ELEMENTARY MECHANICS & THERMODYNAMICS

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Contents

PREFACE

The reason for writing this book was due to the fact that modern introductory textbooks (not only in physics, but also mathematics, psychology, chemistry) are simply not useful to either students or instructors. The typical freshman textbook in physics, and other fields, is over 1000 pages long, with maybe 40 chapters and over 100 problems per chapter. This is overkill! A typical semester is 15 weeks long, giving 30 weeks at best for a year long course. At the *fastest* possible rate, we can "cover" only one chapter per week. For a year long course that is 30 chapters at best. Thus ten chapters of the typical book are left out! 1500 pages divided by 30 weeks is about 50 pages per week. The typical text is quite densed mathematics and physics and it's simply impossible for a student to read all of this in the detail required. Also with 100 problems per chapter, it's not possible for a student to do 100 problems each week. Thus it is impossible for a student to fully read and do all the problems in the standard introductory books. Thus these books are not useful to students or instructors teaching the typical course!

In defense of the typical introductory textbook, I *will* say that their content is usually excellent and very well writtten. They are certainly very fine *reference* books, but I believe they are poor *text* books. Now I know what publishers and authors say of these books. Students and instructors are supposed to only cover a selection of the material. The books are written so that an instructor can pick and choose the topics that are deemed best for the course, and the same goes for the problems. However I object to this. At the end of the typical course, students and instructors are left with a feeling of incompleteness, having usually covered only about half of the book and only about ten percent of the problems. I want a textbook that is self contained. As an instructor, I want to be able to comfortably cover one short chapter each week, and to have each student read the entire chapter and do every problem. I want to say to the students at the beginning of the course that they should read the entire book from cover to cover and do every problem. If they have done that, they will have a good knowledge of introductory physics.

This is why I have written this book. Actually it is based on the introductory physics textbook by Halliday, Resnick and Walker [*Fundamental of Physics*, 5th ed., by Halliday, Resnick and Walker, (Wiley, New York, 1997)], which is an outstanding introductory physics reference book. I had been using that book in my course, but could not cover it all due to the reasons listed above.

Availability of this eBook

At the moment this book is freely available on the world wide web and can be downloaded as a pdf file. The book is still in progress and will be updated and improved from time to time.

INTRODUCTION - What is Physics?

A good way to define physics is to use what philosophers call an ostensive definition, i.e. a way of defining something by pointing out examples.

Physics studies the following general topics, such as:
Motion (this semester)
Thermodynamics (this semester)
Electricity and Magnetism
Optics and Lasers
Relativity
Quantum mechanics
Astronomy, Astrophysics and Cosmology
Nuclear Physics
Condensed Matter Physics
Atoms and Molecules
Biophysics
Solids, Liquids, Gases
Electronics
Geophysics
Acoustics
Elementary particles
Materials science

Thus physics is a very fundamental science which explores nature from the scale of the tiniest particles to the behaviour of the universe and many things in between. Most of the other sciences such as biology, chemistry, geology, medicine rely heavily on techniques and ideas from physics. For example, many of the diagnostic instruments used in medicine (MRI, x-ray) were developed by physicists. All fields of technology and engineering are very strongly based on physics principles. Much of the electronics and computer industry is based on physics principles. Much of the communication today occurs via fiber optical cables which were developed from studies in physics. Also the World Wide Web was invented at the famous physics laboratory called the European Center for Nuclear Research (CERN). Thus anyone who plans to work in any sort of technical area needs to know the basics of physics. This is what an introductory physics course is all about, namely getting to know the basic principles upon which most of our modern technological society is based.

Chapter 1

MOTION ALONG A STRAIGHT LINE

SUGGESTED HOME EXPERIMENT:

Design a simple experiment which shows that objects of different weight fall at the same rate if the effect of air resistance is eliminated.

THEMES:

1. DRIVING YOUR CAR.
2. DROPPING AN OBJECT.

INTRODUCTION:

There are two themes we will deal with in this chapter. They concern DRIVING YOUR CAR and DROPPING AN OBJECT.

When you drive you car and go on a journey there are several things you are interested in. Typically these are **distance** travelled and the **speed** with which you travel. Often you want to know how long a journey will take if you drive at a certain speed over a certain distance. Also you are often interested in the **acceleration** of your car, especially for a very short journey such as a little speed race with you and your friend. You want to be able to accelerate quickly. In this chapter we will spend a lot of time studying the concepts of distance, speed and acceleration.

LECTURE DEMONSTRATION:

1) Drop a ball and hold at different heights; it goes faster at bottom if released from different heights

2) Drop a ball and a pen (different weights - weigh on balance and show they are different weight); both hit the ground at the same time

Another item of interest is what happens when an object is dropped from a certain height. If you drop a ball you know it starts off with zero speed and ends up hitting the ground with a large speed. Actually, if you think about it, that's a pretty amazing phenomenom. WHY did the speed of the ball increase ? You might say gravity. But what's that ? The speed of the ball increased, and therefore gravity provided an *acceleration*. But how ? Why ? When ?

We shall address all of these deep questions in this chapter.

1.1 Motion

Read.

1.2 Position and Displacement

In 1-dimension, *positions* are measured along the x-axis with respect to some *origin*. It is up to us to *define where to put the origin*, because the x-axis is just something we invented to put on top of, say a real landscape.

Example Chicago is 100 *miles* south of Milwaukee and Glendale is 10 *miles* north of Milwaukee.

A. If we define the origin of the x-axis to be at Glendale what is the *position* of someone in Chicago, Milwaukee and Glendale ?

B. If we define the origin of x-axis to be at Milwaukee, what is the *position* of someone in Chicago, Milwaukee and Glendale ?

Solution A. For someone in Chicago, $x = 110\, miles$.

For someone in Milwaukee, $x = 10\, miles$.

For someone in Glendale, $x = 0\, miles$.

B. For someone in Chicago, $x = 100\, miles$.

For someone in Milwaukee, $x = 0\, miles$.

For someone in Glendale, $x = -10\, miles$.

Displacement is defined as a *change in position*. Specifically,

$$\Delta x \equiv x_2 - x_1 \tag{1.1}$$

Note: We *always* write $\Delta anything \equiv anthing_2 - anything_1$ where $anything_2$ is the *final* value and $anything_1$ is the *initial* value. Sometimes you will instead see it written as $\Delta anything \equiv anthing_f - anything_i$ where subscripts f and i are used for the final and initial values instead of the 2 and 1 subscripts.

Example What is the displacement for someone driving from Milwaukee to Chicago ? What is the distance ?

Solution With the origin at Milwaukee, then the initial position is $x_1 = 0\, miles$ and the final position is $x_2 = 100\, miles$, so that $\Delta x = x_2 - x_1 = 100\, miles$. You get the same answer with the origin defined at Gendale. Try it.

The distance is also 100 *miles*.

Example What is the displacement for someone driving from Milwaukee to Chicago and back ? What is the distance ?

Solution With the origin at Milwaukee, then the initial position is $x_1 = 0\, miles$ and the final position is also $x_2 = 0\, miles$, so that $\Delta x = x_2 - x_1 = 0\, miles$. Thus there is *no* displacement if the beginning and end points are the same. You get the same answer with the origin defined at Gendale. Try it.

The distance is $200\, miles$.

Note that the *distance* is what the *odometer* on your car reads. The odometer does not read displacement (except if displacment and distance are the same, as is the case for a one way straight line journey).

Do Checkpoint 1 [from Halliday].

1.3 Average Velocity and Average Speed

Average velocity is *defined* as the ratio of *displacement* divided by the corresponding time interval.

$$\bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{1.2}$$

whereas *average speed* is just the *total distance* divided by the time interval,

$$\bar{s} \equiv \frac{total\ distance}{\Delta t} \tag{1.3}$$

Example What is the average velocity and averge speed for someone driving from Milwaukee to Chicago who takes 2 *hours* for the journey ?

Solution $\Delta x = 100\ miles$ and $\Delta t = 2\ hours$, giving $\bar{v} = \frac{100\ miles}{2\ hours} = 50\frac{miles}{hour} \equiv 50\ miles\ per\ hour \equiv 50\ mph$.

Note that the unit $\frac{miles}{hour}$ has been re-written as *miles per hour*. This is standard. We can always write any fraction $\frac{a}{b}$ as *a per b*. The word *per* just means *divide*.

The average speed is the same as average velocity in this case because the total distance is the same as the displacement. Thus $\bar{s} = 50\ mph$.

Example What is the average velocity and averge speed for someone driving from Milwaukee to Chicago and back to Milwaukee who takes 4 hours for the journey ?

Solution $\Delta x = 0\ miles$ and $\Delta t = 2\ hours$, giving $\bar{v} = 0$!

However the total distance is $200\,miles$ completed in 4 hours giving $\bar{s} = \frac{200\ miles}{4\ hours} = 50\ mph$ again.

A very important thing to understand is how to read graphs of position and time and graphs of velocity and time, and how to interpret such graphs. **It is very important to understand how the average velocity is obtained from a position-time graph**. See Fig. 2-4 in Halliday.

LECTURE DEMONSTRATION:
1) Air track glider standing still
2) Air track glider moving at constant speed.

Let's plot an x, t and v, t graph for
1) Object standing still,
2) Object at constant speed.

Note that the v, t graph is the *slope* of the x, t graph.

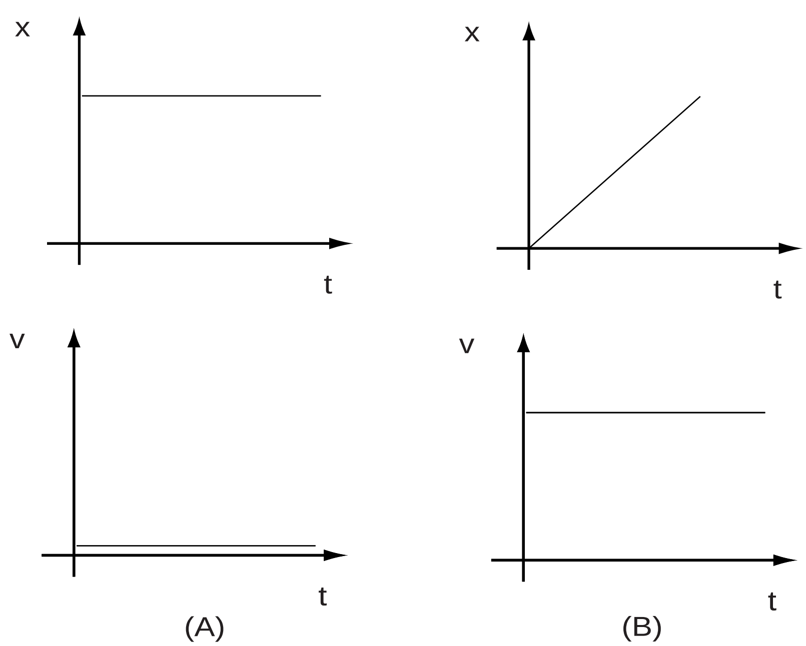


FIGURE 2.1 Position - time and Velocity - time graphs for A) object standing still and B) object moving at constant speed.

Careully study Sample Problems 2-1, 2-2, Checkpoint 2 and Sample Problem 2-3. [from Halliday]

1.4 Instantaneous Velocity and Speed

When you drive to Chicago with an average velocity of 50 *mph* you probably don't drive at this velocity the whole way. Sometimes you might pass a truck and drive at 70 *mph* and when you get stuck in the traffic jams you might only drive at 20 *mph*.

Now when the police use their radar gun and clock you at 70 *mph*, you might legitimately protest to the officer that your average velocity for the whole trip was only 50 *mph* and therefore you don't deserve a speeding ticket. However, as we all know police officers don't care about average velocity or average speed. They only care about your speed at the *instant* that you pass them. Thus let's introduce the concept of *instantaneous velocity* and *instantaneous speed*.

What is an *instant* ? It is nothing more than an extremely short time interval. The way to describe this mathematically is to say that an instant is when the time interval Δt approaches zero, or the *limit* of Δt as $\Delta t \to 0$ (approaches zero). We denote such a tiny time interval as dt instead of Δt. The corresponding distance that we travel over that tiny time interval will also be tiny and we denote that as dx instead of Δx.

Thus *instantaneous velocity* or just *velocity* is defined as

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{1.4}$$

Now such a fraction of one tiny dx divided by a tiny dt has a special name. It is called the *derivative* of x with respect to t.

The instantaneous speed or just speed is defined as simply the magnitude of the instantaneous veloctiy or magnitude of velocity.

Carefully study Sample Problem 2-4 [from Halliday].

1.5 Acceleration

We have seen that velocity tells us how quickly position changes. *Acceleration* tells us how much velocity changes. The *average acceleration* is defined as

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

and the *instantaneous acceleration* or just *acceleration* is defined as

$$a = \frac{dv}{dt}$$

Now because $v = \frac{dx}{dt}$ we can write $a = \frac{d}{dt}v = \frac{d}{dt}\left(\frac{dx}{dt}\right)$ which is often written instead as $\frac{d}{dt}\left(\frac{dx}{dt}\right) \equiv \frac{d^2x}{dt^2}$, that is the *second derivative* of position with respect to time.

Example When driving your car, what is your average acceleration if you are able to reach $20\,mph$ from rest in 5 seconds ?

Solution

$$\begin{aligned} v_2 &= 20 \text{ mph} \qquad & v_1 &= 0 \\ t_2 &= 5 \text{ seconds} \qquad & t_1 &= 0 \end{aligned}$$

$$\begin{aligned} \bar{a} &= \frac{20 \text{ mph} - 0}{5 \text{ sec} - 0} = \frac{20 \text{ miles per hour}}{5 \text{ seconds}} \\ &= 4 \, \frac{\text{miles}}{\text{hour seconds}} = 4 \; mph \; per \; sec \\ &= 4 \, \frac{\text{miles}}{\text{hour } \frac{1}{3600} \text{ hour}} = 14,400 \text{ miles per hour}^2 \end{aligned}$$

LECTURE DEMONSTRATION (previous demo continued):
1) Air track glider standing still
2) Air track glider moving at constant speed.

Now let's also plot an a, t graph for
1) Object standing still,
2) Object at constant speed.

Note that the the a, t graph is the slope of the v, t graph.

