos\theta \frac{2v_0 \sin\theta}{g} = \frac{2v_0^2 \cos\theta \sin\theta}{g} = \frac{v_0^2 \sin 2\theta}{g} \end{aligned}$$

i.e. $\boxed{R = \frac{v_0^2 \sin 2\theta}{g}}$ which is a maximum for $\theta = 45^o$.

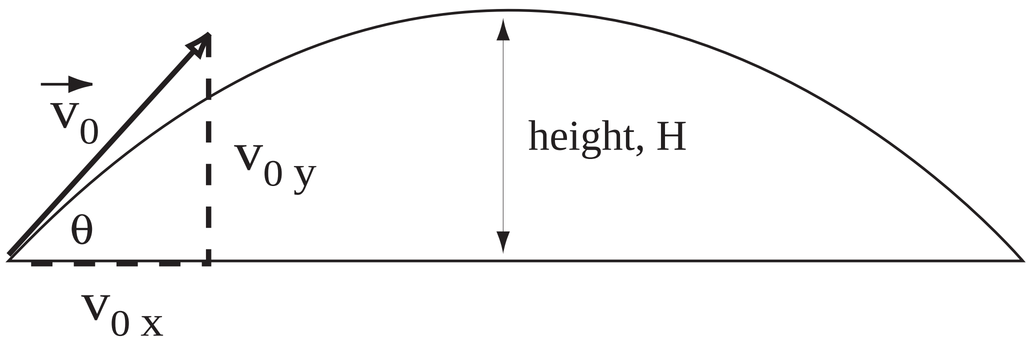
B)

$$\begin{aligned} R &= \frac{(\frac{30\times 10^3 m}{60\times 60 sec})^2 \sin(2\times 15^o)}{9.8 m\, sec^{-2}} = \frac{69.4\times 0.5}{9.8}\frac{m^2}{sec^2\, m\, sec^{-2}} \\ &= 3.5\, m \end{aligned}$$

i.e. $\boxed{R = 3.5\, m}$

2. A projectile is fired with an initial speed v_o at an angle θ with respect to the horizontal. Neglect air resistance and derive a formula for the maximum height H, that the projectile reaches. (Your formula should make no explicit reference to time, t).

<u>SOLUTION</u>



We wish to find the maximum height H. At that point $v_y = 0$. Also in the y direction we have

$a_y = -g$ and $H \equiv y - y_0$.

The approporiate constant acceleration equation is :

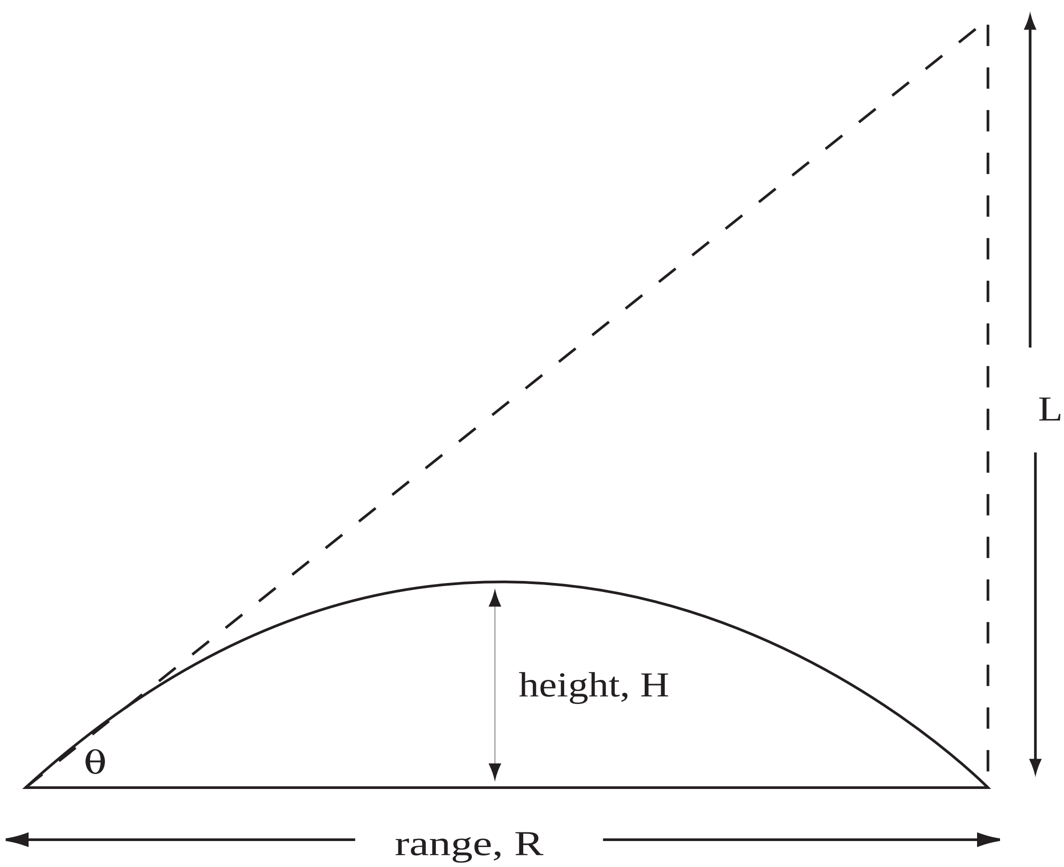
$$\begin{aligned} v_y^2 &= v_{0y}^2 + 2a_y(y - y_0) \\ 0 &= v_0^2 \sin^2\theta - 2gH \\ \Rightarrow H &= \frac{v_0^2 \sin^2\theta}{2g} \end{aligned}$$

which is a maximum for $\theta = 90^o$, as expected.

3. A) If a bulls-eye target is at a horizontal distance R away, derive an expression for the height L, which is the vertical distance above the bulls-eye that one needs to aim a rifle in order to hit the bulls-eye. Assume the bullet leaves the rifle with speed v_0.

   B) How much bigger is L compared to the projectile height H ?
   Note: In this problem use previous results found for the range R and height H, namely $R = \dfrac{v_0^2 \sin 2\theta}{g} = \dfrac{2v_0^2 \sin\theta \cos\theta}{g}$ and $H = \dfrac{v_0^2 \sin^2\theta}{2g}$.

<u>SOLUTION</u>



A) From previous work we found the range $R = \frac{v_0^2 \sin 2\theta}{g} = \frac{2v_0^2 \sin\theta\cos\theta}{g}$. From the diagram we have

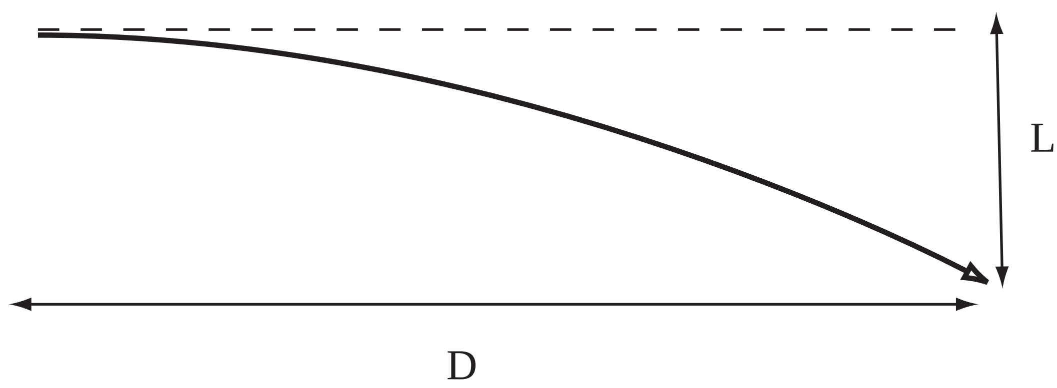
$$\begin{aligned}
\tan\theta &= \frac{L}{R} \\
\Rightarrow L &= R\tan\theta = \frac{2v_0^2 \sin\theta\cos\theta}{g}\,\frac{\sin\theta}{\cos\theta} \\
&= \frac{2v_0^2 \sin^2\theta}{g}
\end{aligned}$$

B) Comparing to our previous formula for the maximum height

$H = \frac{v_0^2 \sin^2\theta}{2g}$ we see that $L = 4H$.

4. Normally if you wish to hit a bulls-eye some distance away you need to aim a certain distance above it, in order to account for the downward motion of the projectile. If a bulls-eye target is at a horizontal distance D away and if you instead aim an arrow directly at the bulls-eye (i.e. directly horiziontally), by what (downward) vertical distance would you miss the bulls-eye ?

SOLUTION



In the x direction we have: $a_x = 0$, $v_{0x} = v_0$, $x - x_0 \equiv R$.

The appropriate constant acceleration equation in the x direction is

$$\begin{aligned} x - x_0 &= v_{0x} + \frac{1}{2} a_x t^2 \\ \Rightarrow D &= v_0 t \\ t &= \frac{D}{v_0} \end{aligned}$$

In the y direction we have: $a_y = -g$, $v_{0y} = 0$.

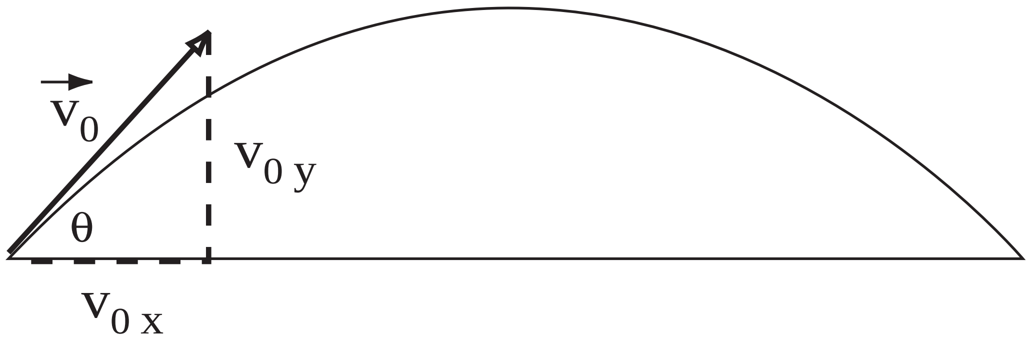
The appropriate constant acceleration equation in the y direction is

$$\begin{aligned} y - y_0 = v_{0y} + \frac{1}{2} a_y t^2 &= 0 - \frac{1}{2} g t^2 \\ &= 0 - \frac{1}{2} g (\frac{D}{v_0})^2 \end{aligned}$$

but $y_0 = 0$, giving $y = -\frac{1}{2} g \left(\frac{D}{v_0}\right)^2$ or $L = \frac{1}{2} g \left(\frac{D}{v_0}\right)^2$.

5. Prove that the trajectory of a projectile is a parabola (neglect air resistance). Hint: the general form of a parabola is given by $y = ax^2 + bx + c$.

SOLUTION



Let $x_0 = y_0 = 0$.

In the x direction we have

$$\begin{aligned} v_x &= v_{0x} + a_x t \\ &= v_{0x} \qquad \textit{because} \quad a_x = 0 \end{aligned}$$

Also

$$\begin{aligned} x - x_0 &= \frac{v_x + v_{0x}}{2} t \\ \Rightarrow x &= v_{0x} t = v_0 \cos\theta t \end{aligned}$$

In the y direction

$$\begin{aligned} y - y_0 &= v_{0y} t + \frac{1}{2} a_y t^2 \\ \Rightarrow y &= v_0 \sin\theta t - \frac{1}{2} g t^2 \qquad \textit{because} \quad a_y = -g \\ &= v_0 \sin\theta \, \frac{x}{v_0 \cos\theta} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta}\right)^2 \\ &= x \tan\theta - \frac{g}{2 v_0^2 \cos^2\theta} x^2 \end{aligned}$$

which is of the form $y = ax^2 + bx + c$, being the general formula for a parabola.

6. Even though the Earth is spinning and we all experience a centrifugal acceleration, we are not flung off the Earth due to the gravitational force. In order for us to be flung off, the Earth would have to be spinning a lot faster.

   A) Derive a formula for the new rotational time of the Earth, such that a person on the equator would be flung off into space. (Take the radius of Earth to be R).

   B) Using $R = 6.4$ *million km*, calculate a numerical anser to part A) and compare it to the actual rotation time of the Earth today.

SOLUTION

A person at the equator will be flung off if the centripetal acceleration a becomes equal to the gravitational acceleration g. Thus

A)

$$
\begin{aligned}
g = a &= \frac{v^2}{R} = \frac{(\frac{2\pi R}{T})^2}{R} = \frac{4\pi^2 R}{T^2} \\
T^2 &= \frac{4\pi^2 R}{g} \\
T &= 2\pi\sqrt{\frac{R}{g}}
\end{aligned}
$$

B)

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{6.4 \times 10^6\ km}{9.81\ m\ sec^{-2}}} \\
&= 2\pi\sqrt{\frac{6.4 \times 10^9\ m}{9.81\ m\ sec^{-2}}} \\
&= 2\pi\sqrt{\frac{6.4 \times 10^9}{9.81}}\ sec \\
&= 2\pi\sqrt{\frac{6.4 \times 10^9}{9.81}}\frac{hour}{60 \times 60\ sec} \\
&= 44.6\ hour
\end{aligned}
$$

i.e. Earth would need to rotate about twice as fast as it does now (24 *hours*).

7. A staellite is in a circular orbit around a planet of mass M and radius R at an altitude of H. Derive a formula for the additional speed that the satellite must acquire to completely escape from the planet. Check that your answer has the correct units.

SOLUTION
The gravitational potential energy is $U = -G\frac{Mm}{r}$ where m is the mass of the satellite and $r = R + H$.
Conservation of energy is

$$U_i + K_i = U_f + K_f$$

To escape to infinity then $U_f = 0$ and $K_f = 0$ (satellite is not moving if it just barely escapes.)

$$\Rightarrow \quad -G\frac{Mm}{r} + \frac{1}{2}mv_i^2 = 0$$

giving the escape speed as

$$v_i = \sqrt{\frac{2GM}{r}}$$

The speed in the circular orbit is obtained from

$$\begin{aligned} F &= ma \\ G\frac{Mm}{r^2} &= m\frac{v^2}{r} \\ \Rightarrow \quad v &= \sqrt{\frac{GMm}{r}} \end{aligned}$$

The additional speed required is

$$\begin{aligned} v_i - v &= \sqrt{\frac{2GM}{r}} - \sqrt{\frac{GM}{r}} \\ &= (\sqrt{2} - 1)\sqrt{\frac{GM}{r}} \end{aligned}$$

Check units:
$F = G\frac{Mm}{r^2}$ and so the units of G are $\frac{Nm^2}{kg^2}$. The units of $\sqrt{\frac{GM}{r}}$ are

$$\sqrt{\frac{N\ m^2\ kg^{-2}\ kg}{m}} = \sqrt{\frac{kg\ m\ sec^{-2}\ m^2\ kg^{-2}\ kg}{m}} = \sqrt{m^2\ sec^{-2}} = m\ sec^{-1}$$

which has the correct units of speed.

8. A mass m is attached to the end of a spring with spring constant k on a frictionless horizontal surface. The mass moves in circular motion of radius R and period T. Due to the centrifugal force, the spring stretches by a certain amount x from its equilibrium position. Derive a formula for x in terms of k, R and T. Check that x has the correct units.

SOLUTION

$$\begin{aligned}\Sigma F &= ma \\ kx &= \frac{mv^2}{r} \\ x &= \frac{mv^2}{kR} = \frac{m(\frac{2\pi R}{T})^2}{kR} = \frac{4\pi^2 mR}{kT^2}\end{aligned}$$

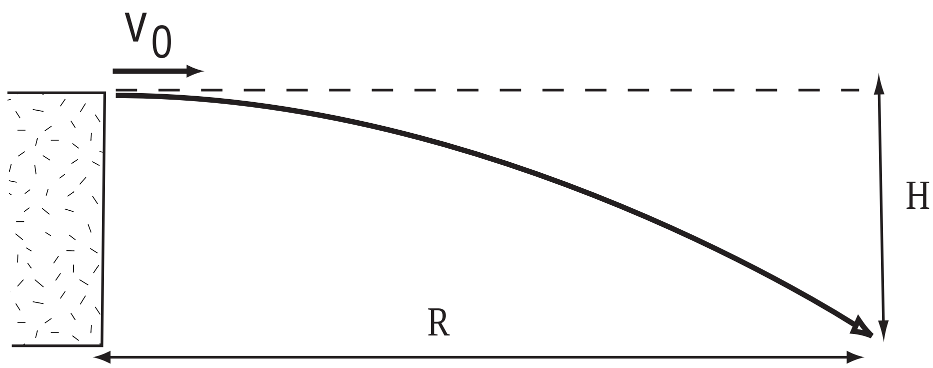
Check units:
The units of k are $N\,m^{-1}$ (because $F = -kx$ for a spring), and $N \equiv \frac{kg\ m}{sec^2}$. Thus $\frac{4\pi^2 mR}{kT^2}$ has units

$$\frac{kg\ m}{N\ m^{-1} sec^2} = \frac{kg\ m}{kg\ m\ sec^{-2}\ \ m^{-1} sec^2} = m$$

which is the correct unit of distance.

9. A cannon ball is fired horizontally at a speed v_0 from the edge of the top of a cliff of height H. Derive a formula for the horizontal distance (i.e. the range) that the cannon ball travels. Check that your answer has the correct units.

SOLUTION



In the x (horizontal) direction

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$$

Now $R = x - x_0$ and $a_x = 0$ and $v_{0x} = v_0$ giving $R = v_0 t$.

We obtain t from the y direction

$$y - y_0 = v_{0y}t + \frac{1}{2}a_y t^2$$

Now $y_0 = 0$, $y = -H$, $v_{0y} = 0$, $a_y = -g$ giving

$$-H = -\frac{1}{2}gt^2 \qquad or \qquad t = \sqrt{\frac{2H}{g}}$$

Substuting we get

$$R = v_0 t = v_0\sqrt{\frac{2H}{g}}$$

Check units:

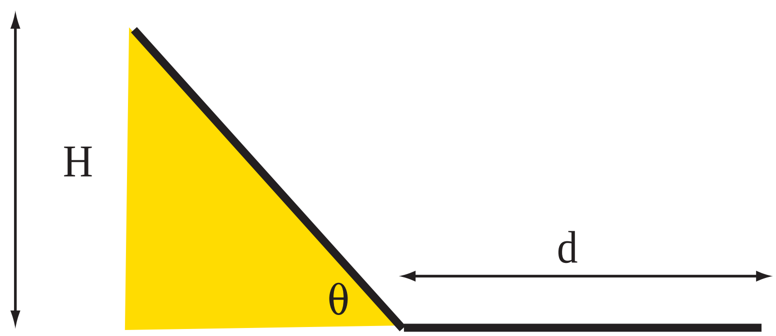
The units of $v_0\sqrt{\frac{2H}{g}}$ are

$$m\ sec^{-1}\sqrt{\frac{m}{m\ sec^{-2}}} = m\ sec^{-1}\sqrt{sec^2} = m\ sec^{-1}\ sec = m$$

which are the correct units for distance.

10. A skier starts from rest at the top of a frictionless ski slope of height H and inclined at an angle θ to the horizontal. At the bottom of the slope the surface changes to horizontal and has a coefficient of kinetic friction μ_k between the horizontal surface and the skis. Derive a formula for the distance d that the skier travels on the horizontal surface before coming to a stop. (Assume that there is a constant deceleration on the horizontal surface). Check that your answer has the correct units.

SOLUTION



The horizontal distance is given by

$$\begin{aligned} v_x^2 &= v_{0x}^2 + 2a_x(x - x_0) \\ 0 &= v_{0x}^2 + 2a_x d \end{aligned}$$

with the final speed $v_x = 0$, $d = x - x_0$, and the deceleration a_x along the horizontal surface is given by

$$\begin{aligned} & F = ma \\ &= -\mu_k N = ma \\ &= -\mu_k mg \\ \Rightarrow \quad & a = -\mu_k g \end{aligned}$$

Substituting gives

$$0 = v_{0x}^2 - 2\mu_k g d$$

$$or \qquad d \quad = \quad \frac{v_{0x}^2}{2\mu_k g}$$

And we get v_{0x} from conservation of energy applied to the ski slope

$$U_i + K_i = U_f + K_f$$
$$mgH + 0 = 0 + \frac{1}{2}mv^2$$
$$\Rightarrow \qquad v = v_{0x} = \sqrt{2gH}$$

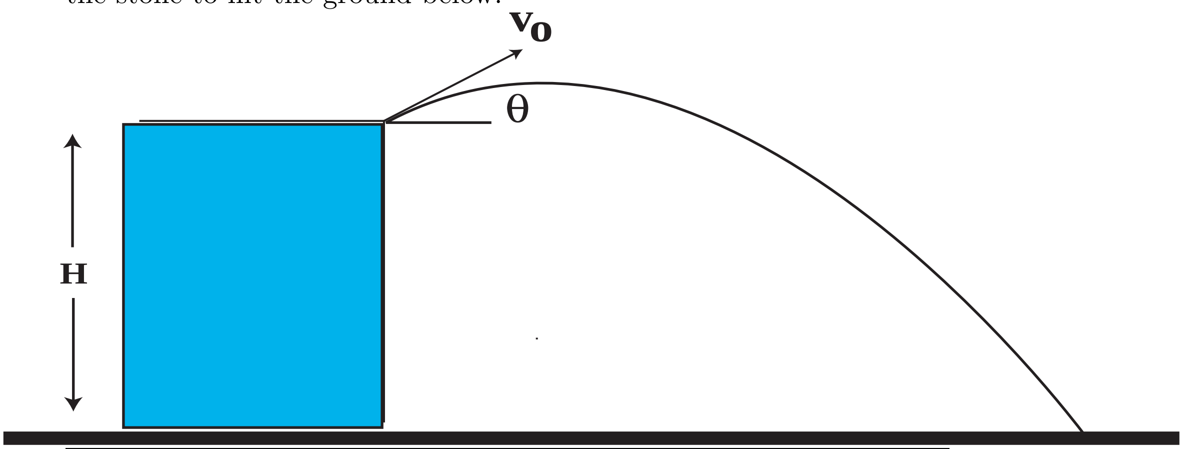
Substituting gives

$$d = \frac{2gH}{2\mu_k g} = \frac{H}{\mu_k}$$

Check units:

μ_k has no units, and so the units of $\frac{H}{\mu_k}$ are m.

11. A stone is thrown from the top of a building upward at an angle θ to the horizontal and with an initial speed of v_0 as shown in the figure. If the height of the building is H, derive a formula for the time it takes the stone to hit the ground below.