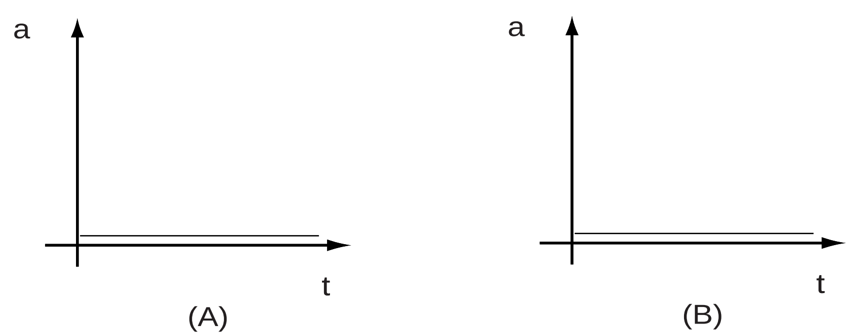


FIGURE 2.2 Acceleration-time graphs for motion depicted in Fig. 2.1.

1.6 Constant Acceleration: A Special Case

Velocity describes changing position and acceleration describes changing velocity. A quantity called *jerk* describes changing acceleration. However, very often the acceleration is *constant*, and we don't consider jerk. When driving your car the acceleration is usually constant when you speed up or slow down or put on the brakes. (When you slow down or put on the brakes the acceleration is constant but negative and is called deceleration.) When you drop an object and it falls to the ground it also has a constant acceleration.

When the acceleration is constant, then we can derive 5 very handy equations that will tell us everything about the motion. Let's derive them and then study some examples.

We are going to use the following symbols:

$$\begin{aligned} t_1 &\equiv 0 \\ t_2 &\equiv t \\ x_1 &\equiv x_0 \\ x_2 &\equiv x \\ v_1 &\equiv v_0 \\ v_2 &\equiv v \end{aligned}$$

and acceleration a is a constant and so $a_1 = a_2 = a$. Thus now

$$\begin{aligned} \Delta t &= t_2 - t_1 = t - 0 = t \\ \Delta x &= x_2 - x_1 = x - x_0 \\ \Delta v &= v_2 - v_1 = v - v_0 \\ \Delta a &= a_2 - a_1 = a - a = 0 \end{aligned}$$

(Δa must be zero because we are only considering *constant* a.)

Also, because acceleration is constant then average acceleration is always the *same* as instantaneous acceleration

$$\bar{a} = a$$

Now use the definition of average acceleration

$$\bar{a} = a = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - 0} = \frac{v - v_0}{t}$$

Thus

$$at = v - v_0$$

or

$$\boxed{v = v_0 + at} \tag{1.5}$$

which is the first of our constant acceleration equations. If you plot this on a v, t graph, then it is a straight line for $a =$ constant. In that case the average velocity is

$$\bar{v} = \frac{1}{2}(v + v_0)$$

From the definition of average velocity

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t}$$

we have

$$\begin{aligned} \frac{x - x_0}{t} &= \frac{1}{2}(v + v_0) \\ &= \frac{1}{2}(v_0 + at + v_0) \end{aligned}$$

giving

$$\boxed{x - x_0 = v_0 t + \frac{1}{2}at^2} \tag{1.6}$$

which is the second of our constant acceleration equations. To get the other three constant acceleration equations, we just combine the first two.

Example Prove that $\boxed{v^2 = v_0^2 + 2a(x - x_0)}$

Solution Obviously t has been eliminated. From (1.5)

$$t = \frac{v - v_0}{a}$$

Substituting into (1.6) gives

$$\begin{aligned} x - x_0 &= v_0 \left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 \\ a(x - x_0) &= v_0 v - v_0^2 + \frac{1}{2}(v^2 - 2vv_0 + v_0^2) \\ &= v^2 - v_0^2 \end{aligned}$$

or

$$v^2 = v_0^2 + 2a(x - x_0)$$

Example Prove that $\boxed{x - x_0 = \frac{1}{2}(v_0 + v)t}$

Solution Obviously a has been eliminated. From (1.5)

$$a = \frac{v - v_0}{t}$$

Substituting into (1.6) gives

$$\begin{aligned} x - x_0 &= v_0 t + \frac{1}{2}\left(\frac{v - v_0}{t}\right)t^2 \\ &= v_0 t + \frac{1}{2}(vt - v_0 t) \\ &= \frac{1}{2}(v_0 + v)t \end{aligned}$$

Exercise Prove that $\boxed{x - x_0 = vt - \frac{1}{2}at^2}$

$\star$ **carefully study Sample Problem 2.8** $\star$ [from Halliday]

1.7 Another Look at Constant Acceleration

(This section is only for students who have studied integral calculus.)

The constant acceleration equations can be derived from integral calculus as follows.

For *constant* acceleration $a \neq a(x)$, $a \neq a(t)$

$$a = \frac{dv}{dt}$$

$$\int_{t_1}^{t_2} a\,dt = \int \frac{dv}{dt} dt$$

$$a \int_{t_1}^{t_2} dt = \int_{v_1}^{v_2} dv$$

$$a(t_2 - t_1) = v_2 - v_1$$

$$a(t - 0) = v - v_0$$

$$\boxed{v = v_0 + at}$$

$$v = \frac{dx}{dt}$$

$$\int v\,dt = \int \frac{dx}{dt} dt$$

v changes $\therefore$ cannot take outside integral

actually $v(t) = v_0 + at$

$$\int_{t_1}^{t_2} (v_0 + at) dt = \int_{x_1}^{x_2} dx$$

$$\left[v_0 t + \frac{1}{2} a t^2 \right]_{t_1}^{t_2} = x_2 - x_1$$

$$= v_0(t_2 - t_1) + \frac{1}{2} a (t_2 - t_1)^2 = x - x_0$$

$$= v_0(t - 0) + \frac{1}{2} a (t - 0)^2$$

$$= v_0 t + \frac{1}{2} a t^2 \qquad \therefore \quad \boxed{x - x_0 = v_0 t + \tfrac{1}{2} a t^2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\int_{x_1}^{x_2} a\, dx = \int v \frac{dv}{dx} dx$$
$$a \int_{x_1}^{x_2} dx = \int_{v_1}^{v_2} v\, dv$$

$$\begin{aligned} a(x_2 - x_1) &= \left[\frac{1}{2}v^2\right]_{v_1}^{v_2} \\ &= \frac{1}{2}\left(v_2^2 - v_1^2\right) \\ a(x - x_0) &= \frac{1}{2}\left(v^2 - v_0^2\right) \end{aligned}$$

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)}$$

One can now get the other equations using algebra.

1.8 Free-Fall Acceleration

If we neglect air resistance, then all falling objects have *same* acceleration

$$a = -g = -9.8 \text{ m/sec}^2$$

($g = 9.8$ m/sec^2).

LECTURE DEMONSTRATION:

1) Feather and penny in vacuum tube

2) Drop a cup filled with water which has a hole in the bottom. Water leaks out if the cup is held stationary. Water does *not* leak out if the cup is dropped.

Carefully study Sample Problems 2-9, 2-10, 2-11. [from Halliday]

Example I drop a ball from a height H, with what speed does it hit the ground ? Check that the units are correct.

Solution

$$v^2 = v_0^2 + 2a(x - x_0)$$

$$\begin{aligned} v_0 &= 0 \\ a &= -g = -9.8 \text{ m/sec}^2 \\ x_0 &= 0 \\ x &= H \end{aligned}$$

$$v^2 = 0 - 2 \times g\,(0 - -H)$$

$$v = \sqrt{2gH}$$

Check units:

The units of g are $m\ sec^{-2}$ and H is in m. Thus $\sqrt{2gH}$ has units of $\sqrt{m\ sec^{-2}\ m} = \sqrt{m^2\ sec^{-2}} = m\ sec^{-1}$. which is the correct unit for speed.

HISTORICAL NOTE

The constant acceleration equations were first discovered by Galileo Galilei (1564 - 1642). Galileo is widely regarded as the "father of modern science" because he was really the first person who went out and actually did expreiments to arrive at facts about nature, rather than relying solely on philosophical argument. Galileo wrote two famous books entitled *Dialogues concerning Two New Sciences* [Macmillan, New York, 1933; QC 123.G13] and *Dialogue concerning the Two Chief World Systems* [QB 41.G1356].

In *Two New Sciences* we find the following [Pg. 173]:

> "THEOREM I, PROPOSITION I : The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a unifrom speed whose value is the mean of the highest speed and the speed just before acceleration began."

In other words this is Galileo's statement of our equation

$$x - x_0 = \frac{1}{2}(v_0 + v)t \tag{1.7}$$

We also find [Pg. 174]:

> "THEOREM II, PROPOSITION II : The spaces described by a falling body from rest with a uniformly accelerated motion are to each other as the squares of the time intervals employed in traversing these distances."

This is Galileo's statement of

$$x - x_0 = v_0 t + \frac{1}{2}at^2 = vt - \frac{1}{2}at^2 \tag{1.8}$$

Galileo was able to test this equation with the simple device shown in Figure 2.3. By the way, Galileo also invented the astronomical telescope !

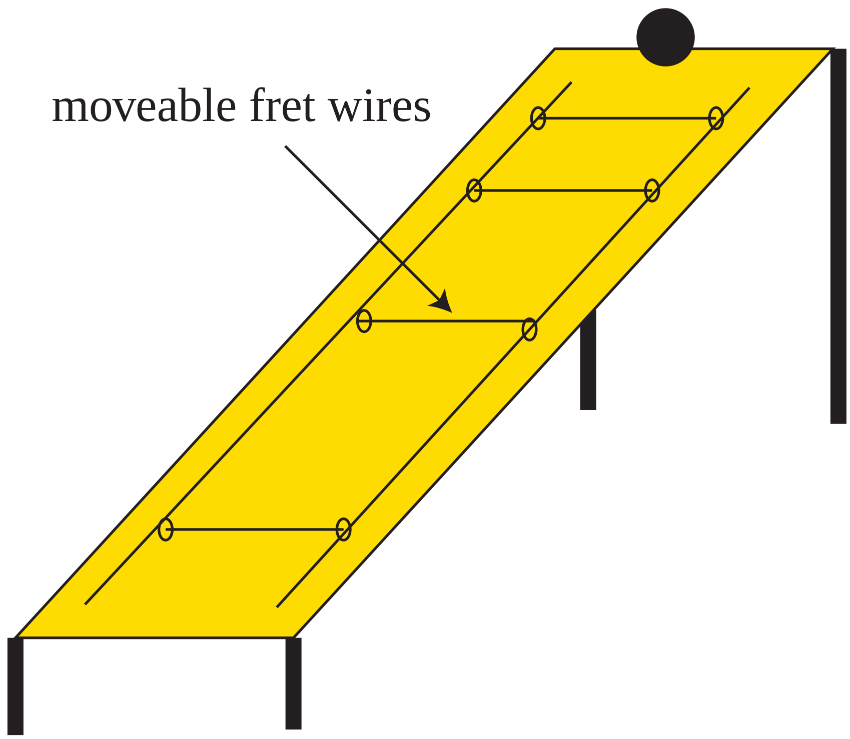


FIGURE 2.3 Galileo's apparatus for verifying the constant acceleration equations.

[from "From Quarks to the Cosmos" Leon M. Lederman and David N. Schramm (Scientific American Library, New York, 1989) QB43.2.L43

1.9 Problems

1. The following functions give the position as a function of time:

   i) $x = A$

   ii) $x = Bt$

   iii) $x = Ct^2$

   iv) $x = D\cos\omega t$

   v) $x = E\sin\omega t$

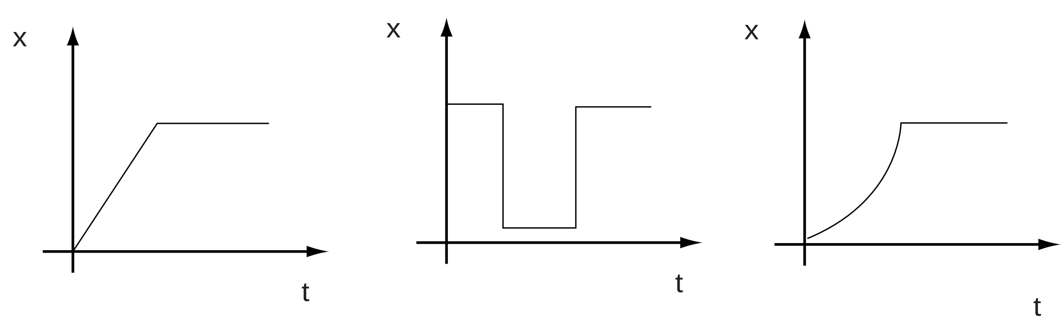
   where A, B, C, D, E, ω are constants.

A) What are the units for A, B, C, D, E, ω?

B) Write down the velocity and acceleration equations as a function of time. Indicate for what functions the acceleration is *constant*.

C) Sketch graphs of x, v, a as a function of time.

2. The figures below show position-time graphs. Sketch the corresponding velocity-time and acceleration-time graphs.



3. If you drop an object from a height H above the ground, work out a formula for the speed with which the object hits the ground.

4. A car is travelling at constant speed v_1 and passes a second car moving at speed v_2. The instant it passes, the driver of the second car decides to try to catch up to the first car, by stepping on the gas pedal and moving at acceleration a. Derive a formula for how long it takes to

catch up. (The first car travels at constant speed v_1 and does not accelerate.)

5. If you start your car from rest and accelerate to 30*mph* in 10 *seconds*, what is your acceleration in *mph per sec* and in *miles per hour*^{2} ?

6. If you throw a ball up vertically at speed V, with what speed does it return to the ground ? Prove your answer using the constant acceleration equations, and neglect air resistance.

Chapter 2

VECTORS

2.1 Vectors and Scalars

When we considered 1-dimensional motion in the last chapter we only had two directions to worry about, namely motion to the Right or motion to the Left and we indicated direction with a + or − sign. We found that the following quantities had a direction (i.e. could take a + or − sign): *displacement, velocity* and *acceleration*. Quantities that don't have a sign were *distance, speed* and *magnitude of acceleration.*

Now in 2 and 3 dimensions we need more than a + or − sign. That's where *vectors* come in.

> Vectors are quantities with both magnitude *and* direction.
>
> Scalars are quantities with magnitude only.
>
> Examples of Vectors are: displacement, velocity, acceleration, force, momentum, electric field
>
> Examples of Scalars are: distance, speed, magnitude of acceleration, time, temperature

Before delving into vectors consider the following problem.

Example Joe and Mary are rowing a boat across a river which is 40 m wide. They row in a direction perpendicular to the bank. However the river is flowing downstream and by the time they reach the other side, they end up 30 m downstream from their starting point. Over what total distance did the boat travel?

Solution Obviously the way to do this is with the triangle in Fig. 3.1, and we deduce that the distance is 50 m.

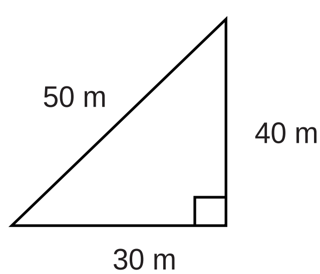


FIGURE 3.1 Graphical solution to river problem.