SOLUTION

$$y - y_0 = v_{0y}t + \frac{1}{2}a_y t^2$$

Choose the origin to be at the top of the building from where the stone is thrown.

$$y_0 = 0, \qquad y = -H, \qquad a_y = -g$$

$$v_{0y} = v_0 \sin\theta$$

$$\Rightarrow -H - 0 = v_0 \sin\theta t - \frac{1}{2}gt^2$$

$$-\frac{1}{2}gt^2 + v_0 \sin\theta t + H = 0$$

or

$$gt^2 - 2v_0 \sin\theta t - 2H = 0$$

which is a quadratic equation with solution

$$\begin{aligned} t &= \frac{2v_0 \sin\theta \pm \sqrt{4(v_0 \sin\theta)^2 + 8gH}}{2g} \\ &= \frac{v_0 \sin\theta \pm \sqrt{(v_0 \sin\theta)^2 + 2gH}}{g} \end{aligned}$$

Chapter 4

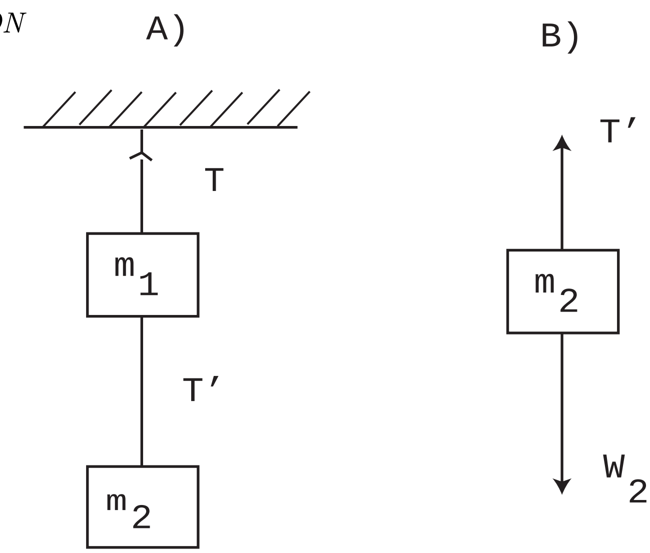
FORCE & MOTION - I

Chapter 5

FORCE & MOTION - II

1. A mass m_1 hangs vertically from a string connected to a ceiling. A second mass m_2 hangs below m_1 with m_1 and m_2 also connected by another string. Calculate the tension in each string.

SOLUTION



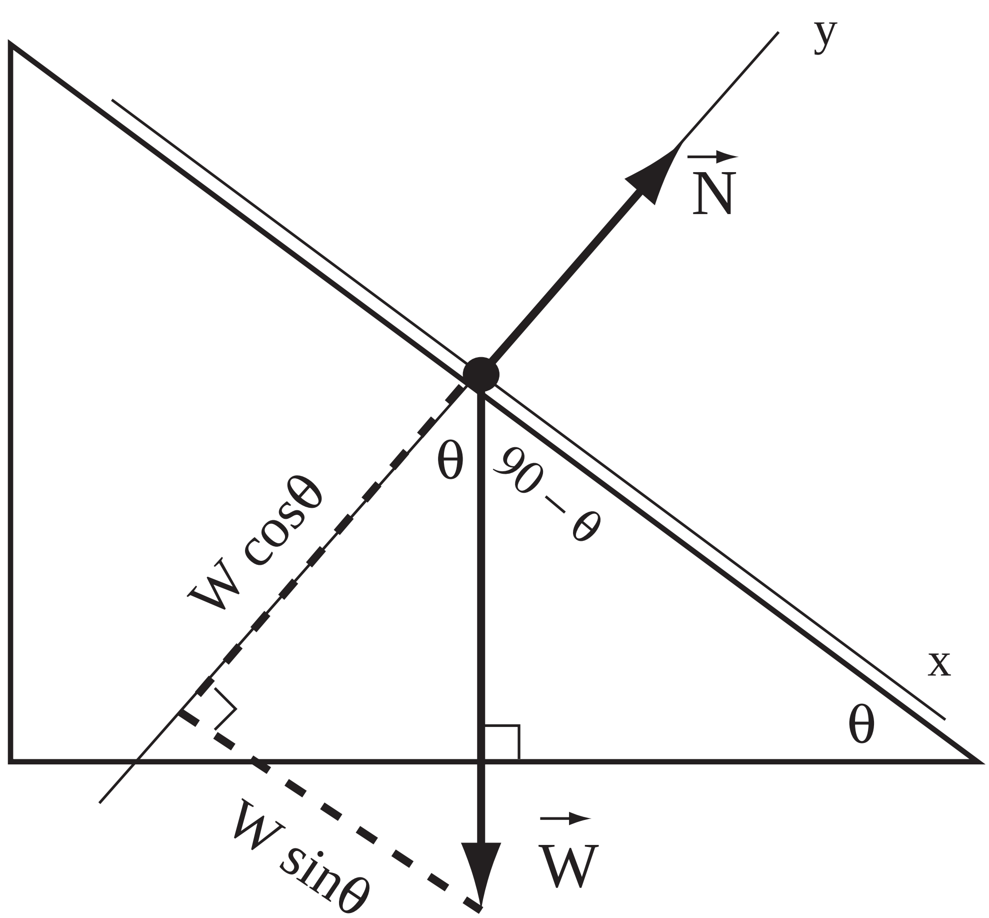
Obviously $T = W_1 + W_2 = (m_1 + m_2)g$. The forces on m_2 are indicated in Figure B. Thus

$$\sum F_y = m_2 a_{2y}$$
$$T' - W_2 = 0$$
$$T' = W_2 = m_2 g$$

2. What is the acceleration of a snow skier sliding down a frictionless ski slope of angle θ ?

   Check that your answer makes sense for $\theta = 0^o$ and for $\theta = 90^o$.

SOLUTION



Newton's second law is

$$\Sigma\vec{F} = m\vec{a}$$

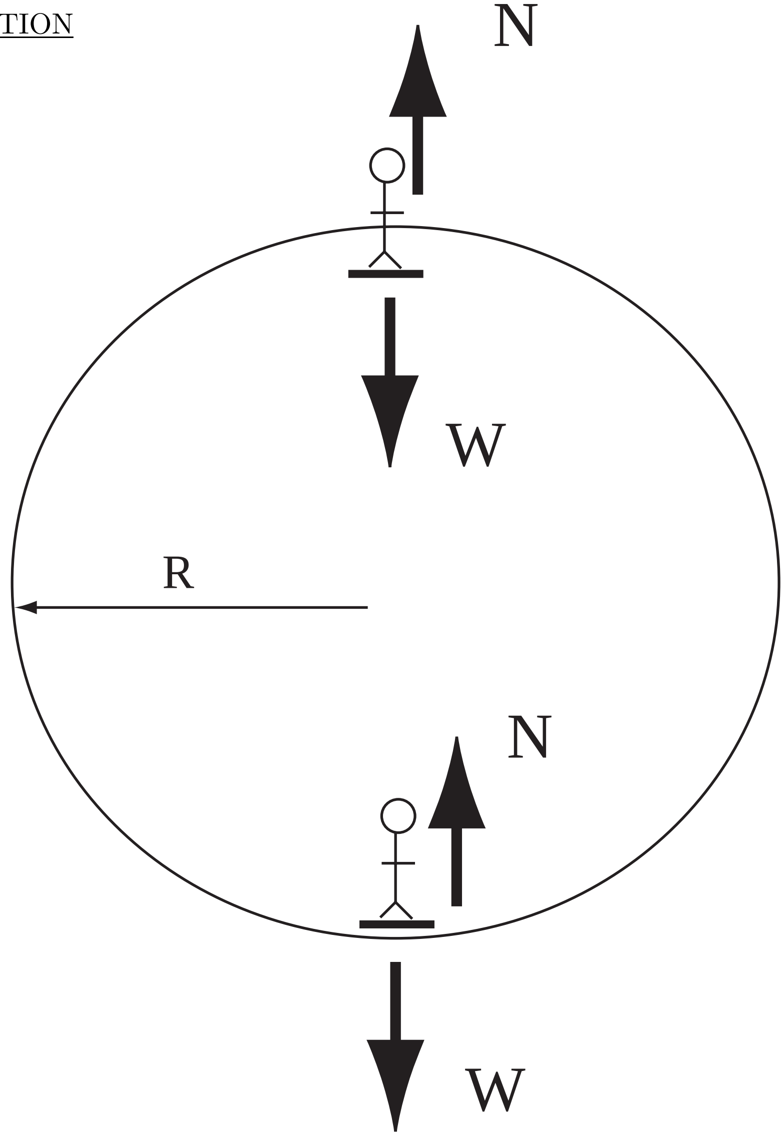
which, broken into components is

$$\begin{aligned} \Sigma F_x &= ma_x \qquad and \qquad \Sigma F_y = ma_y \\ = W\sin\theta &= ma_x \\ = mg\sin\theta &= ma_x \\ \\ \Rightarrow \qquad a_x &= g\sin\theta \end{aligned}$$

when $\theta = 0^o$ then $a_x = 0$ which makes sense, i.e. no motion.
when $\theta = 90^o$ then $a_x = g$ which is free fall.

3. A ferris wheel rotates at constant speed in a vertical circle of radius R and it takes time T to complete each circle. Derive a formula, in terms of m, g, R, T, for the weight that a passenger of mass m feels at the top and bottom of the circle. Comment on whether your answers make sense. (Hint: the weight that a passenger feels is just the normal force.)

<u>SOLUTION</u>



Bottom:

$$
\begin{aligned}
\Sigma F_y &= ma_y \\
N - W &= \frac{mv^2}{R}
\end{aligned}
$$

Top:

$$\Sigma F_y = ma_y$$

$$N - W = -\frac{mv^2}{R}$$

The weight you feel is just N.

$$
\begin{aligned}
N &= W + \frac{mv^2}{R} \\
&= mg + \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2 \\
&= mg + m\frac{4\pi^2 R}{T^2}
\end{aligned}
$$

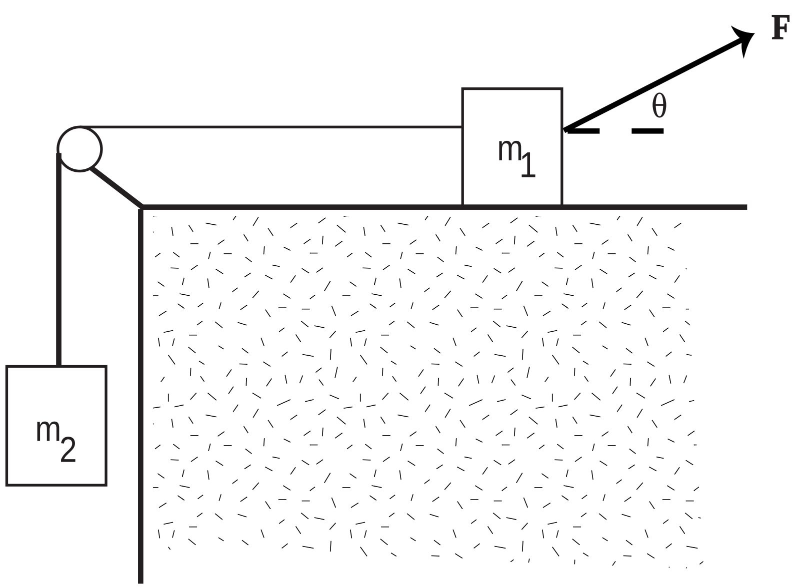
$$N = W - \frac{mv^2}{R}$$

$$= mg - \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2$$

$$= mg - m\frac{4\pi^2 R}{T^2}$$

At the bottom the person feels heavier and at the top the person feels lighter, which is as experience shows !