2.2 Adding Vectors: Graphical Method

Another way to think about the previous problem is with *vectors*, which are little arrows whose *orientation* specifies *direction* and whose *length* specifies magnitude. The displacement *along* the river is represented as

FIGURE 3.2 Displacement along the river.

with a length of 30 m, denoted as $\vec{A}$ and the displacement *across* the river, denoted B,

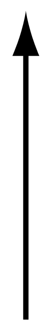


FIGURE 3.3 Displacement across the river.

with length of 40 m. To re-construct the previous triangle, the vectors are added *head-to-tail* as in Fig. 3.4.

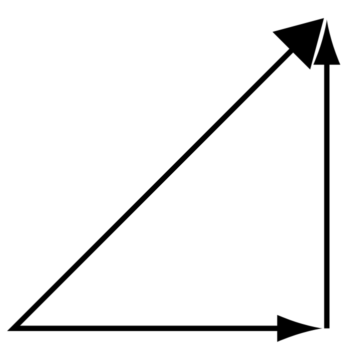


FIGURE 3.4 Vector addition solution to the river problem.

The *resultant* vector, denoted $\vec{C}$, is obtained by filling in the triangle. Mathematically we write $\vec{C} = \vec{A} + \vec{B}$.

The *graphical* method of solving our original problem is to take out a ruler and actually *measure* the length of the resultant vector $\vec{C}$. You would find it to be 50 m.

Summary: When adding any two vectors $\vec{A}$ and $\vec{B}$, we add them head-to-tail.

Students should read the textbook to obtain more details about using the graphical method.

2.3 Vectors and Their Components

The graphical method requires the use of a ruler and protractor for measuring the lengths of vectors and their angles. Thus there is always the problem of inaccuracy in making these measurements. It's better to use analytical methods which rely on pure calculation. To learn this we must learn about components. To do this we need trigonometry.

2.3.1 Review of Trigonometry

Lines are made by connecting two points. Triangles are made by connecting three points. Of all the vast number of different possible triangles, the subject of trigonometry has to do with only a certain, special type of triangle and that is a right-angled triangle, i.e. a triangle where one of the angles is 90°. Let's draw one:

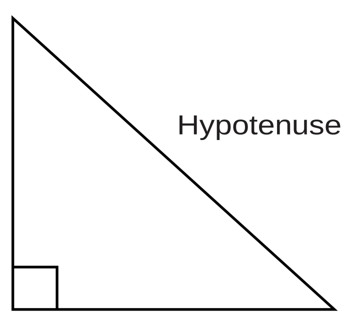


FIGURE 3.5 Right-angled triangle.

The side opposite the right angle is *always* called the Hypotenuse. Consider one of the other angles, say θ.

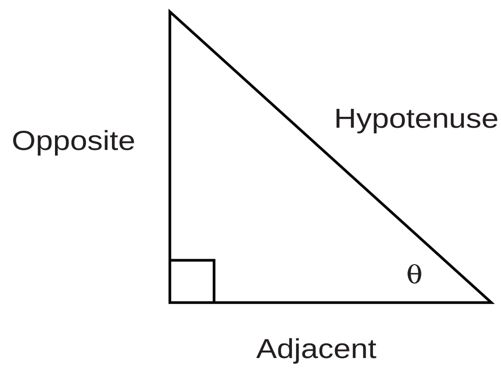


FIGURE 3.6 Right-angled triangle showing sides Opposite and Adjacent to the angle θ.

The side adjacent to θ is called Adjacent and the side opposite θ is called Opposite. Now consider the other angle α. The Opposite and Adjacent sides are switched because the angle is different.

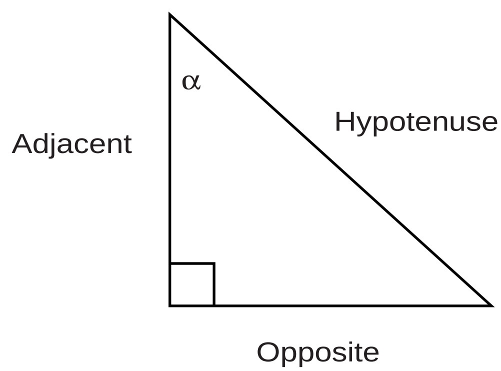


FIGURE 3.7 Right-angled triangle showing sides Opposite and Adjacent to the angle α.

Let's label Hypotenuse as H, Opposite as O and Adjacent as A. Pythagoras' theorem states

$$\boxed{H^2 = A^2 + O^2}$$

This is true no matter how the Opposite and Adjacent sides are labelled, i.e. if Opposite and Adjacent are interchanged, it doesn't matter for Pythagoras' theorem.

Often we are interested in dividing one side by another. Some possible combinations are $\frac{O}{H}$, $\frac{A}{H}$, $\frac{O}{A}$. These special ratios are given special names. $\frac{O}{H}$ is called Sine. $\frac{A}{H}$ is called Cosine. $\frac{O}{A}$ is called Tangent. Remember them by writing SOH, CAH, TOA.

Example Using the previous triangle for the river problem, write down Sine θ, Cosine θ, Tangent θ Sine α, Cosine α, Tangent α

Solution

$$\begin{aligned}
\text{Sine } \theta &= \frac{O}{H} = \frac{40m}{50m} = \frac{4}{5} = 0.8 \\
\text{Cosine } \theta &= \frac{A}{H} = \frac{30m}{50m} = \frac{3}{5} = 0.6 \\
\text{Tangent } \theta &= \frac{O}{A} = \frac{40m}{30m} = \frac{4}{3} = 1.33 \\
\text{Sine } \alpha &= \frac{O}{H} = \frac{30m}{50m} = \frac{3}{5} = 0.6 \\
\text{Cosine } \alpha &= \frac{A}{H} = \frac{40m}{50m} = \frac{4}{5} = 0.8 \\
\text{Tangent } \alpha &= \frac{O}{A} = \frac{30m}{40m} = \frac{3}{4} = 0.75
\end{aligned}$$

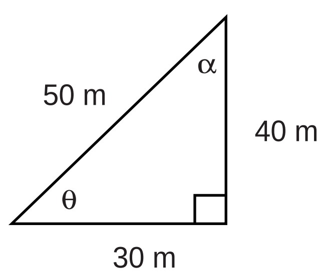


FIGURE 3.8 Triangle for river problem.

Now *whenever* the Sine of an angle is 0.8 the angle is *always* 53.1°. Thus $\theta = 53.1°$. Again *whenever* Tangent of an angle is 0.75 the angle is *always* 36.9°. So if we have calculated any of the ratios, Sine, Cosine or Tangent then we always know what the corresponding angle is.

2.3.2 Components of Vectors

An arbitrary vector has both x and y components. These are like *shadows* on the x and y areas, as shown in Figure 3.9.

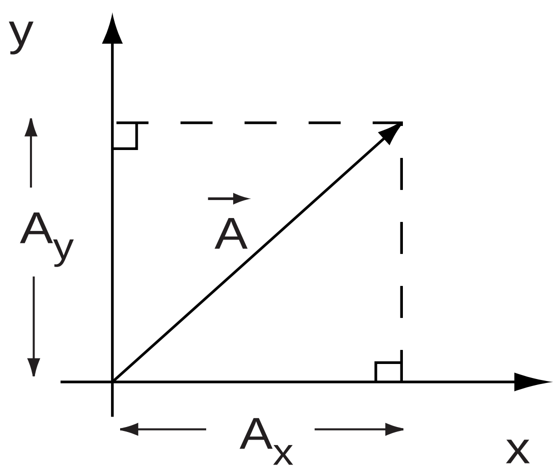


FIGURE 3.9 Components, A_x and A_y, of vector $\vec{A}$.

The components are denoted A_x and A_y and are obtained by dropping a *perpendicular* line from the vector to the x and y axes. *That's why we consider trigonometry and right-angled triangles*!

A physical understanding of components can be obtained. Pull a cart with a rope at some angle to the ground, as shown in Fig. 3.11. The cart will move with a certain acceleration, determined *not* by the force $\vec{F}$, but by the *component* F_x in the x direction. If you change the angle, the acceleration of the cart will change.

LECTURE DEMONSTRATION of Fig. 3.10:

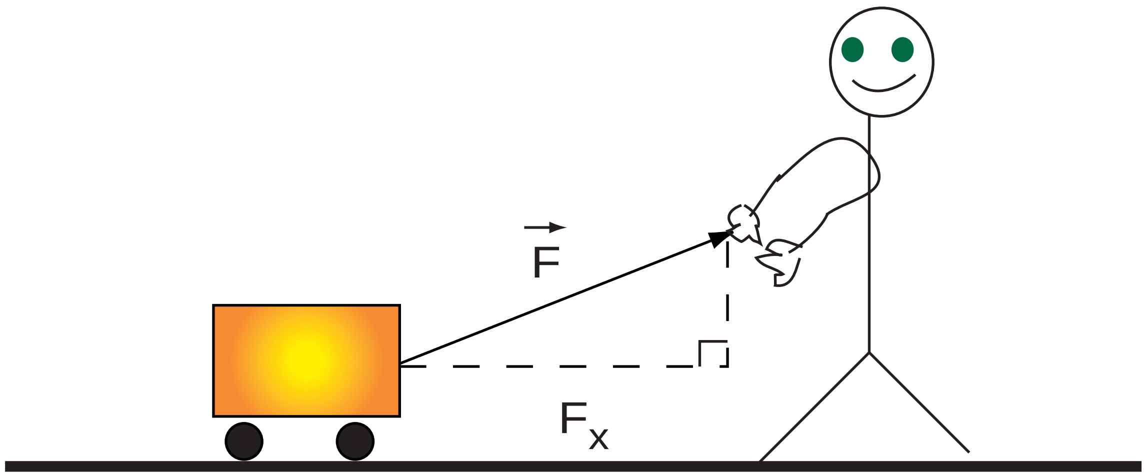


FIGURE 3.10 Pulling a cart with a force $\vec{F}$.

Let's re-draw Figure 3.10, writing $\vec{A}$ instead of $\vec{F}$ as follows:

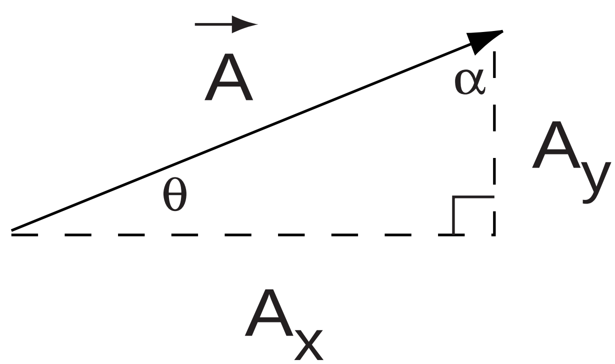


FIGURE 3.11 Components and angles for Fig. 3.10.

Let's denote the magnitude or length of $\vec{A}$ simply as A. Thus Pythagoras' theorem gives

$$A^2 = A_x^2 + A_y^2$$

and also

$$\tan\theta = \frac{A_y}{A_x}$$

and

$$\tan\alpha = \frac{A_x}{A_y}$$

(Also $\sin\theta = \frac{A_y}{A}$, $\cos\theta = \frac{A_x}{A}$, $\sin\alpha = \frac{A_x}{A}$, $\cos\alpha = \frac{A_y}{A}$)

Thus if we have the components, A_x and A_y we can always get the *magnitude* and *direction* of the vector, namely A and θ (or α). Similarly if we start with A and θ (or α) we can always find A_x and A_y.

do Sample Problem 3-3 in Lecture

2.4 Unit Vectors

A vector is *completely specified* by writing down *magnitude* and *direction* (i.e. A and θ) x and y *components* (A_x and A_y).

There's another very useful and compact way to write vectors and that is by using *unit vectors*. The unit vector $\hat{i}$ is *defined* to always have a length of 1 and to always lie in the positive x direction, as in Fig. 3.12. (The symbol $\wedge$ is used to denote these unit vectors.)

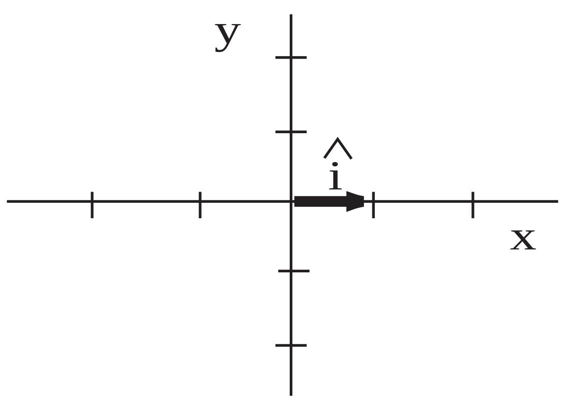


FIGURE 3.12 Unit vector $\hat{i}$.

Similarly the unit vector $\hat{j}$ is defined to always have a length of 1 also but to lie entirely in the positive y direction.

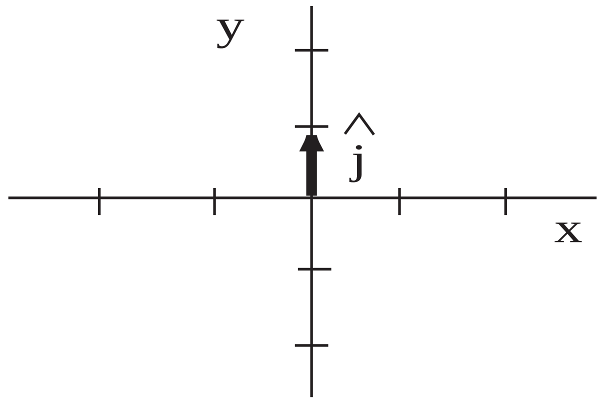


FIGURE 3.13 Unit vector $\hat{j}$.

The unit vector $\hat{k}$ lies in the psoitive z direction.

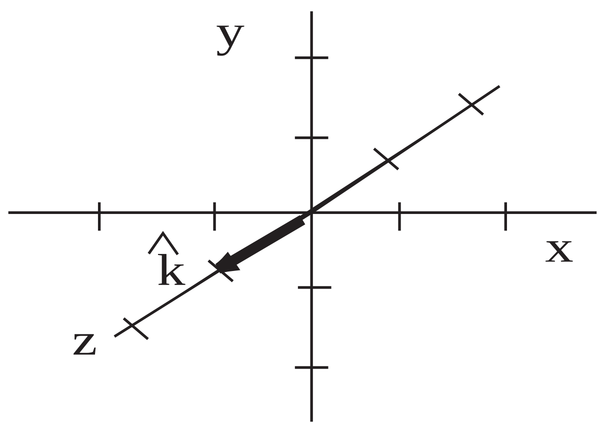


FIGURE 3.14 Unit vector $\hat{k}$.