4. A block of mass m_1 on a rough, horizontal surface is connected to a second mass m_2 by a light cord over a light frictionless pulley as shown in the figure. ('Light' means that we can neglect the mass of the cord and the mass of the pulley.) A force of magnitude F is applied to the mass m_1 as shown, such that m_1 moves to the right. The coefficient of kinetic friction between m_1 and the surface is μ. Derive a formula for the acceleration of the masses. [Serway 5th ed., pg.135, Fig 5.14]



SOLUTION

Let the acceleration of both masses be a. For mass m_2 (choosing $m_2 a$ with the same sign as T):

$$T - W_2 = m_2 a$$
$$T = m_2 a + m_2 g$$

For mass m_1:

$$\sum F_x = m_1 a \qquad \sum F_y = 0$$
$$F\cos\theta - T - F_k = m_1 a \qquad N + F\sin\theta - W_1 = 0$$
$$F\cos\theta - T - \mu N = m_1 a \qquad N = m_1 g - F\sin\theta$$

Substitute for T and N into the left equation

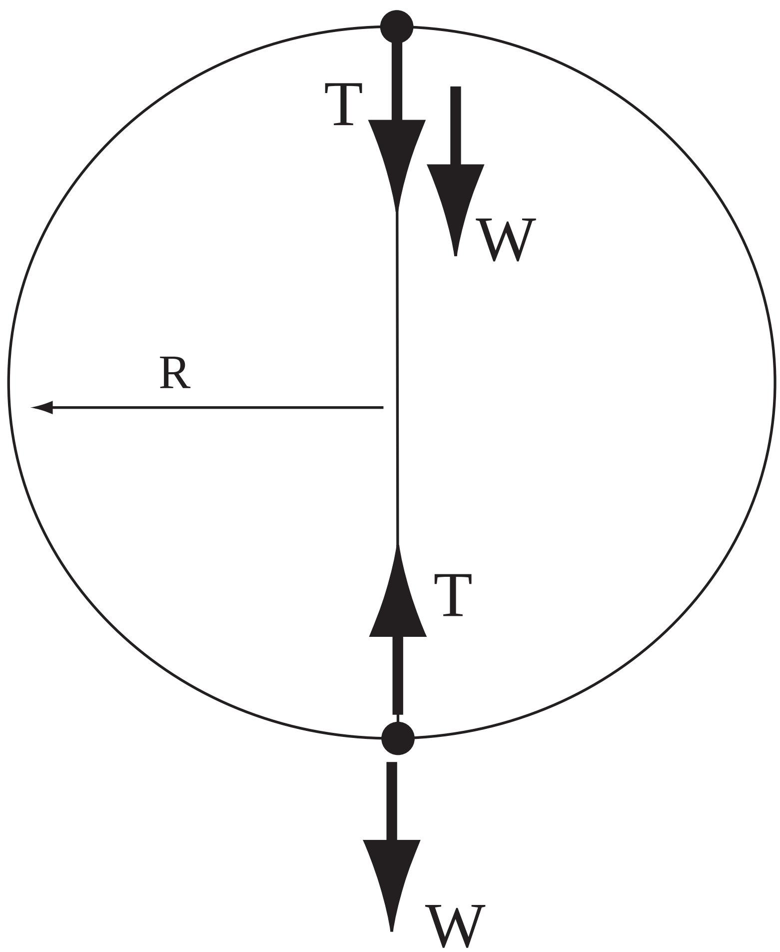
$$F\cos\theta - m_2 a - m_2 g - \mu(m_1 g - F\sin\theta) = m_1 a$$

$$F(\cos\theta + \mu\sin\theta) - g(m_2 + \mu m_1) = m_1 a + m_2 a$$

$$a = \frac{F(\cos\theta + \mu\sin\theta) - g(m_2 + \mu m_1)}{m_1 + m_2}$$

5. If you whirl an object of mass m at the end of a string in a vertical circle of radius R at constant speed v, derive a formula for the tension in the string at the top and bottom of the circle.

<u>SOLUTION</u>



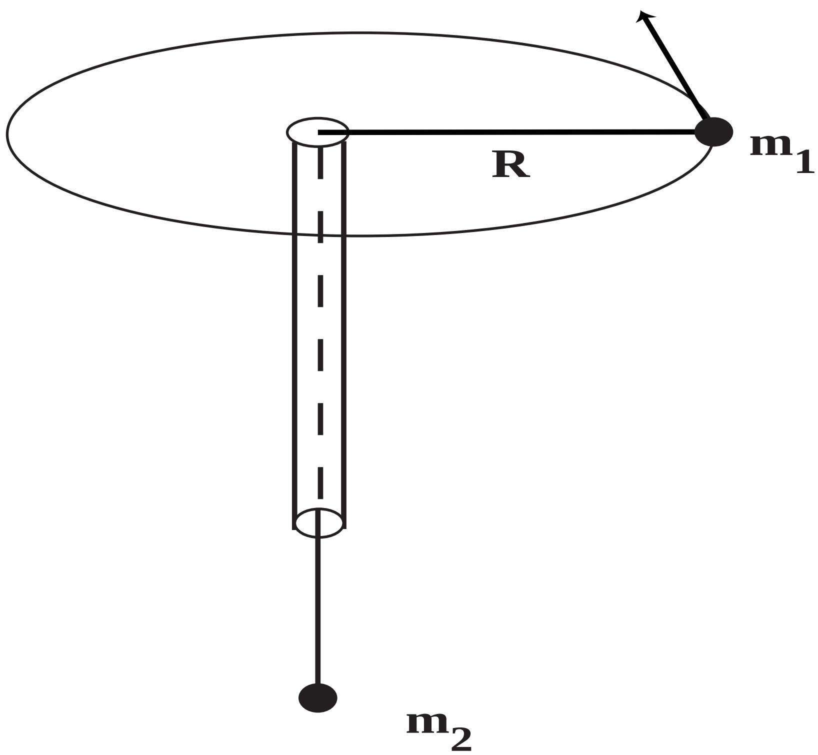
Bottom:

$$\begin{aligned}
\Sigma F_y &= ma_y \\
T - W &= \frac{mv^2}{R} \\
T &= W + \frac{mv^2}{R} \\
T &= mg + \frac{mv^2}{R}
\end{aligned}$$

Top:

$$\Sigma F_y = ma_y$$
$$T + W = \frac{mv^2}{R}$$
$$T = \frac{mv^2}{R} - W$$
$$T = \frac{mv^2}{R} - mg$$

6. Two masses m_1 and m_2 are connected by a string passing through a hollow pipe with m_1 being swung around in a circle of radius R and m_2 hanging vertically as shown in the figure.



Obviously if m_1 moves quickly in the circle then m_2 will start to move upwards, but if m_1 moves slowly m_2 will start to fall.

A) Derive an expression for the tension T in the string.

B) Derive an expression for the acceleration of m_2 in terms of the period t of the circular motion.

C) For what period t, will the mass m_2 be at rest?

D) If the masses are equal, what is the answer to Part C)?

E) For a radius of 9.81 m, what is the numerical value of this period?

SOLUTION

Forces on m_2:

$$\sum F_y = m_2 a_y$$

$$T - W_2 = m_2 a$$

Forces on m_1:

$$\sum F_x = m_1 a_x$$

$$T = m_1 \frac{v^2}{R} = \frac{m_1 (2\pi R/t)^2}{R} = \frac{m_1 4\pi^2 R}{t^2}$$

where we have chosen $m_2 a$ and T with the same sign.

Substituting we obtain

$$\frac{m_1 4\pi^2 R}{T^2} - m_2 g = m_2 a$$

giving the acceleration as

$$a = \frac{m_1}{m_2}\frac{4\pi^2 R}{t^2} - g$$

The acceleration will be zero if

$$\frac{m_1 4\pi^2 R}{m_2 t^2} = g$$

i.e.

$$t^2 = \frac{m_1}{m_2}\frac{4\pi^2 R}{g}$$

or

$$t = 2\pi\sqrt{\frac{m_1}{m_2}\frac{R}{g}}$$

D) If

$$m_1 = m_2 \Rightarrow \boxed{t = 2\pi\sqrt{\frac{R}{g}}}$$

for $R = 9.81$ m

$$\Rightarrow t = 2\pi\sqrt{\frac{9.81 \text{ m}}{9.81 \text{ m sec}^{-2}}} = 2\pi\sqrt{\text{sec}^2} = 2\pi \text{ sec}$$

7. A) What friction force is required to stop a block of mass m moving at speed v_0, assuming that we want the block to stop over a distance d ?

B) Work out a formula for the coefficient of kinetic friction that will achieve this.

C) Evaluate numerical answers to the above two questions assuming the mass of the block is $1000kg$, the initial speed is $60\,km\,per\,hour$ and the braking distance is $200m$.

SOLUTION

A) We have: $v = 0 \quad x_0 = 0$

$$\begin{aligned} v^2 &= v_0^2 + 2a(x - x_0) \\ 0 &= v_0^2 + 2a(d - 0) \\ \Rightarrow v_0^2 &= -2ad \\ \Rightarrow a &= -\frac{v_0^2}{2d} \end{aligned}$$

which gives the force as

$$F = ma = -\frac{mv_0^2}{2d}$$

B) The friction force can also be written

$$F = \mu_k N = \mu_k mg = \frac{mv_0^2}{2d}$$
$$\Rightarrow \mu_k = \frac{v_0^2}{2dg}$$

C) The force is

$$\begin{aligned}
F &= -\frac{mv_0^2}{2d} \\
&= -\frac{1000kg \times (60 \times 10^3 m\ hour^{-1})^2}{2 \times 200m} \\
&= -\frac{1000kg \times (60 \times 10^3 m)^2}{2 \times 200m \times (60 \times 60 sec)^2} \\
&= -694\ \frac{kg\ m}{sec^2} \\
&= -694\ Newton
\end{aligned}$$

The coefficient of kinetic friction is

$$\begin{aligned}
\mu_k &= \frac{v_0^2}{2dg} \\
&= \frac{(60 \times 10^3\ m\ hour^{-1})^2}{2 \times 200\ m \times 9.81\ m\ sec^{-2}} \\
&= \frac{(60 \times 10^3\ m)^2}{2 \times 200\ m \times 9.81\ m^2\ sec^{-2} \times (60 \times 60\ sec)^2} \\
&= 0.07
\end{aligned}$$

which has no units.

Chapter 6

KINETIC ENERGY & WORK

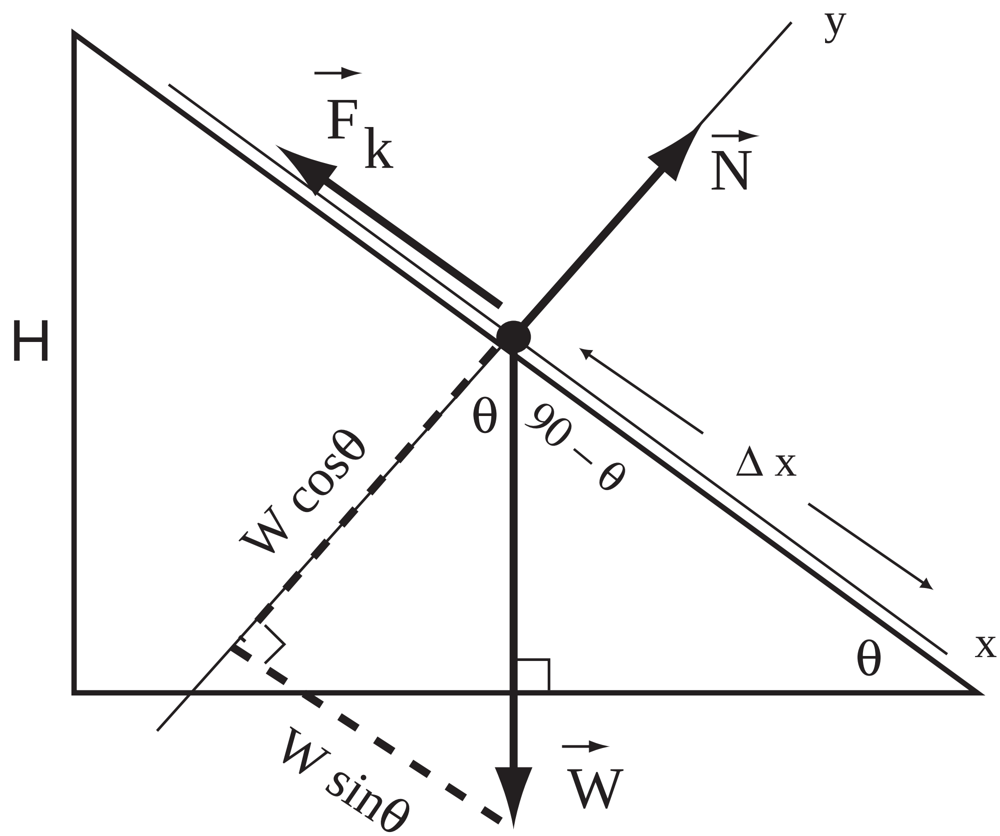
Chapter 7

POTENTIAL ENERGY & CONSERVATION OF ENERGY

1. A block of mass m slides down a rough incline of height H and angle θ to the horizontal. Calculate the speed of the block when it reaches the bottom of the incline, assuming the coefficient of kinetic friction is μ_k.