Thus any arbitrary vector $\vec{A}$ is now written as

$$\boxed{\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}}$$

(Think about this and make sure you understand.)

2.5 Adding Vectors by Components

Finally we will now see the *use* of components and unit vectors. Remember how we discussed adding vectors graphically using a ruler and protractor. A better method is with the use of components, because then we can get our answers by pure calculation.

In Fig. 3.16 we have shown two vectors $\vec{A}$ and $\vec{B}$ added to form $\vec{C}$, but we have also indicated all the components.

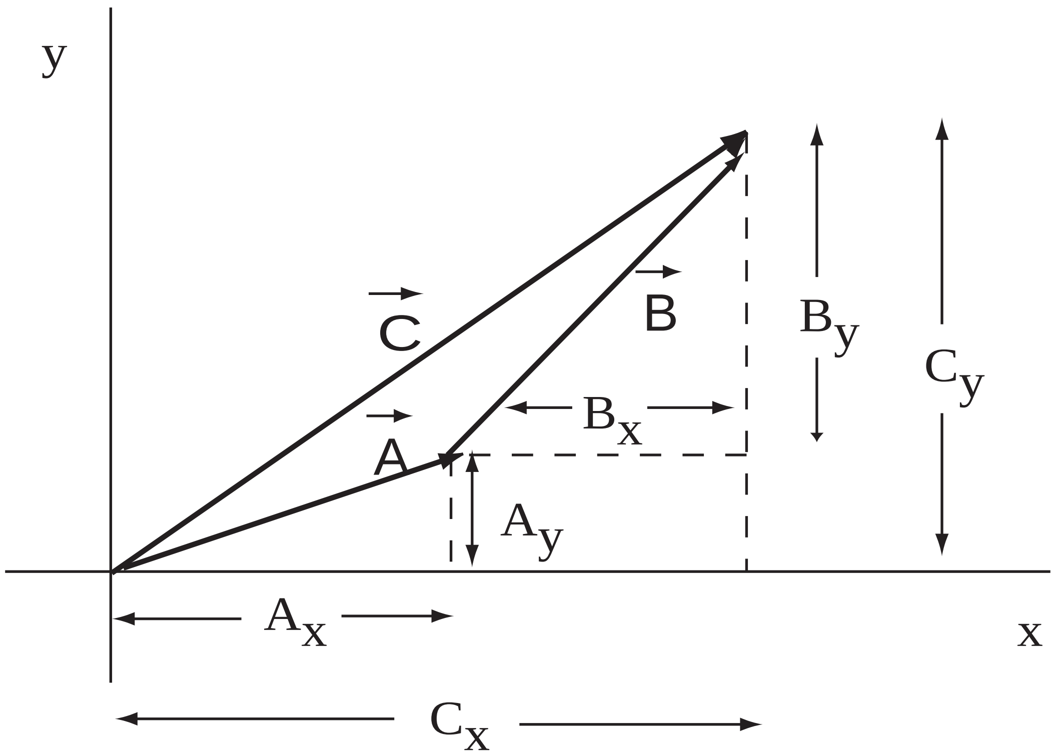


FIGURE 3.15 Adding vectors by components.

By carefully looking at the figure you can see that

$$\boxed{\begin{aligned} C_x &= A_x + B_x \\ C_y &= A_y + B_y \end{aligned}}$$

This is a *very* important result.

Now let's back-track for a minute. When we write

$$\vec{C} = \vec{A} + \vec{B}$$

you should say, "Wait a minute! What does the + sign mean?" We are used to adding *numbers* such as $5 = 3 + 2$, but in the above equation $\vec{A}$, $\vec{B}$ and $\vec{C}$ are *not* numbers. They are these strange arrow-like objects called vectors which are "add" by putting head-to-tail. We should really write

$$\vec{C} = \vec{A} \oplus \vec{B}$$

where $\oplus$ is a *new* type of "addition", totally unlike adding numbers. However A_x, B_x, A_y, B_y, C_x, C_y *are* ordinary numbers and the + sign we used above *does* denote ordinary addition. Thus $\vec{C} = \vec{A} \oplus \vec{B}$ *actually means* $C_x = A_x + B_x$ and $C_y = A_y + B_y$. The statement $\vec{C} = \vec{A} \oplus \vec{B}$ is really shorthand for two ordinary addition statements. *Whenever* anyone writes something like $\vec{D} = \vec{F} + \vec{E}$ it *actually means two* things, namely $D_x = F_x + E_x$ and $D_y = F_y + E_y$.

All of this is much more obvious with the use of unit vectors. Write $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$ and $\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath}$. Now

$$\vec{C} = \vec{A} + \vec{B}$$

is simply

$$\begin{aligned} C_x\hat{\imath} + C_y\hat{\jmath} &= A_x\hat{\imath} + A_y\hat{\jmath} + B_x\hat{\imath} + B_y\hat{\jmath} \\ &= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} \end{aligned}$$

and equating coefficients of $\hat{\imath}$ and $\hat{\jmath}$ gives

$$C_x = A_x + B_x$$

and

$$C_y = A_y + B_y$$

Example Do the original river problem using components.

Solution

$$\begin{aligned}
\vec{A} &= 30\hat{i} \quad \vec{B} = 40\hat{j} \\
\vec{C} &= \vec{A} + \vec{B} \\
C_x\hat{i} &+ C_y\hat{j} = A_x\hat{i} + A_y\hat{j} + B_x\hat{i} + B_y\hat{j} \\
& \qquad A_y = 0 \quad B_x = 0 \\
C_x\hat{i} &+ C_y\hat{j} = 30\hat{i} + 40\hat{j} \\
& C_x = 30 \quad C_y = 40 \\
or \quad & C_x = A_x + B_x = 30 + 0 = 30 \\
& C_y = A_y + B_y = 0 + 40 = 40 \\
& C^2 = C_x^2 + C_y^2 = 30^2 + 40^2 = 900 + 1600 = 2500 \\
& \qquad \therefore\ C = 50
\end{aligned}$$

carefully study Sample Problems 3-4, 3-5

2.6 Vectors and the Laws of Physics

2.7 Multiplying Vectors

2.7.1 The Scalar Product (often called dot product)

We know how to add vectors. Now let's learn how to multiply them.

When we add vectors we always get a new vector, namely $\vec{c} = \vec{a} + \vec{b}$. When we multiply vectors we get *either* a scalar *or* vector. There are two types of vector multiplication called scalar products or vector product. (Sometimes also called dot product or cross product).

The scalar product is *defined* as

$$\boxed{\vec{a} \cdot \vec{b} \equiv ab\cos\phi} \tag{2.1}$$

where a and b are the magnitude of $\vec{a}$ and $\vec{b}$ respectively and ϕ is the angle between $\vec{a}$ and $\vec{b}$. The whole quantity $\vec{a} \cdot \vec{b} = ab\cos\phi$ is a scalar, i.e. it has magnitude only. As shown in Fig. 3-19 of Halliday the scalar product is the product of the magnitude of one vector times the component of the other vector along the first vector.

Based on our definition (2.1) we can work out the scalar products of all of the unit vectors.

Example Evaluate $\hat{i} \cdot \hat{i}$

Solution $\hat{i} \cdot \hat{i} = ii \cos \phi$

but i is the magnitude of $\hat{i}$ which is 1, and the angle ϕ is $0°$. Thus

$\hat{i} \cdot i = 1$

Example Evaluate $\hat{i} \cdot \hat{j}$

Solution $\hat{i} \cdot \hat{j} = ij \cos 90° = 0$

Thus we have $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = \hat{j} \cdot \hat{i} = \hat{k} \cdot \hat{i} = \hat{k} \cdot \hat{j} = 0$. (see Problem 38)

Now *any* vector can be written in terms of unit vectors as $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$. Thus the scalar product of any two arbitrary vectors is

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= ab \cos \phi \\
&= (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) \\
&= a_x b_x + a_y b_y + a_z b_z
\end{aligned}$$

Thus we have a new formula for scalar product, namely

$$\boxed{\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z} \tag{2.2}$$

(see Problem 46) which has been derived from the original definition (2.1) using unit vectors.

What's the good of all this? Well for one thing it's now easy to figure out the *angle* between vectors, as the next example shows.

do Sample Problem 3-6 in Lecture

2.7.2 The Vector Product

In making up the definition of vector product we have to define its magnitude *and* direction. The symbol for vector product is $\vec{a} \times \vec{b}$. Given that the result is a vector let's write $\vec{c} \equiv \vec{a} \times \vec{b}$. The magnitude is defined as

$$c = ab \sin \phi$$

and the direction is defined to follow the right hand rule. ($\vec{c}$ = thumb, $\vec{a}$ = forefinger, $\vec{b}$ = middle finger.)

(Do a few examples finding direction of cross product)

Example Evaluate $\hat{i} \times \hat{j}$

Solution $|\hat{i} \times \hat{j}| = ij \sin 90° = 1$
direction same as $\hat{k}$
Thus $\hat{i} \times \hat{j} = \hat{k}$

Example Evaluate $\hat{k} \times \hat{k}$

Solution $|\hat{k} \times \hat{k}| = kk \sin 0 = 0$
Thus $\hat{k} \times \hat{k} = 0$

Thus we have

$$\begin{array}{lll} \hat{i} \times \hat{j} = \hat{k} & \hat{j} \times \hat{k} = \hat{i} & \hat{k} \times \hat{i} = \hat{j} \\ \hat{j} \times \hat{i} = -\hat{k} & \hat{k} \times \hat{j} = -\hat{i} & \hat{i} \times \hat{k} = -\hat{j} \end{array}$$

and

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

(see Problem 39) Thus the vector product of any two arbitrary vectors is

$$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

which gives a new formula for vector product, namely

$$\boxed{\begin{array}{c} \vec{a} \times \vec{b} = (a_yb_z - a_zb_y)\hat{i} + (a_zb_x - a_xb_z)\hat{j} \\ +(a_xb_y - a_yb_x)\hat{k} \end{array}}$$

(see Problem 49). Study Sample Problem 3-7 and 3-8.

2.8 Problems

1. Calculate the angle between the vectors $\vec{r} = \hat{i} + 2\hat{j}$ and $\vec{t} = \hat{j} - \hat{k}$.

2. Evaluate $(\vec{r} + 2\vec{t}).\vec{f}$ where $\vec{r} = \hat{i} + 2\hat{j}$ and $\vec{t} = \hat{j} - \hat{k}$ and $\vec{f} = \hat{i} - \hat{j}$.

3. Two vectors are defined as $\vec{u} = \hat{j} + \hat{k}$ and $\vec{v} = \hat{i} + \hat{j}$. Evaluate:

   A) $\vec{u} + \vec{v}$

   B) $\vec{u} - \vec{v}$

   C) $\vec{u}.\vec{v}$

   D) $\vec{u} \times \vec{v}$

Chapter 3

MOTION IN 2 & 3 DIMENSIONS

<u>SUGGESTED HOME EXPERIMENT</u>:

Design a simple experiment which shows that the range of a projectile depends upon the angle at which it is launched. Have your experiment show that the maximum range is achieved when the launch angle is 45^o.

<u>THEMES</u>:

1. FOOTBALL.

3.1 Moving in Two or Three Dimensions

In this chapter we will go over everything we did in Chapter 2 concerning motion, except that now the entire discussion will use the formation of vectors.

3.2 Position and Displacement

In Chapter 2 we used the coordinate x alone to denote position. However for 3-dimensions position is generally described with the *position vector*

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Now in Chapter 2, displacement was defined as a change in position, namely displacement $= \Delta x = x_2 - x_1$. In 3-dimensions, displacement is defined as the change in position vector,

$$\begin{aligned} \text{displacement} &= \Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \\ &= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k} \\ &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \end{aligned}$$

Thus *displacement* is a *vector*.

Sample Problem 4-1

3.3 Velocity and Average Velocity

In 1-dimension, the average velocity was defined as displacement divided by time interval or $\bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$. Similarly, in 3-dimensions average velocity is defined as

$$\begin{aligned} \bar{\vec{v}} &\equiv \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} \\ &= \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} \\ &= \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k} \\ &= \bar{v}_x\hat{i} + \bar{v}_y\hat{j} + \bar{v}_z\hat{k} \end{aligned}$$

For 1-dimension, the instantaneous velocity, or just velocity, was defined as $v \equiv \frac{dx}{dt}$. In 3-dimensions we define velocity as

$$\begin{aligned} \vec{v} &\equiv \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k}) \\ &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \\ &= v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \end{aligned}$$

Thus *velocity* is a *vector.*

Point to note: The instantaneous velocity of a particle is always *tangent* to the path of the particle. (carefully read about this in Halliday, pg. 55)

3.4 Acceleration and Average Acceleration

Again we follow the definitions made for 1-dimension. In 3-dimensions, the average acceleration is defined as

$$\bar{\vec{a}} \equiv \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

and acceleration (instantaneous acceleration) is defined as

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Constant Acceleration Equations

In 1-dimension, our basic definitions were

$$\begin{aligned} \bar{v} &= \frac{\Delta x}{\Delta t} \\ v &= \frac{dx}{dt} \\ \bar{a} &= \frac{\Delta v}{\Delta t} \\ a &= \frac{dv}{dt} \end{aligned}$$

We found that *if* the acceleration is constant, then from these equations we can *prove* that

$$v = v_o + at$$

$$\begin{aligned} v^2 &= v_o^2 + 2a(x - x_o) \\ x - x_o &= \frac{v_o + v}{2} t \\ x - x_o &= v_o t + \frac{1}{2} a t^2 \\ &= vt - \frac{1}{2} a t^2 \end{aligned}$$

which are known as the 5 constant acceleration equations.