SOLUTION

The situation is shown in the figure.



The work-energy theorem is

$$\Delta U + \Delta K = W_{NC}$$
$$= U_f - U_i + K_f - K_i$$

but $U_f = 0$ and $K_i = 0$ giving

$$K_F = U_i + W_{NC}$$

Obviously W_{NC} must be negative so that $K_f < U_i$

$$\begin{aligned} K_F &= U_i - F_k \Delta x \qquad where \qquad \Delta x = \frac{H}{\sin\theta} \\ \frac{1}{2}mv^2 &= mgH - \mu_k N \frac{H}{\sin\theta} \end{aligned}$$

where we have used $F_k = \mu_k N$. To get N use Newton's law

$$\begin{aligned} F &= ma \\ N - W\cos\theta &= 0 \\ N &= W\cos\theta \\ &= mg\cos\theta \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{1}{2}mv^2 &= mgH - \mu_k mg\cos\theta \frac{H}{\sin\theta} \\ v^2 &= 2gH - 2\mu_k g \frac{H}{\tan\theta} \\ &= 2gH(1 - \frac{\mu_k}{\tan\theta}) \end{aligned}$$

$$v = \sqrt{2gH(1 - \frac{\mu_k}{\tan\theta})}$$

Chapter 8

SYSTEMS OF PARTICLES

1. A particle of mass m is located on the x axis at the position $x = 1$ and a particle of mass $2m$ is located on the y axis at position $y = 1$ and a third particle of mass m is located off-axis at the position $(x, y) = (1, 1)$. What is the location of the center of mass?

<u>SOLUTION</u>

The position of the center of mass is

$$\vec{r}_{\rm cm} = \frac{1}{M}\sum_i m_i \vec{r}_i$$

with $M \equiv \sum_i m_i$. The x and y coordinates are

$$\begin{aligned} x_{\rm cm} &= \frac{1}{M}\sum_i m_i x_i \\ &= \frac{1}{m+2m+m}\times \\ &\times(m\times 1 + 2m\times 0 + m\times 1) \\ &= \frac{1}{4m}(m+0+m) = \frac{2m}{4m} \\ &= \frac{1}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} y_{\rm cm} &= \frac{1}{M}\sum_i m_i y_i \\ &= \frac{1}{m+2m+m}\times \\ &\times(m\times 0 + 2m\times 1 + m\times 1) \\ &= \frac{1}{4m}(0+2m+m) = \frac{3m}{4m} \\ &= \frac{3}{4} \end{aligned}$$

Thus the coordinates of the center of mass are

$$(x_{\rm cm}, y_{\rm cm}) = \left(\frac{1}{2}, \frac{3}{4}\right)$$

2. Consider a square flat table-top. Prove that the center of mass lies at the center of the table-top, assuming a constant mass density.

<u>SOLUTION</u>

Let the length of the table be L and locate it on the x–y axis so that one corner is at the origin and the x and y axes lie along the sides of the table. Assuming the table has a constant area mass density σ, locate the position of the center of mass.

$$\begin{aligned}
x_{\text{cm}} &= \frac{1}{M}\sum_i m_i x_i = \frac{1}{M}\int x\,dm \\
&= \frac{1}{M}\int x\,\sigma dA \quad \text{with } \sigma = \frac{dm}{dA} = \frac{M}{A} \\
&= \frac{\sigma}{M}\int x\,dA \quad \text{if } \sigma \text{ is constant} \\
&= \frac{1}{A}\int_0^L \int_0^L x\,dx\,dy \quad \text{with } A = L^2 \\
&= \frac{1}{A}\left[\frac{1}{2}x^2\right]_0^L [y]_0^L \\
&= \frac{1}{A}\frac{1}{2}L^2 \times L = \frac{L^3}{2A} = \frac{L^3}{2L^2} \\
&= \frac{1}{2}L
\end{aligned}$$

and similarly for

$$y_{\text{cm}} = \frac{\sigma}{M}\int y\,dA = \frac{1}{2}L$$

Thus

$$(x_{\text{cm}}, y_{\text{cm}}) = \left(\frac{1}{2}L, \frac{1}{2}L\right) \quad \text{as expected}$$

3. A child of mass m_c is riding a sled of mass m_s moving freely along an icy frictionless surface at speed v_0. If the child falls off the sled, derive a formula for the change in speed of the sled. (Note: energy is not conserved !) WRONG WRONG WRONG ??????????????
speed of sled remains same - person keeps moving when fall off ???????

<u>SOLUTION</u>

Conservation of momentum in the x direction is

$$\sum p_{ix} = \sum p_{fx}$$

$$(m_c + m_s)v_0 = m_s v$$

where v is the new final speed of the sled, or

$$v = \left(1 + \frac{m_c}{m_s}\right) v_0$$

the change in speed is

$$v - v_0 = \frac{m_c}{m_s} v_0$$

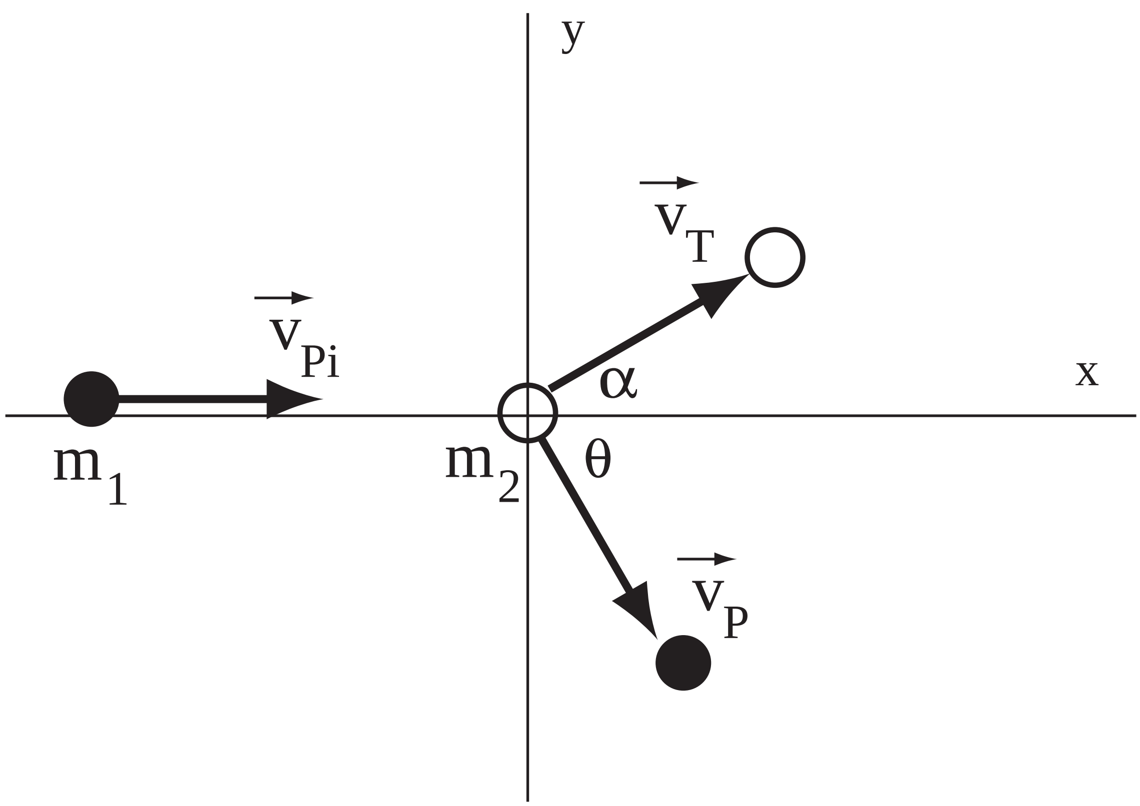
which will be large for small m_s or large m_c.

Chapter 9

COLLISIONS

1. In a game of billiards, the player wishes to hit a stationary target ball with the moving projectile ball. After the collision, show that the sum of the scattering angles is 90^o. Ignore friction and rolling motion and assume the collision is elastic. Also both balls have the same mass.

SOLUTION The collision occurs as shown in the figure. We have $m_1 = m_2 \equiv m$.



Momentum conservation is:

$$\vec{p}_{Pi} = \vec{p}_P + \vec{p}_T$$

and we break this down into the x and y directions. Momentum conservation in the y direction is:

$$\begin{aligned} 0 &= m\, v_T \sin\alpha - m\, v_P \sin\theta \\ v_P \sin\theta &= v_T \sin\alpha \end{aligned}$$

Momentum conservation in the x direction is:

$$\begin{aligned} m\, v_{Pi} &= m\, v_T \cos\alpha + m\, v_P \cos\theta \\ v_{Pi} &= v_T \cos\alpha + v_P \cos\theta \end{aligned}$$

Energy conservation is:

$$\begin{aligned} \frac{1}{2} m\, v_{Pi}^2 &= \frac{1}{2} m\, v_P^2 + \frac{1}{2} m\, v_T^2 \\ v_{Pi}^2 &= v_P^2 + v_T^2 \end{aligned}$$

We now have 3 simultaneous equations which can be solved. This involves a fair amount of algebra. We can do the problem much quicker by using the *square* of the momentum conservation equation. Use the notation $\vec{A}.\vec{A} \equiv A^2$

$$\begin{aligned} \vec{p}_{Pi} &= \vec{p}_P + \vec{p}_T \\ \Rightarrow \quad p_{Pi}^2 = (\vec{p}_P + \vec{p}_T)^2 & \\ &= (\vec{p}_P + \vec{p}_T).(\vec{p}_P + \vec{p}_T) \\ &= p_P^2 + p_T^2 + 2 p_T p_P \cos(\theta + \alpha) \end{aligned}$$

but the masses cancel out, giving

$$v_{Pi}^2 = v_P^2 + v_T^2 + 2 v_P v_T \cos(\theta + \alpha)$$

which, from energy conservation, *also* equals

$$v_{Pi}^2 = v_P^2 + v_T^2$$

implying that

$$\cos(\theta + \alpha) = 0$$

which means that

$$\theta + \alpha = 90^o$$

Chapter 10

ROTATION

1. Show that the ratio of the angular speeds of a pair of coupled gear wheels is in the inverse ratio of their respective radii. [WS 13-9]

SOLUTION

2. Consider the *point of contact* of the two coupled gear wheels. At that point the tangential velocity of a point on each (touching) wheel must be the same.

$$v_1 = v_2$$

$$\Rightarrow \quad r_1\omega_1 = r_2\omega_2$$
$$\Rightarrow \quad \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}$$

3. Show that the magnitude of the total linear acceleration of a point moving in a circle of radius r with angular velocity ω and angular acceleration α is given by $a = r\sqrt{\omega^4 + \alpha^2}$ [WS 13-8]

SOLUTION

The total linear acceleration is given by a vector sum of the radial and tangential accelerations

$$a = \sqrt{a_t^2 + a_r^2}$$

where the radial (centripetal) aceleration is

$$a_r = \frac{v^2}{r} = \omega^2 r$$

and

$$a_t = r\alpha$$

so that

$$a = \sqrt{r^2\alpha^2 + \omega^4 r^2} = r\sqrt{\omega^4 + \alpha^2}$$

4. The turntable of a record player rotates initially at a rate of 33 revolutions per minute and takes 20 seconds to come to rest. How many rotations does the turntable make before coming to rest, assuming constant angular deceleration ?

SOLUTION

$$\begin{aligned}\omega_0 &= 33\frac{rev}{min} = 33\frac{2\pi\ radians}{min} = 33\frac{2\pi\ rad}{60\ sec} \\ &= 3.46\ rad\ sec^{-1}\end{aligned}$$

$$\begin{aligned}\omega &= 0 \\ t &= 20\ sec\end{aligned}$$

$$\begin{aligned}\Delta\theta &= \frac{\omega + \omega_0}{2}\, t = \frac{3.46\ rad\ sec^{-1}}{2}\ 20\ sec \\ &= 34.6\ radian\end{aligned}$$

$$number\ of\ rotations = \frac{34.6\ radian}{2\pi radian} = 5.5$$

5. A cylindrical shell of mass M and radius R rolls down an incline of height H. With what speed does the cylinder reach the bottom of the incline ? How does this answer compare to just dropping an object from a height H ?

<u>SOLUTION</u>

Conservation of energy is

$$K_i + U_i = K_f + U_f$$
$$0 + mgH = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

For a cylindrical shell $I = mR^2$. Thus

$$mgH = \frac{1}{2}mv^2 + \frac{1}{2}mR^2\omega^2$$

and $v = r\omega$ giving (with m cancelling out)

$$\begin{aligned} gH &= \frac{1}{2}v^2 + \frac{1}{2}R^2(\frac{v}{R})^2 \\ &= \frac{1}{2}v^2 + \frac{1}{2}v^2 \\ &= v^2 \\ \Rightarrow \quad v &= \sqrt{gH} \end{aligned}$$

If we just drop an object then $mgH = \frac{1}{2}mv^2$ and $v = \sqrt{2gH}$. Thus the dropped object has a speed $\sqrt{2}$ times greater than the rolling object. This is because some of the potential energy has been converted into rolling kinetic energy.

6. Four point masses are fastened to the corners of a frame of negligible mass lying in the xy plane. Two of the masses lie along the x axis at positions $x = +a$ and $x = -a$ and are both of the same mass M. The other two masses lie along the y axis at positions $y = +b$ and $y = -b$ and are both of the same mass m.

A) If the rotation of the system occurs about the y axis with an angular velocity ω, find the moment of inertia about the y axis and the rotational kinetic energy about this axis.

B) Now suppose the system rotates in the xy plane about an axis through the origin (the z axis) with angular velocity ω. Calculate the moment of inertia about the z axis and the rotational kinetic energy about this axis. [Serway, 3rd ed., pg. 151]

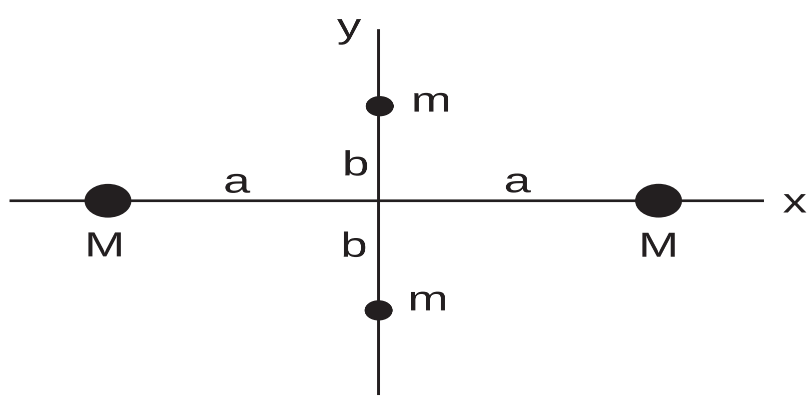
SOLUTION

A) The masses are distributed as shown in the figure. The rotational inertia about the y axis is

$$I_y = \sum_i r_i^2 m_i = a^2 M + (-a)^2 M = 2Ma^2$$

(The m masses don't contribute because their distance from the y axis is 0.) The kinetic energy about the y axis is

$$K_y = \frac{1}{2} I\omega^2 = \frac{1}{2} 2Ma^2\omega^2 = Ma^2\omega^2$$



B) The rotational inertia about the z axis is

$$\begin{aligned} I_z &= \sum_i r_i^2 m_i \\ &= a^2 M + (-a)^2 M + b^2 m + (-b)^2 m \\ &= 2Ma^2 + 2mb^2 \end{aligned}$$

The kinetic energy about the z axis is

$$\begin{aligned} K_z &= \frac{1}{2} I \omega^2 \\ &= \frac{1}{2}(2Ma^2 + 2mb^2)\omega^2 \\ &= (Ma^2 + mb^2)\omega^2 \end{aligned}$$

7. A uniform object with rotational inertia $I = \alpha m R^2$ rolls without slipping down an incline of height H and inclination angle θ. With what speed does the object reach the bottom of the incline? What is the speed for a hollow cylinder ($I = mR^2$) and a solid cylinder ($I = \frac{1}{2}MR^2$)? Compare to the result obtained when an object is simply dropped from a height H.

SOLUTION

The total kinetic energy is (with $v = \omega R$)

$$\begin{aligned} K &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}\alpha m R^2 \left(\frac{v}{R}\right)^2 = (1+\alpha)\frac{1}{2}mv^2 \end{aligned}$$

Conservation of energy is

$$K_i + U_i = K_f + U_f$$

$$O + mgH = (1+\alpha)\frac{1}{2}mv^2 + O$$

$$\Rightarrow v = \sqrt{\frac{2gH}{1+\alpha}}$$

For a hollow cylinder $I = mR^2$, i.e. $\alpha = 1$ and $v = \sqrt{gH}$.

For a solid cylinder $I = \frac{1}{2}mR^2$, i.e. $\alpha = \frac{1}{2}$ and $v = \sqrt{\frac{4}{3}gH}$

When $\alpha = 0$, we get the result for simply dropping an object, namely $v = \sqrt{2gH}$.

8. A pencil of length L, with the pencil point at one end and an eraser at the other end, is initially standing vertically on a table with the pencil point on the table. The pencil is let go and falls over. Derive a formula for the speed with which the eraser strikes the table, assuming that the pencil point does not move. [WS 324]

SOLUTION

The center of mass of the pencil (of mass m) is located half-way up at a height of $L/2$. Using conservation of energy

$$\frac{1}{2}I\omega^2 = mg\;L/2$$

where ω is the final angular speed of the pencil. We need to calculate I for a uniform rod (pencil) about an axis at one end. This is

$$I = \int r^2 dm = \int r^2\,\rho\,dV$$

where $dV = Adr$ with A being the cross-sectional area of the rod (pencil). Thus

$$\begin{aligned} I &= \rho \int r^2 A dr = \rho A \int_0^L r^2 dr \\ &= \rho A \left[\frac{1}{3}r^3\right]_0^L = \rho A\, L^3/3 \end{aligned}$$

The density is $\rho = \frac{m}{V} = \frac{m}{AL}$ giving

$$I = \frac{m}{AL}A\frac{L^3}{3} = \frac{1}{3}mL^2$$

We put this into the conservation of energy equation

$$\begin{aligned} \frac{1}{2}\frac{1}{3}mL^2\omega^2 &= mg\frac{L}{2} \\ \Rightarrow\;\; \frac{1}{3}L\omega^2 &= g \end{aligned}$$

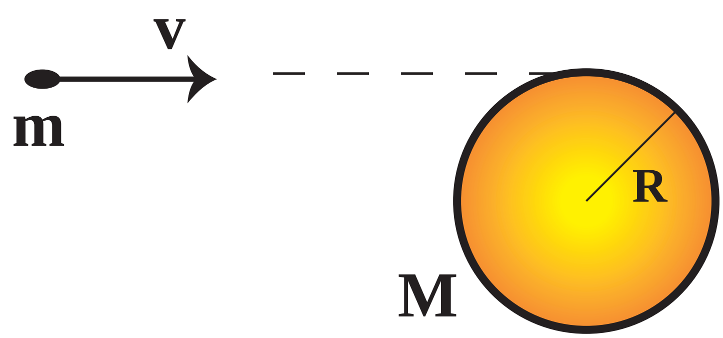
Now for the eraser $v = L\omega$, so that

$$\begin{aligned} \frac{1}{3}L\frac{v^2}{L^2} &= g \\ \Rightarrow\;\; \frac{v^2}{3L} &= g \\ \Rightarrow\;\; v &= \sqrt{3gL} \end{aligned}$$

Chapter 11

ROLLING, TORQUE & ANGULAR MOMENTUM

1. A bullet of mass m travelling with a speed v is shot into the rim of a solid circular cylinder of radius R and mass M as shown in the figure. The cylinder has a fixed horizontal axis of rotation, and is originally at rest. Derive a formula for the angular speed of the cylinder after the bullet has become imbedded in it. (Hint: The rotational inertia of a solid cylinder about the center axis is $I = \frac{1}{2}MR^2$). [WS354-355]



SOLUTION

Conservation of angular momentum is

$$L_i = L_f$$

The initial angular momentum is just that of the bullet, with magnitude $L_i = mvR$. Thus

$$mvR = I\omega$$

where the final rotational inertial I is due to the spinning cylinder *and* the bullet, namely

$$I = \frac{1}{2}MR^2 + mR^2$$

Thus

$$mvR = \left(\frac{1}{2}M + m\right)R^2\omega$$

giving

$$\omega = \frac{mv}{\left(\frac{1}{2}M + m\right)R}$$

Chapter 12

OSCILLATIONS

1. An object of mass m oscillates on the end of a spring with spring constant k. Derive a formula for the time it takes the spring to stretch from its equilibrium position to the point of maximum extension. Check that your answer has the correct units.

SOLUTION

The frequency of a spring, with mass m on one end is

$$\omega = \sqrt{\frac{k}{m}} \qquad and \qquad \omega = \frac{2\pi}{T}$$

The time for one complete cycle is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The time for a quarter cycle is

$$\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{m}{k}}$$

Check units:

The units of k are $N\,m^{-1}$ (because $F = -kx$ for a spring). Thus the units of $\sqrt{\frac{m}{k}}$ are

$$\sqrt{\frac{kg}{N\ m^{-1}}} = \sqrt{\frac{kg}{kg\ m\ sec^{-2}\ m^{-1}}} = \sqrt{sec^2} = sec$$

which are the correct units for the time $\frac{T}{4}$.

2. An object of mass m oscillates at the end of a spring with spring constant k and amplitude A. Derive a formula for the speed of the object when it is at a distance d from the equilibrium position. Check that your answer has the correct units.

SOLUTION

Conservation of energy is

$$U_i + K_i = U_f + K_f$$

with $U = \frac{1}{2}kx^2$ for a spring. At the point of maximum extension $x = A$ and $v = 0$ giving

$$\begin{aligned} \frac{1}{2}kA^2 + 0 &= \frac{1}{2}kd^2 + \frac{1}{2}mv^2 \\ mv^2 &= k(A^2 - d^2) \\ v &= \sqrt{\frac{k}{m}(A^2 - d^2)} \end{aligned}$$

Check units:

The units of k are $N\,m^{-1}$ (because $F = -kx$ for a spring). Thus the units of $\sqrt{\frac{k}{m}(A^2 - d^2)}$ are

$$\sqrt{\frac{N\ m^{-1}\ m^2}{kg}} = \sqrt{\frac{kg\ m\ sec^{-2}\ m^{-1}\ m^2}{kg}} = \sqrt{m^2\ sec^{-2}} = m\ sec^{-1}$$

which are the correct units for speed v.

3. A block of mass m is connected to a spring with spring constant k, and oscillates on a horizontal, frictionless surface. The other end of the spring is fixed to a wall. If the amplitude of oscillation is A, derive a formula for the speed of the block as a function of x, the displacement from equilibrium. (Assume the mass of the spring is negligible.)

SOLUTION

The position as a function of time is

$$x = A\cos\omega t$$

with $\omega = \sqrt{\frac{k}{m}}$. The speed is

$$v = \frac{dx}{dt} = -A\omega\sin\omega t$$

giving the total energy

$$\begin{aligned} E &= K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}mA^2\omega^2\sin^2\omega t + \frac{1}{2}kA^2\cos^2\omega t \\ &= \frac{1}{2}mA^2\frac{k}{m}\sin^2\omega t + \frac{1}{2}kA^2\cos^2\omega t \\ &= \frac{1}{2}kA^2(\sin^2\omega t + \cos^2\omega t) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

(Alternative derivation:
$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$; when $v = 0$, $x = A \Rightarrow E = \frac{1}{2}kA^2$).