In 3-dimensions we had

$$\bar{\vec{v}} \equiv \frac{\Delta \vec{r}}{\Delta t}$$

or

$$\bar{v}_x \hat{i} + \bar{v}_y \hat{j} + \bar{v}_z \hat{k} = \frac{\Delta x}{\Delta t} \hat{i} + \frac{\Delta y}{\Delta t} \hat{j} + \frac{\Delta z}{\Delta t} \hat{k}$$

or

$$\bar{v}_x = \frac{\Delta x}{\Delta t}, \quad \bar{v}_y + \frac{\Delta y}{\Delta t}, \quad \bar{v}_z = \frac{\Delta z}{\Delta t}$$

These 3 equations are the *meaning* of the first vector equation $\bar{\vec{v}} \equiv \frac{\Delta \vec{r}}{\Delta t}$. Similarly

$$\vec{v} \equiv \frac{d\vec{r}}{dt}$$

or

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

Similarly

$$\bar{\vec{a}} \equiv \frac{\Delta \vec{v}}{\Delta t}$$

or

$$\bar{a}_x = \frac{\Delta x}{\Delta t}, \quad \bar{a}_y = \frac{\Delta y}{\Delta t}, \quad \bar{a}_z = \frac{\Delta z}{\Delta t}$$

and

$$\vec{a} \equiv \frac{d\vec{v}}{dt}$$

or

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}$$

So we see that in 3-dimensions the equations are the same as in 1-dimension except that we have 3 sets of them; one for each dimension. Thus if the 3-dimensional acceleration vector $\vec{a}$ is now constant, then a_x, a_y and a_z must all be constant. Thus we will have 3 sets of constant acceleration equations, namely

$$\begin{aligned} v_x &= v_{ox} + a_x t \\ v_x^2 &= v_{ox}^2 + 2a_x(x - x_o) \\ x - x_o &= \frac{v_{ox} + v_x}{2} t \\ x - x_o &= v_{ox} t + \frac{1}{2} a_x t^2 \\ &= v_x t - \frac{1}{2} a_x t^2 \end{aligned}$$

and

$$\begin{aligned} v_y &= v_{oy} + a_y t \\ v_y^2 &= v_{oy}^2 + 2a_y(y - y_o) \\ y - y_o &= \frac{v_{oy} + v_y}{2} t \\ y - y_o &= v_{oy} t + \frac{1}{2} a_y t^2 \\ &= v_y t - \frac{1}{2} a_y t^2 \end{aligned}$$

and

$$\begin{aligned} v_z &= v_{oz} + a_z t \\ v_z^2 &= v_{oz}^2 + 2a_z(z - z_o) \\ z - z_o &= \frac{v_{oz} + v_z}{2} t \\ z - z_o &= v_{oz} t + \frac{1}{2} a_z t^2 \\ &= v_z t - \frac{1}{2} a_z t^2 \end{aligned}$$

These 3 sets of constant acceleration equations are easy to remember. They are the same as the old ones in 1-dimension except now they have subscripts for x, y, z.

3.5 Projectile Motion

Read.

3.6 Projectile Motion Analyzed

Most motion in 3-dimensions actually only occurs in 2-dimensions. The classic example is kicking a football off the ground. It follows a 2-dimensional curve, as shown in Fig. 4.1. Thus we can ignore all motion in the z direction and just analyze the x and y directions. Also we shall ignore air resistance.

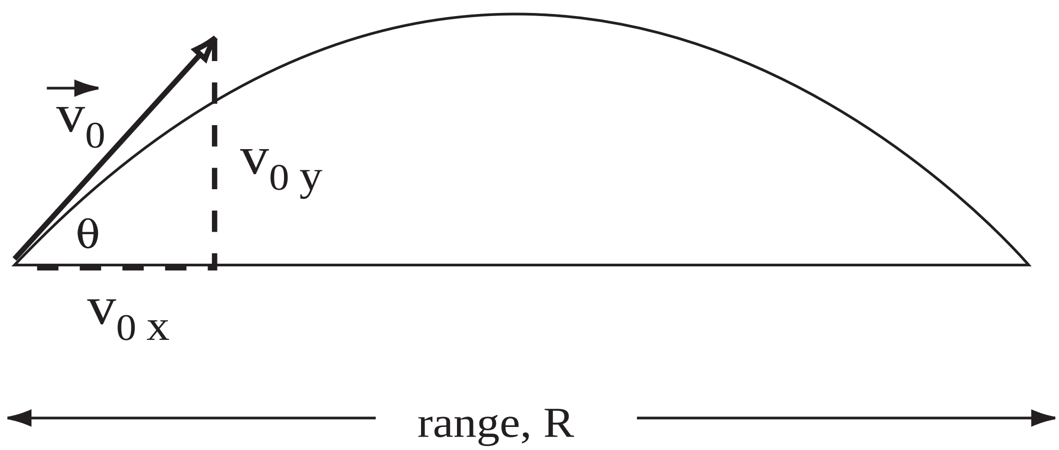


FIGURE 4.1 Projectile Motion.

Example A football is kicked off the ground with an initial velocity of $\vec{v}_o$ at an angle θ to the ground. Write down the x constant acceleration equation in simplified form. (Ignore air resistance)

Solution The x direction is easiest to deal with, because there is *no acceleration in the x direction* after the ball has been kicked, i.e. $a_x = 0$. Thus the constant acceleration equations in the x direction become

$$\begin{aligned} v_x &= v_{ox} \\ v_x^2 &= v_{ox}^2 \\ x - x_o &= \frac{v_{ox} + v_x}{2} t = v_{ox} t = v_x t \\ x - x_o &= v_{ox} t \\ &= v_x t \end{aligned} \tag{3.1}$$

The first equation ($v_x = v_{ox}$) makes perfect sense because if $a_x = 0$ then the speed in the x direction is *constant*, which means $v_x = v_{ox}$. The second equation just says the same thing. If $v_x = v_{ox}$ then of course also $v_x^2 = v_{ox}^2$. In the third equation we also use $v_x = v_{ox}$ to get $\frac{v_{ox}+v_x}{2} = \frac{v_{ox}+v_{ox}}{2} = v_{ox}$ or $\frac{v_{ox}+v_x}{2} = \frac{v_x+v_x}{2} = v_x$. The fourth and fifth equations are also consistent with $v_x = v_{ox}$, and simply say that distance = speed × time when the acceleration is 0.

Now, what is v_{ox} in terms of $v_o \equiv |\vec{v}_o|$ and θ? Well, from Fig. 4.1 we see that $v_{ox} = v_o \cos\theta$ and $v_{oy} = v_o \sin\theta$. Thus (3.1) becomes

$$x - x_o = v_o \cos\theta \, t$$

Example What is the form of the y-direction constant acceleration equations from the previous example ?

Solution Can we also simplify the constant acceleration equations for the y direction? No. In the y direction the acceleration is constant $a_y = -g$ but not zero. Thus the y direction equations don't simplify at all, except that we know that the value of a_y is $-g$ or -9.8 m/sec^2.

Also we can write $v_{oy} = v_o \sin\theta$. Thus the equations for the y direction are

$$\begin{aligned} v_y &= v_o \sin\theta - gt \\ v_y^2 &= (v_o \sin\theta)^2 - 2g(y - y_o) \\ y - y_o &= \frac{v_o \sin\theta + v_y}{2} t \\ y - y_o &= v_o \sin\theta\, t - \frac{1}{2} g t^2 \end{aligned}$$

An important thing to notice is that t *never* gets an x, y or z subscript. This is because t is the same for all 3 components, i.e. $t = t_x = t_y = t_z$. (You should do some thinking about this.)

LECTURE DEMONSTRATIONS

1) Drop an object: it accerates in y direction.
   Air track: no acceleration in x direction.

2) Push 2 objects off table at same time. One falls in vertical path and the other on parabolic trajectory but both hit ground at same time.

3) Monkey shoot.

Example The total horizontal distance (called the Range) that a football will travel when kicked, depends upon the initial speed and angle that it leaves the ground. Derive a formula for the Range, and show that the maximum Range occurs for $\theta = 45°$. (Ignore air resistance and the spin of the football.)

Solution The Range, R is just

$$\begin{aligned} R = x - x_o &= v_{ox}t \\ &= v_o \cos\theta \, t \end{aligned}$$

Given v_o and θ we could calculate the range if we had t. We get this the y direction equation. From the previous example we had

$$y - y_o = v_o \sin\theta \, t - \frac{1}{2}gt^2$$

But for this example, we have $y - y_o = 0$. Thus

$$\begin{aligned} 0 &= v_o \sin\theta \, t - \frac{1}{2}gt^2 \\ 0 &= v_o \sin\theta \; - \frac{1}{2}gt \\ \Rightarrow t &= \frac{2v_o \sin\theta}{g} \end{aligned}$$

Substituting into our Range formula above gives

$$\begin{aligned} R &= v_o \cos\theta \, t \\ &= \frac{2v_o^2 \sin\theta \cos\theta}{g} \\ &= \frac{v_o^2 \sin 2\theta}{g} \end{aligned}$$

using the formula $\sin 2\theta = 2\sin\theta\cos\theta$. Now R will be largest when $\sin 2\theta$ is largest which occurs when $2\theta = 90^o$. Thus $\theta = 45^o$.

COMPUTER SIMULATION (Interactive Physics): Air Drop.

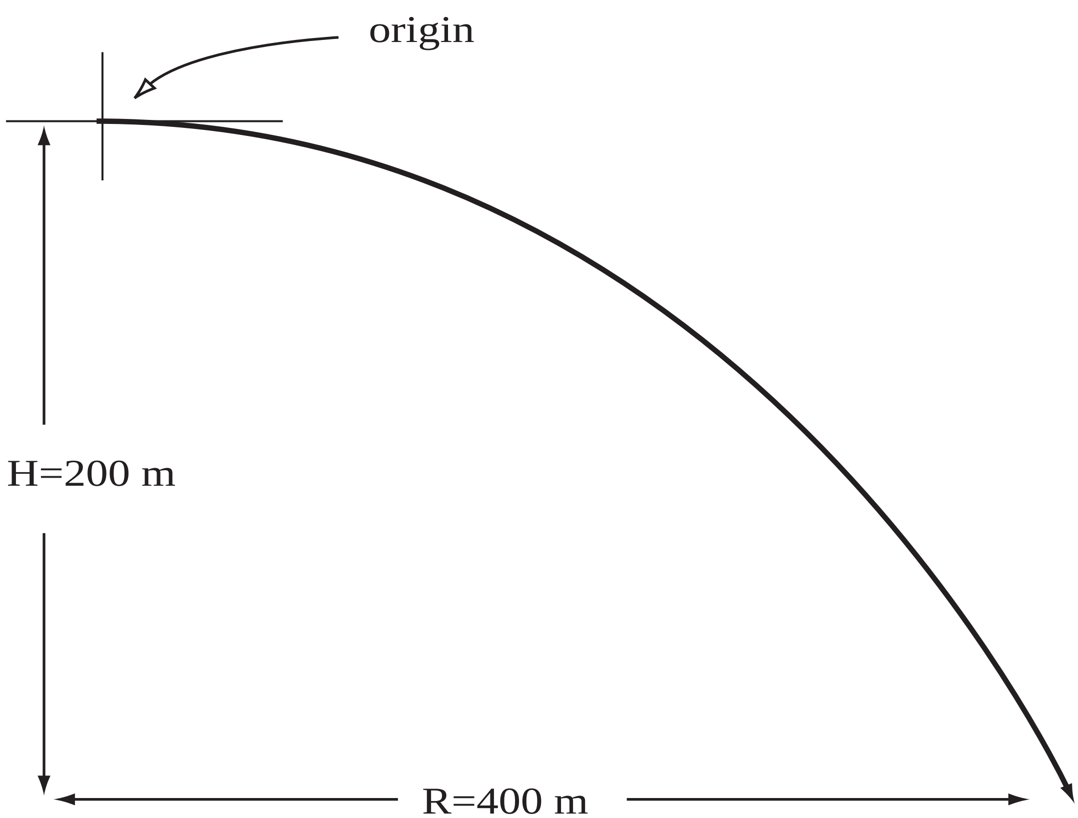


FIGURE 4.2 Air Drop.

Example A rescue plane wants to drop supplies to isolated mountain climbers on a rocky ridge a distance H below. The plane is travelling horizontally at a speed of v_{ox}. The plane releases supplies a horizontal distance of R in advance of the mountain climbers. Derive a formula in terms of H, v_{0x},R and g, for the vertical velocity (up or down) that the supplies should be given so they land exactly at the climber's position. If $H = 200\ m$, $v_{0x} = 250\ km/hr$ and $R = 400m$, calculate a numerical value for this speed.(See Figure 4.2.)

Solution Let's put the origin *at* the plane. See Fig. 4.2. The initial speed of supplies when released is $v_{ox} = +250$ km/hour

$$\begin{aligned} x - x_o &= R - 0 = R \\ a_y &= -g \\ y - y_o &= 0 - H = -H \quad (\textit{note the minus sign}\,!) \end{aligned}$$

We want to find the initial vertical velocity of the supplies, namely v_{oy}. We can get this from

$$\begin{aligned} y - y_o &= v_{oy}t + \frac{1}{2}a_y t^2 = -H \\ &= v_{oy}t - \frac{1}{2}gt^2 \end{aligned}$$

or

$$v_{oy} = \frac{-H}{t} + \frac{1}{2}gt$$

and we get t from the x direction, namely

$$x - x_o = v_{ox}t = R$$

$$\Rightarrow t = \frac{R}{v_{ox}}$$

giving

$$v_{oy} = \frac{-H\ v_{ox}}{R} + \frac{1}{2}g\frac{R}{v_{ox}}$$

which is the formula we seek. Let's now put in numbers:

$$\begin{aligned} &= -\frac{200\,\text{m} \times 250\,\text{km}\,\text{hour}^{-1}}{400\,\text{m}} \\ &\quad + \frac{1}{2}9.8\frac{\text{m}}{\text{sec}^2} \times \frac{400\ \text{km}}{250\ \text{km hour}^{-1}} \\ &= -125\frac{\text{km}}{\text{hour}} + 7.84\frac{\text{m}^2\text{hour}}{\text{sec}^2\text{km}} \\ &= -125\frac{1000\ \text{m}}{60 \times 60\ \text{sec}} + 7.85\frac{\text{m}^2 \times 60 \times 60\ \text{sec}}{\text{sec}^2 1000\ \text{m}} \\ &= -34.722\ \text{m/sec} + 28.22\ \text{m/sec} \\ &= -6.5\ \text{m/sec} \end{aligned}$$

Thus the supplies must be thrown in the *down* direction (not up) at 6.5 m/sec.

3.7 Uniform Circular Motion

In today's world of satellites and spacecraft circular motion is very important to understand because many satellites have circular orbits. Also circular motion is a classic example where we have a definite non-zero acceleration even though the speed of a satellite is constant. This occurs because the direction of velocity is constantly changing for the satellite even though the magnitude of velocity (i.e. speed) is constant. This is shown in Fig. 4-19 of Halliday. The word "uniform" means that speed is constant.