The energy is constant and always has this value. Thus

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{1}{2}kA^2 - \frac{1}{2}kx^2 \\ v^2 &= \frac{k}{m}(A^2 - x^2) \\ v &= \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \end{aligned}$$

4. A particle that hangs from a spring oscillates with an angular frequency ω. The spring-particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car), as the car descends at a constant speed v. The car then stops suddenly. Derive a formula for the amplitude with which the particle oscillates. (Assume the mass of the spring is negligible.) [Serway, 5th ed., pg. 415, Problem 14]

SOLUTION

The total energy is

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

When $v = 0$, $x = A$ giving

$$E = \frac{1}{2}kA^2$$

which is a constant and is the constant value of the total energy always. For the spring in the elevator we have the speed $= v$ when $x = 0$. Thus

$$\begin{aligned} E = \frac{1}{2}kA^2 &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}mv^2 + O \end{aligned}$$

Thus

$$A^2 = \frac{m}{k}v^2$$

but $\omega = \sqrt{\frac{k}{m}}$ giving $\omega^2 = \frac{k}{m}$ or $\frac{m}{k} = \frac{1}{\omega^2}$

$$\begin{aligned} A^2 &= \frac{v^2}{\omega^2} \\ A &= \frac{v}{\omega} \end{aligned}$$

5. A large block, with a second block sitting on top, is connected to a spring and executes horizontal simple harmonic motion as it slides across a frictionless surface with an angular frequency ω. The coefficient of static friction between the two blocks is μ_s. Derive a formula for the maximum amplitude of oscillation that the system can have if the upper block is not to slip. (Assume that the mass of the spring is negligible.) [Serway, 5th ed., pg. 418, Problem 54]

SOLUTION

Consider the upper block (of mass m),

$$\begin{aligned} F &= ma \\ &= \mu_s N \\ &= \mu_s m g \end{aligned}$$

so that the maximum acceleration that the upper block can experience without slipping is

$$a = \mu_s g$$

the acceleration of the whole system is (with the mass of the lower block being M)

$$\begin{aligned} F &= (M+m)a \\ &= -kx \end{aligned}$$

The maximum acceleration occurs when x is maximum,
i.e. x = amplitude = A, giving the magnitude of a as

$$a = \frac{kA}{M+m}$$

But $\omega = \sqrt{\frac{k}{M+m}}$ giving $a = A\omega^2 = \mu_s g$, i.e.

$$A = \frac{\mu_s g}{\omega^2}$$

6. A simple pendulum consists of a ball of mass M hanging from a uniform string of mass m, with $m \ll M$ (m is much smaller than M). If the period of oscillation for the pendulum is T, derive a formula for the speed of a transverse wave in the string when the pendulum hangs at rest. [Serway, 5th ed., pg. 513, Problem 16]

SOLUTION

The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum. The speed of a transverse wave on a string is

$$v = \sqrt{\frac{\tau}{\mu}}$$

where τ is the tension and μ is the mass per unit length. Newton's law gives (neglecting the mass of the string m)

$$F = Ma$$

$$\tau - Mg = 0$$

$$\tau = Mg$$

and the mass per unit length is

$$\mu = \frac{m}{L}$$

Thus

$$v = \sqrt{\frac{Mg}{m/L}} = \sqrt{\frac{MgL}{m}}$$

but $T^2 = 4\pi^2\frac{L}{g}$ or $L = \frac{T^2 g}{4\pi^2}$ giving

$$v = \sqrt{\frac{MgT^2g}{m4\pi^2}} = \frac{gT}{2\pi}\sqrt{\frac{M}{m}}$$

Chapter 13

WAVES - I

Chapter 14

WAVES - II

1. A uniform rope of mass m and length L is suspended vertically. Derive a formula for the time it takes a transverse wave pulse to travel the length of the rope.

   (Hint: First find an expression for the wave speed at any point a distance x from the lower end by considering the tension in the rope as resulting from the weight of the segment below that point.) [Serway, 5th ed., p. 517, Problem 59]

SOLUTION

Consider a point a distance x from the lower end, assuming the rope has a uniform linear mass density $\mu = \frac{m}{L}$. The mass below the point is

$$m = \mu x$$

and the weight of that mass will produce tension T in the rope above

$$T = mg = \mu x g$$

(This agrees with our expectation. The tension at the bottom of the rope ($x = 0$) is $T = 0$, and at the top of the rope ($x = L$) the tension is $T = \mu L g = mg$.)

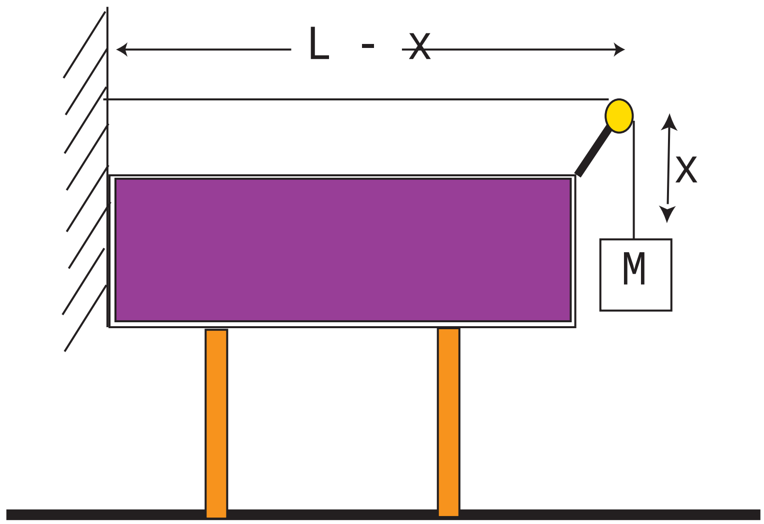
The wave speed is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\mu x g}{\mu}} = \sqrt{xg}$$

The speed is defined as $v \equiv \frac{dx}{dt}$ and the time is $dt = \frac{dx}{v}$. Integrate this to get the total time to travel the length of the rope

$$\begin{aligned} t &= \int_0^t dt = \int_0^L \frac{dx}{v} = \frac{1}{\sqrt{g}} \int_0^L \frac{dx}{\sqrt{x}} \\ &= \frac{1}{\sqrt{g}} \left[2x^{1/2} \right]_0^L \\ &= \frac{1}{\sqrt{g}} 2\sqrt{L} \\ &= 2\sqrt{\frac{L}{g}} \end{aligned}$$

2. A uniform cord has a mass m and a length L. The cord passes over a pulley and supports an object of mass M as shown in the figure. Derive a formula for the speed of a wave pulse travelling along the cord. [Serway, 5 ed., p. 501]



SOLUTION

The tension T in the cord is equal to the weight of the mass M or

$$\sum F = Ma$$

$$T - Mg = 0$$

$$T = Mg$$

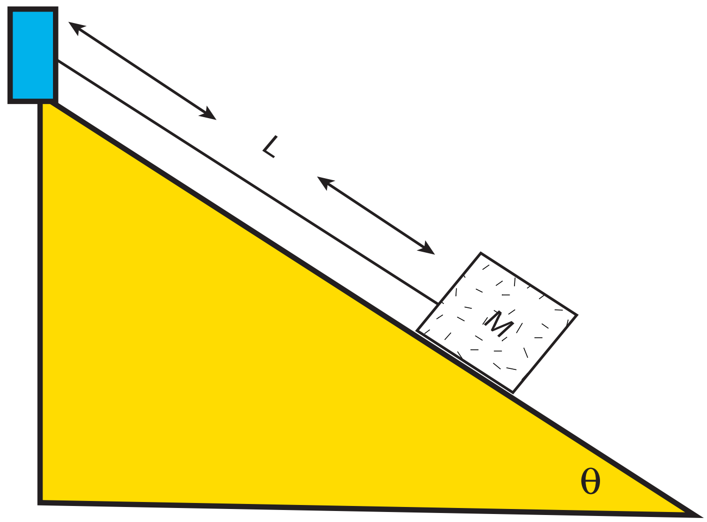
The wave speed is $v = \sqrt{\frac{T}{\mu}}$ where μ is the mass per unit length

$$\mu = \frac{m}{L}$$

Thus

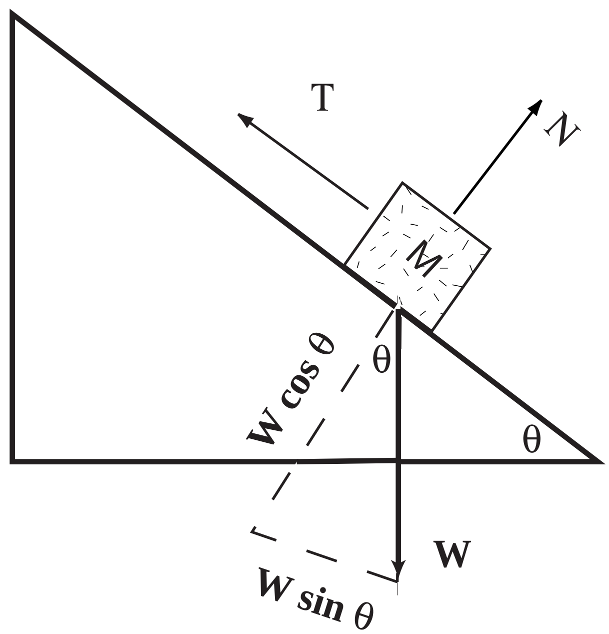
$$v = \sqrt{\frac{Mg}{m/L}} = \sqrt{\frac{MgL}{m}}$$

3. A block of mass M, supported by a string, rests on an incline making an angle θ with the horizontal. The string's length is L and its mass is $m \ll M$ (i.e. m is negligible compared to M). Derive a formula for the time it takes a transverse wave to travel from one end of the string to the other. [Serway, 5th ed., p. 516, Problem 53]



SOLUTION

The wave speed is given by $v = \sqrt{\frac{T}{\mu}}$ where T is the tension in the string and μ is the mass per unit length $\mu = \frac{m}{L}$. To get the tension, use Newton's laws as shown in the figure below.



Choose the x direction along the edge

$$\sum F_x = Ma_x$$

$$T - W\sin\theta = 0$$

$$T = W\sin\theta = Mg\sin\theta$$

where we have used the fact that $m \ll M$ so that the mass of the string does not affect the tension. Thus the wave speed is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Mg\sin\theta}{m/L}} = \sqrt{\frac{MgL\sin\theta}{m}}$$

To get the time t for the wave to travel from one end to the other, simply use $v = \frac{L}{t}$ giving

$$t = \frac{L}{v} = L\sqrt{\frac{m}{MgL\sin\theta}} = \sqrt{\frac{mL}{Mg\sin\theta}}$$

4. A stationary train emits a whistle at a frequency f. The whistle sounds higher or lower in pitch depending on whether the moving train is approaching or receding. Derive a formula for the *difference* in frequency Δf, between the approaching and receding train whistle in terms of u, the speed of the train, and v, the speed of sound. [Serway, 5th ed., p. 541, Problem 54]

SOLUTION The Doppler effect is summarized by

$$f' = f\frac{v \pm v_D}{v \mp v_s}$$

where f is the stationary frequency, f' is the observed frequency, v_D is the speed of the detector, v_s is the speed of the source and v is the speed of sound.

In this example $v_D = 0$. If the train is approaching the frequency increases, with $v_s \equiv u$, i.e.

$$f' = f\frac{v}{v-u}$$

and if the train recedes then the frequency decreases, i.e.

$$f'' = f\frac{v}{v+u}$$

The difference in frequencies is then

$$\begin{aligned}
\Delta f &= f' - f'' = f\left[\frac{v}{v-u} - \frac{v}{v+u}\right] \\
&= f\frac{v(v+u) - v(v-u)}{(v-u)(v+u)} = f\frac{v^2 + vu - v^2 + vu}{v^2 - u^2} \\
&= f\frac{2vu}{v^2 - u^2} \\
&= f\frac{2vu/v^2}{v^2/v^2 - u^2/v^2} = f\frac{2(u/v)}{1 - (u/v)^2}
\end{aligned}$$

Let $\beta \equiv \frac{u}{v}$. Thus

$$\Delta f = \frac{2\beta}{1-\beta^2}f$$

Chapter 15

TEMPERATURE, HEAT & 1ST LAW OF THERMODYNAMICS

1. The coldest that any object can ever get is 0 K (or -273 C). It is rare for physical quantities to have an upper or lower possible limit. Explain why temperature has this lower limit.