In circular motion, there is a well defined radius which we will call r. Also the time it takes for the satellite to complete 1 orbit is called the *period* T. If the speed is constant then it is given by

$$\boxed{v = \frac{\Delta s}{\Delta t} = \frac{2\pi r}{T}} \tag{3.2}$$

Here I have written $\frac{\Delta s}{\Delta t}$ instead of $\frac{\Delta x}{\Delta t}$ or $\frac{\Delta y}{\Delta t}$ because Δs is the total distance *around* the circle which is a mixture of x and y. $\frac{2\pi r}{T}$ is just the distance of 1 orbit (circumference) divided by the time of 1 orbit (period).

What about the acceleration? Well that's just $a = \frac{\Delta v}{\Delta t}$ but how do we work it out? Look at Figure 4.3, where the displacement and velocity vectors are drawn for a satellite at two different positions P_1 and P_2.

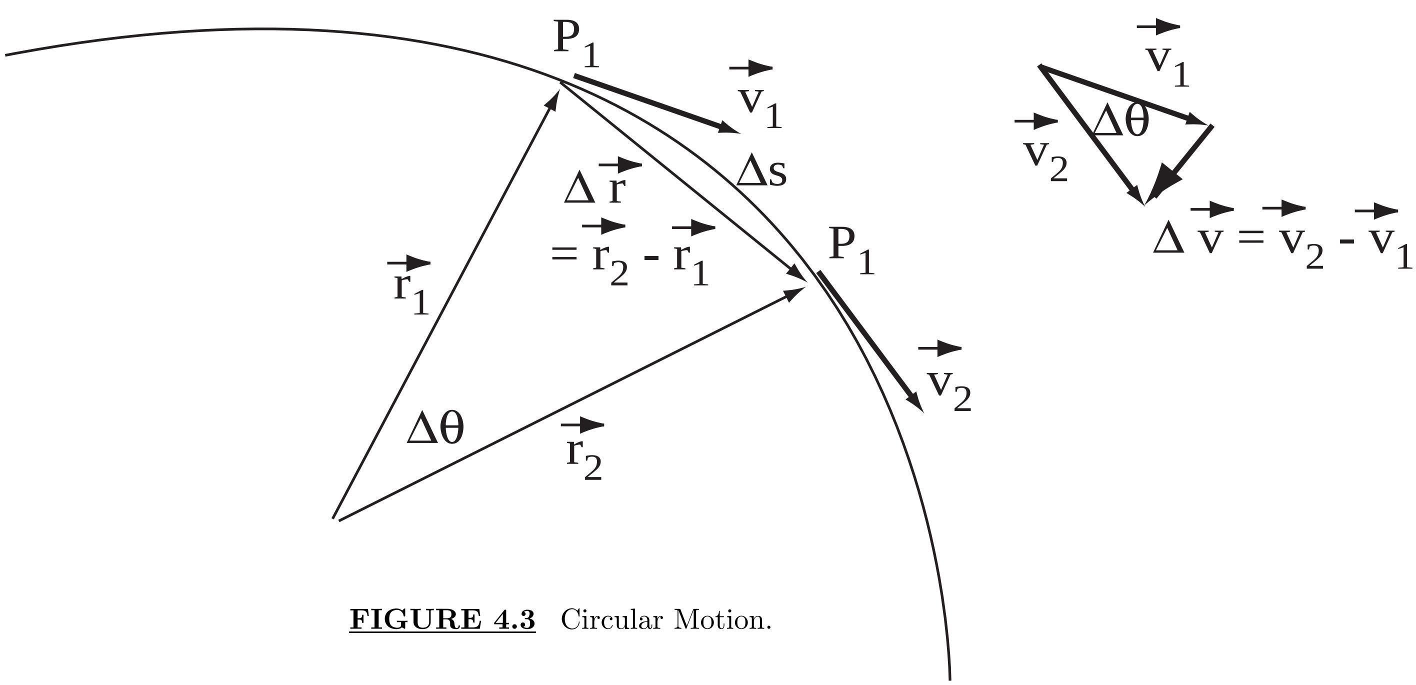


FIGURE 4.3 Circular Motion.

Now angle $\Delta\theta$ is *defined* as (with $|\vec{r}_1| = |\vec{r}_2| \equiv r$)

$$\Delta\theta \equiv \frac{\Delta s}{r} = \frac{v\Delta t}{r} \tag{3.3}$$

The velocity vectors can be re-drawn as in the bottom part of the figure. The triangle is *similar* to the top triangle in that the angle $\Delta\theta$ is the same. Also the speed v is constant, meaning that

$$|\vec{v}_2| = |\vec{v}_1| \equiv v. \tag{3.4}$$

Writing $\Delta v \equiv |\Delta\vec{v}|$ the bottom figure also gives

$$\Delta\theta = \frac{\Delta v}{v} \tag{3.5}$$

Now the magnitude of acceleration is

$$a = \frac{\Delta v}{\Delta t} \tag{3.6}$$

Combining the above two equations for $\Delta\theta$ gives $\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$, i.e.

$$\boxed{a \equiv \frac{\Delta v}{\Delta t} = \frac{v^2}{r}} \tag{3.7}$$

This is a very important equation. *Whenever* we have uniform circular motion we *always* know the *actual value* of acceleration if we know v and r. We have worked out the *magnitude* of the acceleration. What about its direction? I will show you a VIDEO in class (Mechanical Universe video #9 showing vectors for circular motion) which will clearly show that the direction of acceleration is *always* towards the *center* of the circle. For this reason it is called *centripetal* acceleration.

One final thing. When you drive your car around in a circle then you, as the driver, feel as though you are getting pushed against the door. In reality it is the car that is being accelerated around in the circle, and because of your inertia, the car pushes on you. This "acceleration" that you feel is the same as the car's acceleration. The "acceleration" you feel is called the centrifugal acceleration. The same idea occurs when you spin-dry clothes in a washing machine.

Example Future spacecraft will be made to spin in order to provide artificial gravity for the astronauts. Suppose the spacecraft is a cylinder of L in length. Derive a formula for the rotation period would it need to spin in order to simulate the gravity on Earth. If $L = 1\ km$ what is the numerical value foe the period ?

Solution The centifugal acceleration is a and we want it to equal g. Thus

$$g = \frac{v^2}{r} = \frac{(2\pi r/T)^2}{r} = \frac{4\pi^2 r}{T^2}$$

Thus

$$T^2 = \frac{4\pi^2 r}{g}$$

giving

$$T = 2\pi\sqrt{\frac{L}{g}}$$

which is the formula we seek. Putting in numbers:

$$\begin{aligned} T &= 2\pi\sqrt{\frac{1000 \text{ m}}{9.8 \text{ m sec}^{-2}}} \\ &= 2\pi\sqrt{102.04 \text{ sec}^{-2}} \\ &= 2\pi \times 10.1 \text{ sec} \\ &= 63.5 \text{ sec} \end{aligned}$$

i.e. about once every minute!

Example The Moon is 1/4 million miles from Earth. How fast does the Moon travel in its orbit ?

Solution The period of the Moon is 1 month. Thus

$$\begin{aligned} v &= \frac{2\pi r}{T} = \frac{2\pi \times 250,000 \text{ miles}}{30 \times 24 \text{ hours}} \\ &= 2,182 \text{ mph} \end{aligned}$$

i.e. about 2000 mph!

3.8 Problems

1. A) A projectile is fired with an initial speed v_o at an angle θ with respect to the horizontal. Neglect air resistance and derive a formula for the horizontal range R, of the projectile. (Your formula should make no explicit reference to time, t). At what angle is the range a maximum ?

   B) If $v_0 = 30\, km/hour$ and $\theta = 15^o$ calculate the numerical value of R.

2. A projectile is fired with an initial speed v_o at an angle θ with respect to the horizontal. Neglect air resistance and derive a formula for the maximum height H, that the projectile reaches. (Your formula should make no explicit reference to time, t).

3. A) If a bulls-eye target is at a horizontal distance R away, derive an expression for the height L, which is the vertical distance above the bulls-eye that one needs to aim a rifle in order to hit the bulls-eye. Assume the bullet leaves the rifle with speed v_0.

   B) How much bigger is L compared to the projectile height H ?
   Note: In this problem use previous results found for the range R and height H, namely $R = \frac{v_0^2 \sin 2\theta}{g} = \frac{2v_0^2 \sin\theta\cos\theta}{g}$ and $H = \frac{v_0^2 \sin^2\theta}{2g}$.

4. Normally if you wish to hit a bulls-eye some distance away you need to aim a certain distance above it, in order to account for the downward motion of the projectile. If a bulls-eye target is at a horizontal distance D away and if you instead aim an arrow directly at the bulls-eye (i.e. directly horiziontally), by what (downward) vertical distance would you miss the bulls-eye ?

5. Prove that the trajectory of a projectile is a parabola (neglect air resistance). Hint: the general form of a parabola is given by $y = ax^2 + bx + c$.

6. Even though the Earth is spinning and we all experience a centrifugal acceleration, we are not flung off the Earth due to the gravitational force. In order for us to be flung off, the Earth would have to be spinning a lot faster.

   A) Derive a formula for the new rotational time of the Earth, such that a person on the equator would be flung off into space. (Take the radius of Earth to be R).

B) Using $R = 6.4$ *million km*, calculate a numerical anser to part A) and compare it to the actual rotation time of the Earth today.

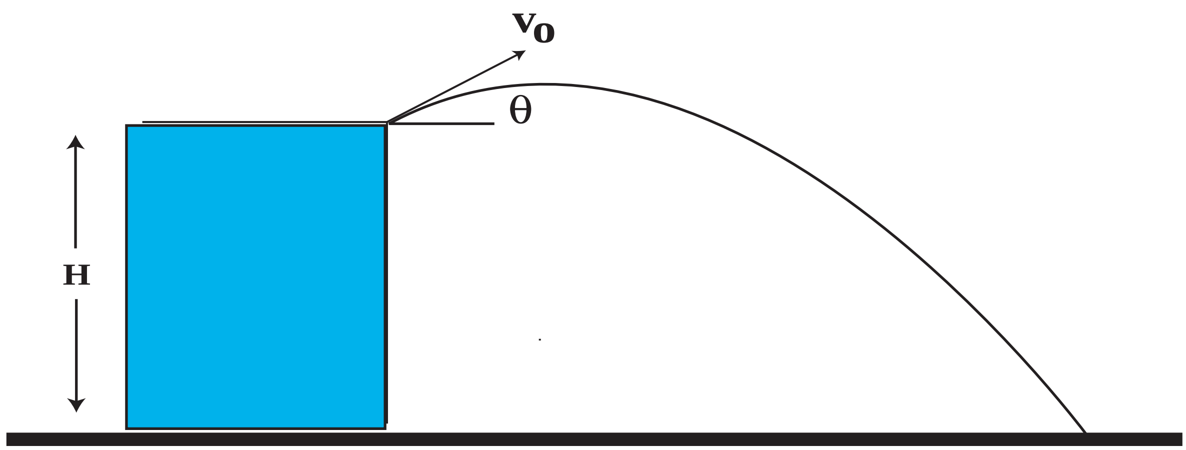
7. A staellite is in a circular orbit around a planet of mass M and radius R at an altitude of H. Derive a formula for the additional speed that the satellite must acquire to completely escape from the planet. Check that your answer has the correct units.

8. A mass m is attached to the end of a spring with spring constant k on a frictionless horizontal surface. The mass moves in circular motion of radius R and period T. Due to the centrifugal force, the spring stretches by a certain amount x from its equilibrium position. Derive a formula for x in terms of k, R and T. Check that x has the correct units.

9. A cannon ball is fired horizontally at a speed v_0 from the edge of the top of a cliff of height H. Derive a formula for the horizontal distance (i.e. the range) that the cannon ball travels. Check that your answer has the correct units.

10. A skier starts from rest at the top of a frictionless ski slope of height H and inclined at an angle θ to the horizontal. At the bottom of the slope the surface changes to horizontal and has a coefficient of kinetic friction μ_k between the horizontal surface and the skis. Derive a formula for the distance d that the skier travels on the horizontal surface before coming to a stop. (Assume that there is a constant deceleration on the horizontal surface). Check that your answer has the correct units.

11. A stone is thrown from the top of a building upward at an angle θ to the horizontal and with an initial speed of v_0 as shown in the figure. If the height of the building is H, derive a formula for the time it takes the stone to hit the ground below.



v0
θ
H

Chapter 4

FORCE & MOTION - I

THEMES:

1. HOW STRONG A ROPE DO I NEED ?

4.1 What Causes an Acceleration?

So far we have studied some things about acceleration but we never considered what *causes* things to accelerate. The answer is *force*. The gravitational force causes objects to fall (i.e. accelerate downwards). Friction force causes cars to slow down (decelerate), etc.

Fundamental classical physics is all about finding the *force*. Once you know that you can get acceleration as we shall see. Once you have the acceleration, you can find velocity, displacement and time as we have studied previously.

4.2 Newton's First Law

A body remains in a state of rest, or uniform motion in a straight line, unless acted upon by a force.

LECTURE DEMONSTRATION: Tablecloth

4.3 Force

Read

4.4 Mass

Read

4.5 Newton's Second Law

Newton's second law of motion is *not* something we can derive from other equations. Rather it is a fundamental postulate of physics. It was introduced by Isaac Newton to describe the cause of acceleration. The law is

$$\Sigma\vec{F} = m\vec{a}$$

$\Sigma\vec{F}$ represents the *sum* (Σ) of all forces ($\vec{F}$) acting on a single body of mass m. The body then undergoes an acceleration given by $\vec{a}$. One of the key activities in classical physics is to find all the forces $\Sigma\vec{F}$. Once you have them then you have the acceleration via $\vec{a} = \frac{\Sigma\vec{F}}{m}$ and once you have that you can get velocity, displacement and time.

Now Newton's second law is a *vector* equation. Thus its actual meaning is given by 3 equations, namely

$$\Sigma F_x = ma_x \quad \Sigma F_y = ma_y \quad \Sigma F_z = ma_z$$

Once we have ΣF_x, ΣF_y, ΣF_z we just divide by m to give the accelerations a_x, a_y, a_z. If they are constant, just plug them into the constant acceleration equations and solve for the other quantities you are interested in.

One extra point is the units. The units of a are m/sec^2. The units of m are kg and thus the units of F are kg m/sec^2. This is given a special name called Newton (N). Thus

$$N \equiv \text{kg m/sec}^2$$

In the English system of units a Pound (lb) is a unit of *force*. The mass unit is called slug. The units of acceleration are foot/sec^2. thus

$$\text{Pound (lb)} \equiv \text{slug foot/sec}^2$$

4.6 Some Particular Forces

Weight

If you stand on a set of scales you measure your weight. If you stand on the same scales on the moon your weight will be less because the moon's gravity is small, even though your mass is the same.

Weight is defined as

$$W \equiv mg$$

where g is the acceleration due to gravity. (It's 9.8 m/sec^2 on Earth, but only 1.7 m/sec^2 on the Moon.) Weight is a *force* which pulls you *down*.

Normal Force

You are sitting still in your chair. The sum of all forces in the x and z direction are zero ($\Sigma F_x = 0$, $\Sigma F_z = 0$) which means that $a_x = a_z = 0$. Now you also know that $a_y = 0$. (You are not moving.) Yet there is a weight force W pulling down.

If your $a_y = 0$ then there must be another force pushing *up* to balance the weight force. We call this up force the Normal force N. Thus

$$\begin{aligned} \Sigma F_y &= ma_y \\ N - W &= 0 \end{aligned}$$

The N has a $+$ sign (up) and W has a $-$ sign (down) and they both balance out to give zero acceleration. That's how we know that the chair must push up on the person sitting on it. The heavier the person, the bigger N must be.

The Normal force is called "Normal" because it always acts *perpendicular* (normal *means* perpendicular) to the surface (of the chair).

Friction

Friction is another force that we will study shortly.

Tension

Finally another important type of force is tension, which is the force in a rope or cable when under a stress.

Carefully study Sample Problem 5-4

4.7 Newton's Third Law

Every action has an equal and opposite reaction.

LECTURE DEMONSTRATION: Fire extinguisher rocket

4.8 Applying Newton's Laws

⋆ Carefully study Sample Problems 5-5, 5-6, 5-7, 5-8, 5-9, 5-10, 5-11.⋆

Example A chandelier of mass m is hanging from a single cord in the ceiling. Derive a formula for the tension in the cord. If $m = 50\ kg$ evaluate a numerical answer for the tension.

Solution Carefully draw a *diagram* showing *all* forces, as seen in Fig. 5.1. Then solve $\vec{\Sigma}F = m\vec{a}$. Thus

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z$$

but all forces and acceleration in the x and z directions are zero and so the only interesting equation is

$$\Sigma F_y = ma_y.$$

Now the forces are tension $(+T)$ in the up direction and weight $(-W)$ in the down direction. You don't want the chandelier to move, so $a_y = 0$. Thus

$$T - W = 0$$

$$\begin{aligned} \Rightarrow T &= W \\ &= \text{mg} \end{aligned}$$

which is the formula we seek. Putting in numbers:

$$T = 50 \text{ kg} \times 9.8 \text{ m/sec}^2 = 490 \text{ kg m/sec}^2 = 490\ N$$

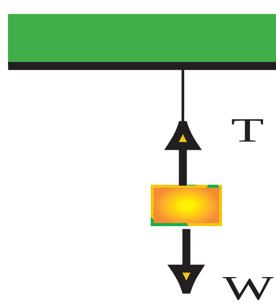


FIGURE 5.1 Chandelier hanging from ceiling.

Example A chandelier of mass m is now suspended by two cords, one at an angle of α to the ceiling and the other at θ. Derive a formula for is the tension in each cord. If $m = 50kg$ and $\alpha = 60^o$ and $\theta = 30^o$ evaluate a numerical answer for each tension.

Solution Again carefully draw a figure showing all forces. See Fig. 5.2.

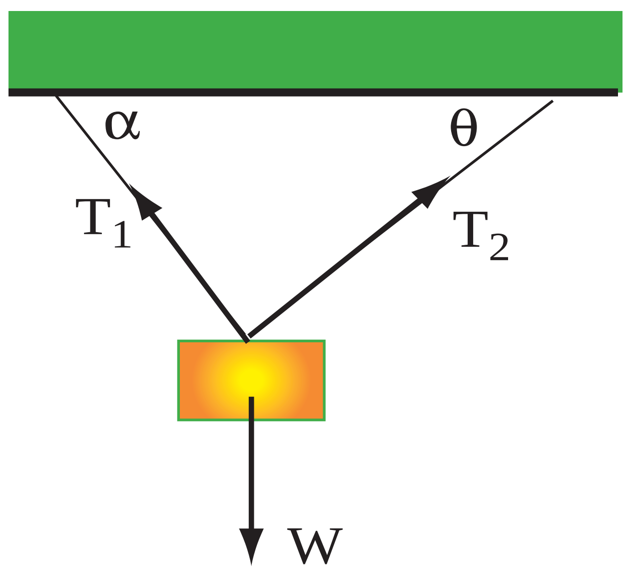


FIGURE 5.2 Chandelier suspended by 2 cables.

In the z direction all forces and acceleration are zero. We need to consider the x and y directions (both with $a_x = a_y = 0$), namely,

$$\Sigma F_x = ma_x \quad \text{and} \quad \Sigma F_y = ma_y$$

$$T_{2x} - T_{1x} = 0 \quad \text{and} \quad T_{2y} + T_{1y} - W = 0$$

Now

$$T_{2x} = T_2 \cos\theta, \qquad T_{1x} = T_1 \cos\alpha$$

$$T_{2y} = T_2 \sin\theta, \qquad T_{1y} = T_1 \sin\alpha$$

giving

$$T_2 \cos\theta - T_1 \cos\alpha = 0 \text{ and } T_2 \sin\theta + T_1 \sin\alpha = W$$

The x equation gives $T_2 = \frac{T_1 \cos\alpha}{\cos\theta}$ which is substituted into the y equation giving

$$\frac{T_1 \cos\alpha}{\cos\theta} \sin\theta + T_1 \sin\alpha = W$$

or

$$\begin{aligned} T_1 &= \frac{W}{\cos\alpha \tan\theta + \sin\alpha} \\ &= \frac{mg}{\cos\alpha \tan\theta + \sin\alpha} \end{aligned}$$

and upon substitution

$$\begin{aligned} T_2 &= \frac{T_1 \cos\alpha}{\cos\theta} \\ &= \frac{mg}{\sin\theta + \tan\alpha \cos\theta} \end{aligned}$$

which are the formulas we seek. Putting in numbers gives:

$$\begin{aligned} W &= \text{mg} \\ &= 50 \text{ kg} \times 9.8 \text{ m/sec}^2 = 490 \ N \end{aligned}$$

Thus

$$T_1 = \frac{490N}{\cos 60 \tan 30 + \sin 60} = 426\,N.$$

Now put back into

$$T_2 = \frac{T_1 \cos 60}{\cos 30} = \frac{426N \cos 60}{\cos 30} = 246 \ N$$

Example If you normally have a weight of W, how much will a weight scale read if you are standing on it in an elevator moving up at an acceleration of a ?

Solution The *reading* on the scale will just be the *Normal force*. Thus

$$\Sigma F = ma$$

$$\begin{aligned} N - W &= ma \\ N &= W + ma \end{aligned}$$

The answer makes sense. You would expect the scale to read a higher value.

Example A block of mass m slides down a frictionless incline of angle θ.

A) What is the normal force?

B) What is the acceleration of the block?

Solution In Fig. 5.3 the forces are drawn. Notice that I have chosen the orientation of the y axis to lie along the normal force. You could make other choices, but this will make things easier to work out.

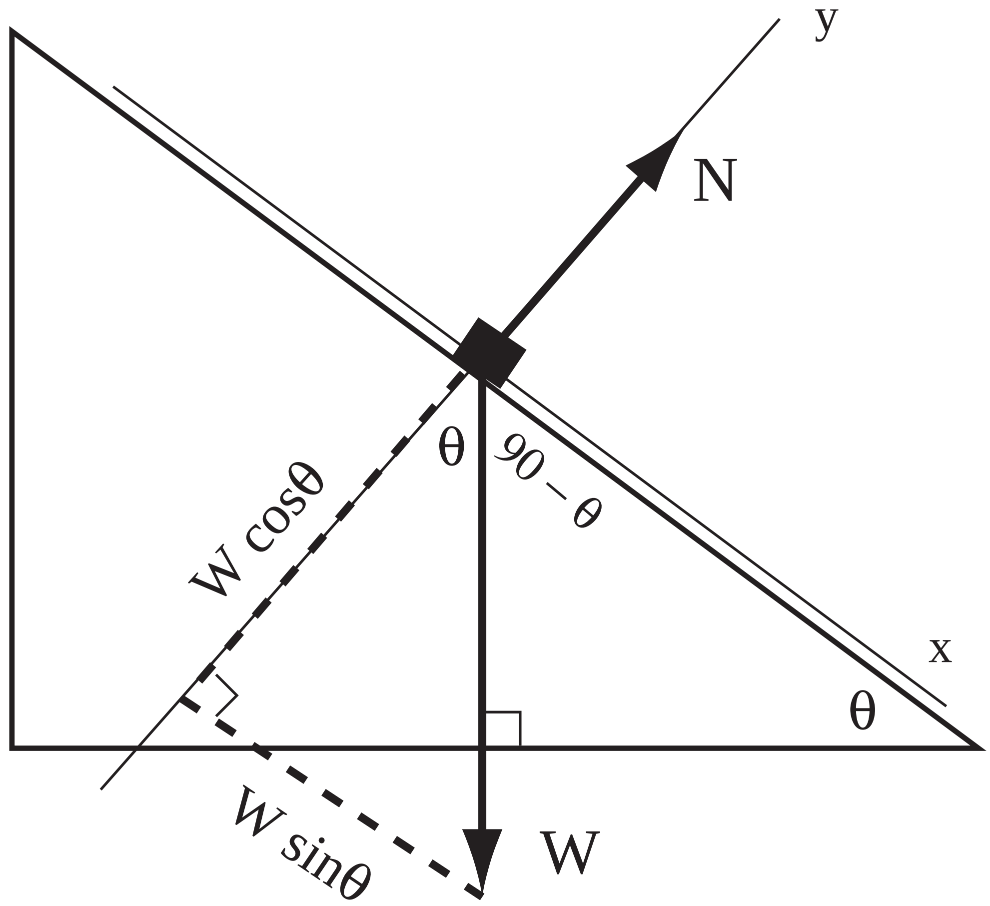


FIGURE 5.3 Block sliding down frictionless incline.

A) Analyzing the y direction,

$$\Sigma F_y = ma_y$$

$$N - W\cos\theta = 0$$

because the block has zero acceleration in the y direction. Thus

$$N = W\cos\theta \quad = \quad \text{mg}\cos\theta$$

B) Analyzing the x direction,

$$\Sigma F_x = ma_x$$

$$W\sin\theta = ma_x$$

$$a_x \quad = \quad \frac{W\sin\theta}{m} = \frac{mg\sin\theta}{m} = g\sin\theta$$

Example Derive a formula for the acceleration of the block system shown in Fig. 5.4 (Atwood machine). Assume the pulley is frictionless and the tension T is the same throughout the rope.

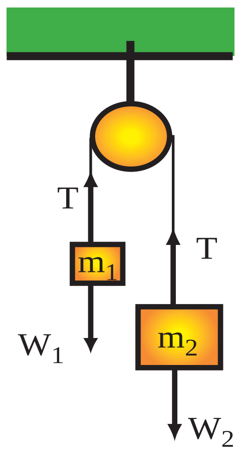


FIGURE 5.4 Atwood machine.

Solution The tension is the same throughout the rope; thus $T_1 = T_2 = T$. Analyze forces in y direction on m_1;

$$\begin{aligned} \Sigma F_y &= m_1 a_1 \\ T - W_1 &= m_1 a \end{aligned} \tag{4.1}$$

with $a_1 \equiv a$. Analyze forces in y direction on m_2;

$$\begin{aligned} \Sigma F_y &= m_2 a_2 \\ T - W_2 &= m_2 a_2 \end{aligned}$$

but if $a_1 = a$ then $a_2 = -a$ giving

$$T - W_2 = -m_2 a \tag{4.2}$$

Subtracting eqn. (4.2) from eqn. (4.1) gives

$$\begin{aligned} T &- W_1 - (T - W_2) = m_1 a - (-m_2 a) \\ &- W_1 + W_2 = m_1 a + m_2 a \\ & a = \frac{W_2 - W_1}{m_1 + m_2} = \frac{m_2 - m_1}{m_1 + m_2} g \end{aligned}$$

Thus a is positive if $m_2 > m_1$ and negative if $m_2 < m_1$.

HISTORICAL NOTE

Isaac Newton is widely regarded as the greatest physicist of all time. One of his major works was *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy.) [University of California Press, Berkeley, California, ed. by F. Cajori; 1934; QA 803 .A45 1934]. Very early on in the book we find the section entitled *Axioms, or Laws of Motion.* The laws are stated as follows:

> *"LAW I: Every body continues in a state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*
>
> *LAW II: The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*
>
> *LAW III: To every action there is always opposed an equal re-action; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."*

After the axioms are stated, the *Principia* is then divided into two major books, namely *Book I: The Motion of Bodies* and *Book II: The Motion of Bodies (in resisting mediums).* In these books we find discussion of such toipics as centripetal forces, conic sections, orbits, rectilinear motion, oscillating pendulum, attractive force of spherical bodies, motion of bodies in fluids, fluid dynamics, hydrostatics, etc. This makes for wonderful reading and is highly recommended.

By the way Newton also invented calculus and the reflecting telescope !

4.9 Problems

Chapter 5

FORCE & MOTION - II

<u>SUGGESTED HOME EXPERIMENT</u>:
Measure the coefficient of static friction between 2 surfaces.

<u>THEMES</u>:
FRICTION.

5.1 Friction

There are two types of friction — static and kinetic. When two surfaces are in relative motion then the friction is kinetic, such as when you slam the brakes on in your car and the car skids along the road. Eventually, kinetic friction will cause the car to stop.

If you put a coin on top of a book and tilt the book at a small angle, the coin will remain stationary. Static friction prevents the coin from sliding. Tilt the book a bit more and still the coin does not slide. The static friction has *increased* to keep the coin in place. Eventually however, static friction will be overcome and the coin will slide down the book (with kinetic friction operating). Notice that the *maximum* amount of static friction occurred just before the coin started to slide.

(LECTURE DEMONSTRATION of above.)

5.2 Properties of Friction

If you press down hard on the coin, then the friction force will increase. When you press down you are causing the normal force N to get bigger. Thus friction is proportional to N. The proportionality constant is called the coefficient of friction μ.

The kinetic friction force f_k is given by

$$f_k \equiv \mu_k N$$

where μ_k is the coefficient of kinetic friction. We saw that static friction varies. However the maximum value of the static friction force $f_{s,\max}$ is

$$f_{s,\max} \equiv \mu_s N$$

Both of these equations can be regarded as *definitions* for μ_k and μ_s.

(Carefully study Samples Problems 6-1, 6-2, 6-3, 6-4).

Example The coefficient of static friction is just the tangent of the angle where two objects start to slide relative to each other. Show that $\mu_s = \tan\theta$.