SOLUTION

From the kinetic theory of gases, the temperature (or pressure) depends on the speed with which the gas molecules are moving. The slower the molecules move, the lower the temperature. We can easily imagine the situation where the molecules are completely at rest and not moving at all. This corresponds to the coldest possible temperature (0 K), and the molecules obviously cannot get any colder.

2. Suppose it takes an amount of heat Q to make a cup of coffee. If you make 3 cups of coffee how much heat is required?

SOLUTION

The heat required is

$$Q = mc\Delta T$$

For fixed c and ΔT we have

$$Q \propto m$$

Thus if m increases by 3, then so will Q. Thus the heat required is $3Q$ (as one would guess).

3. How much heat is required to make a cup of coffee? Assume the mass of water is 0.1 kg and the water is initially at 0°C. We want the water to reach boiling point.
   Give your answer in Joule and calorie and Calorie.

   (1 cal = 4.186 J; 1 Calorie = 1000 calorie.

   For water: $c = 1\frac{\text{cal}}{\text{gC}} = 4186\frac{\text{J}}{\text{kg C}}$; $L_v = 2.26 \times 10^6\frac{\text{J}}{\text{kg}}$; $L_f = 3.33 \times 10^5\frac{\text{J}}{\text{kg}}$)

   SOLUTION

   The amount of heat required to change the temperature of water from 0°C to 100°C is

$$
\begin{aligned}
Q &= mc\,\Delta T \\
&= 0.1\,\text{kg} \times 4186\frac{\text{J}}{\text{kg}} \times 100\,\text{C} \\
&= 41860\,\text{J} = 41860\frac{1\,\text{cal}}{4.186} = 10,000\,\text{cal} \\
&= 10\,\text{Calorie}
\end{aligned}
$$

4. How much heat is required to change a 1 kg block of ice at −10°C to steam at 110°C ?
Give your answer in Joule and calorie and Calorie.

(1 cal = 4.186 J; 1 Calorie = 1000 calorie.
$c_{\text{water}} = 4186 \frac{\text{J}}{\text{kg C}}$; $c_{\text{ice}} = 2090 \frac{\text{J}}{\text{kg C}}$; $c_{\text{steam}} = 2010 \frac{\text{J}}{\text{kg C}}$
For water, $L_v = 2.26 \times 10^6 \frac{\text{J}}{\text{kg}}$; $L_f = 3.33 \times 10^5 \frac{\text{J}}{\text{kg}}$)

SOLUTION

To change the ice at −10°C to ice at 0°C the heat is

$$Q = mc\Delta T = 1\,\text{kg} \times 2090 \frac{\text{J}}{\text{kg C}} \times 10\text{C} = 20900\text{J}$$

To change the ice at 0°C to water at 0°C the heat is

$$Q = mL_f = 1\,\text{kg} \times 3.33 \times 10^5 \frac{\text{J}}{\text{kg}} = 3.33 \times 10^5\,\text{J}$$

To change the water at 0°C to water at 100°C the heat is

$$Q = mc\Delta T = 1\,\text{kg} \times 4186 \frac{\text{J}}{\text{kg C}} \times 100 = 418600\,\text{J}$$

To change the water at 100°C to steam at 100°C the heat is

$$Q = mL_v = 1\,\text{kg} \times 2.26 \times 10^6 \frac{\text{J}}{\text{kg}} = 2.26 \times 10^6\,\text{J}$$

To change the steam at 100°C to steam at 110°C the heat is

$$Q = mC\Delta T = 1\,\text{kg} \times 2010 \frac{\text{J}}{\text{kg C}} \times 10\,\text{C} = 20100\,\text{J}$$

The total heat is
$(20900 + 3.33 \times 10^5 + 418600 + 2.26 \times 10^6 + 20100)\text{J} = 3.0526 \times 10^6$ J

$$= 3.0526 \times 10^6 \frac{1\,\text{cal}}{4.186} = 7.29 \times 10^5\,\text{cal} = 729\,\text{Cal}$$

Chapter 16

KINETIC THEORY OF GASES

1.

A) If the number of molecules in an ideal gas is doubled, by how much does the pressure change if the volume and temperature are held constant?

B) If the volume of an ideal gas is halved, by how much does the pressure change if the temperature and number of molecules is constant?

C) If the temperature of an ideal gas changes from 200 K to 400 K, by how much does the volume change if the pressure and number of molecules is constant.

D) Repeat part C) if the temperature changes from 200 C to 400 C.

SOLUTION

The ideal gas law is

$$PV = nRT$$

where n is the number of moles and T is the temperature in Kelvin. This can also be written as

$$PV = NkT$$

where N is the number of molecules, k is Boltzmann's constant and T is still in Kelvin.

A) For V and T constant, then $P \propto N$. Thus P is doubled.

B) For T and N constant, then $P \propto \frac{1}{V}$. Thus P is doubled.

C) In the idea gas law T is in Kelvin. Thus the Kelvin temperature has doubled. For P and N constant, then $V \propto T$. Thus V is doubled.

D) We must first convert the Centigrade temperatures to Kelvin. The conversion is

$$K = C + 273$$

where K is the temperature in Kelvin and C is the temperature in Centigrade. Thus

$$200C = 473K$$

$$400C = 673K$$

Thus the Kelvin temperature changes by $\frac{673}{473}$. As in part C, we have $V \propto T$. Thus V changes by $\frac{673}{473} = 1.4$

2. If the number of molecules in an ideal gas is doubled and the volume is doubled, by how much does the pressure change if the temperature is held constant ?

SOLUTION

The ideal gas law is

$$PV = NkT$$

If T is constant then

$$P \propto \frac{N}{V}$$

If N is doubled and V is doubled then P does not change.

3. If the number of molecules in an ideal gas is doubled, and the absolute temperature is doubled and the pressure is halved, by how much does the volume change ?
(Absolute temperature is simply the temperature measured in Kelvin.)

SOLUTION

The ideal gas law is

$$PV = NkT$$

which implies

$$V \propto \frac{NT}{P}$$

If $N \to 2N$, $T \to 2T$ and $P \to \frac{1}{2}P$ then $V \to \frac{2\times 2}{1/2}V = 8V$.

Thus the volume increases by a factor of 8.

Chapter 17

Review of Calculus

1. Calculate the derivative of $y(x) = 5x + 2$.

SOLUTION

$y(x) = 5x + 2$
$y(x + \Delta x) = 5(x + \Delta x) + 2 = 5x + 5\Delta x + 2$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{5x + 5\Delta x + 2 - (5x + 2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} 5 \\
&= 5 \quad \text{as expected because the slope} \\
&\qquad \text{of the straight line } y = 5x + 2 \text{ is } 5.
\end{aligned}$$

2. Calculate the slope of the curve $y(x) = 3x^2 + 1$ at the points $x = -1$, $x = 0$ and $x = 2$.

SOLUTION

$$\begin{aligned} y(x) &= 3x^2 + 1 \\ y(x + \Delta x) &= 3(x + \Delta x)^2 + 1 \\ &= 3(x^2 + 2x\Delta x + \Delta x^2) + 1 \\ &= 3x^2 + 6x\Delta x + 3(\Delta x)^2 + 1 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 + 1 - (3x^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \to 0} (6x + 3\Delta x) \\ &= 6x \end{aligned}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -6$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 12$$

3. Calculate the derivative of x^4 using the formula $\frac{dx^n}{dx} = nx^{n-1}$. Verify your answer by calculating the derivative from $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x)-y(x)}{\Delta x}$.

SOLUTION

$$\begin{aligned} \frac{dx^n}{dx} &= nx^{n-1} \\ \therefore \qquad \frac{dx^4}{dx} &= 4x^{4-1} = 4x^3 \end{aligned}$$

Now let's verify this.

$$\begin{aligned} y(x) &= x^4 \\ y(x+\Delta x) &= (x+\Delta x)^4 \\ &= x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \to 0} [4x^3 + 6x^2\Delta x + 4x(\Delta x)^2 + (\Delta x)^3] \\ &= 4x^3 \quad \text{which agrees with above} \end{aligned}$$

4. Prove that $\frac{d}{dx}(3x^2) = 3\frac{dx^2}{dx}$.

SOLUTION

$y(x) = 3x^2$
$y(x + \Delta x) = 3(x + \Delta x)^2 = 3x^2 + 6x\Delta x + 3(\Delta x)^2$

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}(3x^2) &= \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 - 3x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 6x + 3\Delta x \\
&= 6x
\end{aligned}$$

Now take

$$y(x) = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

Thus

$$\begin{aligned}
\frac{d}{dx}(3x^2) &= 6x \\
&= 3\frac{d}{dx}x^2
\end{aligned}$$

5. Prove that $\frac{d}{dx}(x+x^2) = \frac{dx}{dx} + \frac{dx^2}{dx}$.

SOLUTION

Take $y(x) = x + x^2$

$$y(x+\Delta x) = x + \Delta x + (x+\Delta x)^2$$
$$= x + \Delta x + x^2 + 2x\Delta x + (\Delta x)^2$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x+x^2) = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x + \Delta x + x^2 + 2x\Delta x + (\Delta x)^2 - (x+x^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (1 + 2x + \Delta x) \\
&= 1 + 2x \\
\frac{dx}{dx} &= 1 \\
\frac{dx^2}{dx} &= 2x
\end{aligned}$$

$$\therefore \frac{d}{dx}(x+x^2) = \frac{dx}{dx} + \frac{dx^2}{dx}$$

6. Verify the chain rule and product rule using some examples of your own.

SOLUTION

your own examples

7. Where do the extremum values of $y(x) = x^2 - 4$ occur? Verify your answer by plotting a graph.

SOLUTION

$$
\begin{aligned}
&y(x) = x^2 - 4 \\
&0 = \frac{dy}{dx} = 2x \\
&\therefore x = 0 \\
&y(0) = 0 - 4 = -4 \\
&\therefore \text{extreme occurs at } (x, y) = (0, -4)
\end{aligned}
$$

The graph below shows this is a *minimum*.

8. Evaluate $\int x^2 dx$ and $\int 3x^3 dx$.

SOLUTION

$y = \int f\, dx$ with $f(x) \equiv \frac{dy}{dx}$ A) the derivative function is $f(x) = x^2 = \frac{dy}{dx}$. Thus the original function must be $\frac{1}{3}x^3 + c$. Thus

$$\int x^2 dx = \frac{1}{3}x^3 + c$$

B) the derivative function is $f(x) = 3x^3 = \frac{dy}{dx}$. Thus the original function must be $3\left(\frac{1}{4}x^4 + c\right)$. Thus

$$\begin{aligned} \int 3x^3 dx &= \frac{3}{4}x^4 + 3c \\ \text{or} &= \frac{3}{4}x^4 + c' \end{aligned}$$

where I have written $c' \equiv 3c$.

9. What is the area under the curve $f(x) = x$ between $x_1 = 0$ and $x_2 = 3$? Work out your answer i) graphically and ii) with the integral.

<u>SOLUTION</u>

$f(x) = x$

The area of the triangle between $x_1 = 0$ and $x_1 = 3$ is $\frac{1}{2}\times$ Base $\times$ Height $= \frac{1}{2} \times 3 \times 3 = 4.5$

$$\begin{aligned} \int_0^3 x\,dx = \left[\frac{1}{2}x^2 + c\right]_0^3 &= \left(\frac{1}{2}3^2 + c\right) - \left(\frac{1}{2}0^2 + c\right) \\ &= \left(\frac{9}{2} + c\right) - c \\ &= \frac{9}{2} = 4.5 \end{aligned}$$

in agreement with the graphical method.