Solution A force diagram is shown in Fig. 6.1.

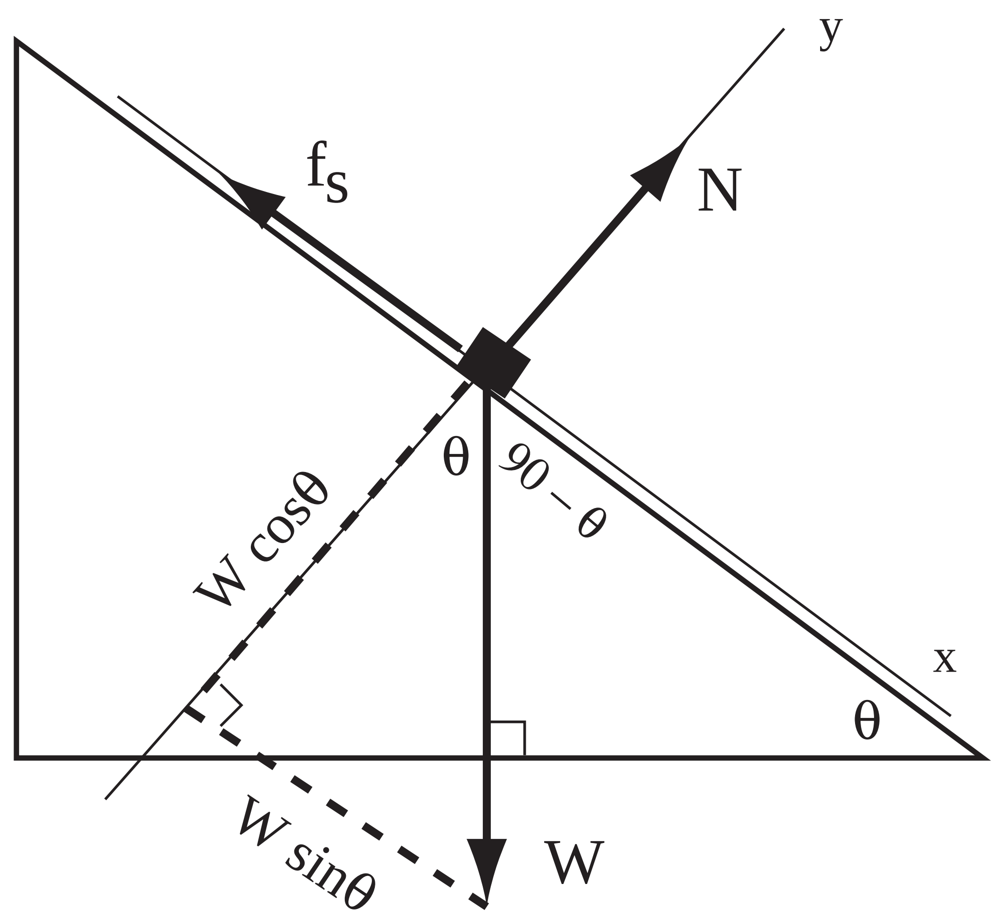


FIGURE 6.1 Block sliding down incline with friction.

Analyze forces in y direction

$$\Sigma F_y = ma_y$$

$$N - W\cos\theta = 0$$

In x direction

$$\Sigma F_x = ma_x$$

$$f_s - W\sin\theta = 0$$

$$\mu_s N - W\sin\theta = 0$$

$$\mu_s = \frac{W\sin\theta}{N}$$

where $a_x = 0$ just before object starts to slide. Now we get N from y equation above ($N = \cos\theta$). Thus

$$\mu_s = \frac{W\sin\theta}{W\cos\theta}$$

or

$$\mu_s = \tan\theta$$

5.3 Drag Force and Terminal Speed

Read

5.4 Uniform Circular Motion

In the case of circular motion we *always* know that the acceleration is $a = \frac{v^2}{r}$. Thus we always know the right hand side of Newton's second law, namely

$$\begin{aligned} \Sigma F &= ma \\ &= \frac{mv^2}{r} \end{aligned}$$

The *forces* that *produce* circular motion get put into the *left hand side.*

Example In designing a curved road, engineers consider the speed v of a car and the coefficient of friction between the car tires and the road. The radius of curvature of the road bend is chosen to be large enough so that the car will be able to derive around smoothly in a part-circle. Work out a formula for the radius of curvature in terms of the speed of the car and the coefficient of friction.

Solution Force diagrams are shown in Fig. 6.2. The top part of the figure shows that *static friction alone keeps the car in circular motion*. (The forward motion of the car involves moving kinetic friction, but the sideways motion involves static friction.)

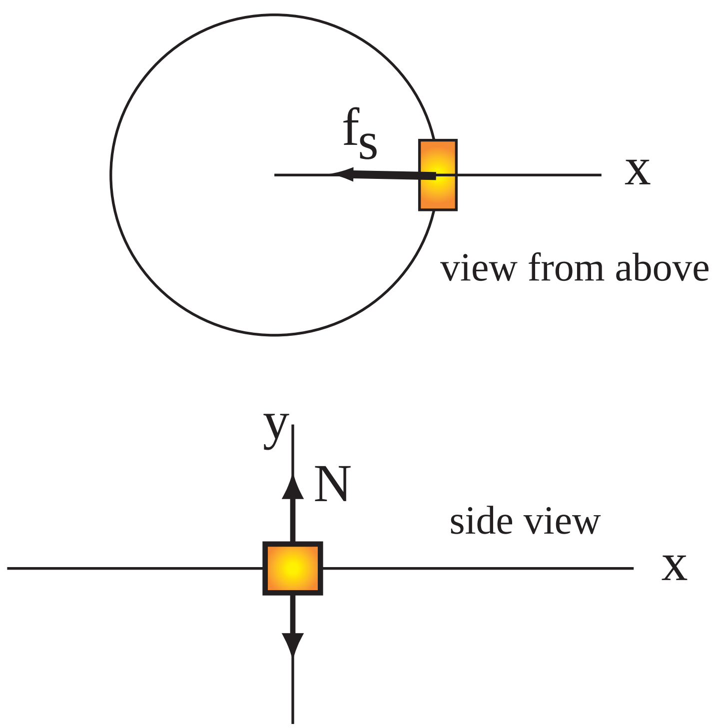


FIGURE 6.2 Car rounding a curve.

In the x direction

$$\begin{aligned}\Sigma F_x &= ma_x \\ f_s &= m\frac{v^2}{r} \\ \mu_s N &= \frac{mv^2}{r}\end{aligned}$$

We get N from the y direction,

$$\Sigma F_y = ma_y$$

$$N - W = 0$$

$$\begin{aligned}N &= W \\ &= mg\end{aligned}$$

Substituting into the x equation gives

$$\mu_s mg = \frac{mv^2}{r}$$

or

$$r = \frac{v^2}{\mu_s g}$$

This formula tells an engineer how large to make the radius of curvature of the road for a given car speed v (say 5 times the speed limit) and a coefficient of friction μ_s.

5.5 Problems

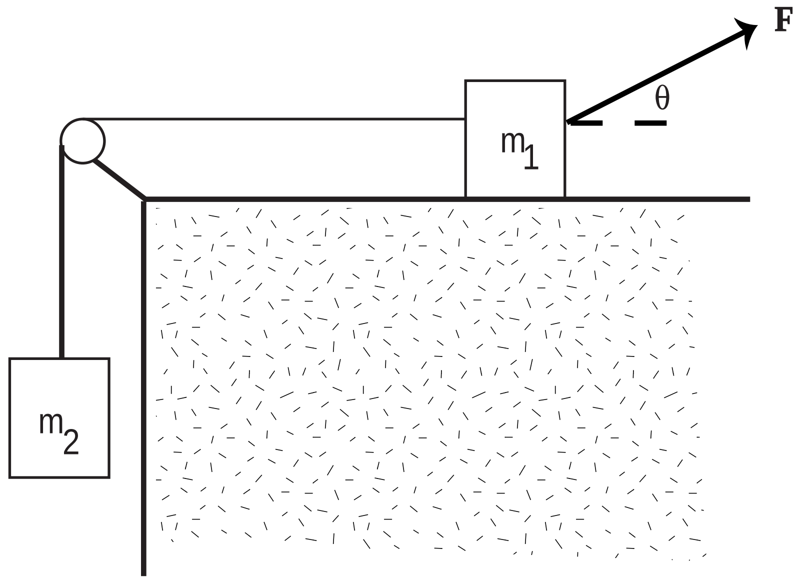
1. A mass m_1 hangs vertically from a string connected to a ceiling. A second mass m_2 hangs below m_1 with m_1 and m_2 also connected by another string. Calculate the tension in each string.

2. What is the acceleration of a snow skier sliding down a frictionless ski slope of angle θ ?

   Check that your answer makes sense for $\theta = 0^o$ and for $\theta = 90^o$.

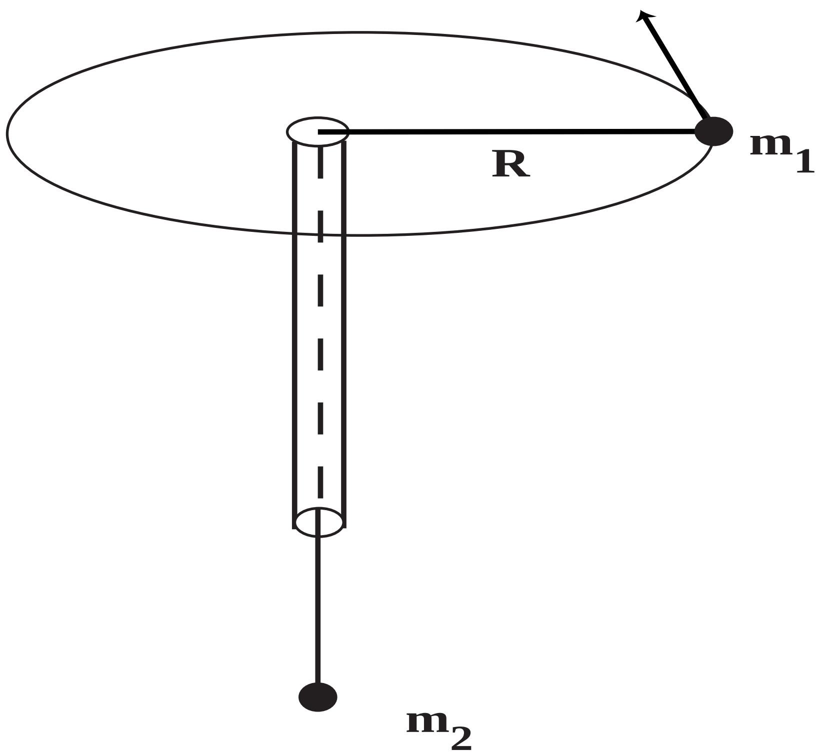
3. A ferris wheel rotates at constant speed in a vertical circle of radius R and it takes time T to complete each circle. Derive a formula, in terms of m, g, R, T, for the weight that a passenger of mass m feels at the top and bottom of the circle. Comment on whether your answers make sense. (Hint: the weight that a passenger feels is just the normal force.)

4. A block of mass m_1 on a rough, horizontal surface is connected to a second mass m_2 by a light cord over a light frictionless pulley as shown in the figure. ('Light' means that we can neglect the mass of the cord and the mass of the pulley.) A force of magnitude F is applied to the mass m_1 as shown, such that m_1 moves to the right. The coefficient of kinetic friction between m_1 and the surface is μ. Derive a formula for the acceleration of the masses. [Serway 5th ed., pg.135, Fig 5.14]



5. If you whirl an object of mass m at the end of a string in a vertical circle of radius R at constant speed v, derive a formula for the tension in the string at the top and bottom of the circle.

6. Two masses m_1 and m_2 are connected by a string passing through a hollow pipe with m_1 being swung around in a circle of radius R and m_2 hanging vertically as shown in the figure.



Obviously if m_1 moves quickly in the circle then m_2 will start to move upwards, but if m_1 moves slowly m_2 will start to fall.

A) Derive an expression for the tension T in the string.

B) Derive an expression for the acceleration of m_2 in terms of the period t of the circular motion.

C) For what period t, will the mass m_2 be at rest?

D) If the masses are equal, what is the answer to Part C)?

E) For a radius of 9.81 m, what is the numerical value of this period?

7. A) What friction force is required to stop a block of mass m moving at speed v_0, assuming that we want the block to stop over a distance d ?

   B) Work out a formula for the coefficient of kinetic friction that will achieve this.

   C) Evaluate numerical answers to the above two questions assuming the mass of the block is $1000kg$, the initial speed is $60\,km\,per\,hour$ and the braking distance is $200m$.

Chapter 6

POTENTIAL ENERGY & CONSERVATION OF ENERGY

SUGGESTED HOME EXPERIMENT:

Design any experiment which illustrates that energy is conserved.

THEMES:

MACHINES.

In this chapterI am going to include the discussion of Chapter 7 and 8 [from Halliday] all together and try to present a single unified approach to the whole topic of work and energy. The textbook by Halliday should be read very carefully for specific illustrations of my unified approach.

In our study of mechanics so far our approach has been to identify all the forces, divide by mass to get acceleration and then solve for velocity, displacement, time, etc. There is an alternative formulation of mechanics which does not rely heavily on force, but rather is based on the concepts of work and energy. The work-energy formulation of mechanics is worthwhile since sometimes it is easier to work with and involves only scalar quantities. Also it leads to a better physical understanding of mechanics. However the key reason for introducing work-energy is because *energy is conserved.* This great discovery simplified a great deal of physics and we shall study it in detail.

6.1 Work

The basic concept of *work* is that it is *force times distance.* You do work on an object by applying a force over a certain distance. When you lift an object you apply a lifting force over the height that you lift the object.

Machines are objects that allow us to do work more efficiently. For example, a *ramp* is what is called a *simple machine.* If you load objects into a truck, then a large ramp (large distance) allows you to apply less force to achieve the same work.

All students should read my handout on simple machines. There it is clearly explained why work is *defined* as force × distance.

LECTURE DEMONSTRATION: SIMPLE MACHINES

Actually the proper physical definition of work is more complicated. The proper definition is

$$\boxed{W \equiv \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r}}$$

Writing

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

and

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

gives

$$\begin{aligned} W &= \int (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \\ &= \int F_x dx + F_y dy + F_z dz \end{aligned}$$

Let's first look at the 1-dimensional case

$$W = \int_{x_i}^{x_f} F_x dx$$

If the force F_x is *constant* then it can be taken outside the integral to give

$$\begin{aligned} W &= F_x \int_{x_i}^{x_f} dx = F_x\,[x]_{x_i}^{x_f} \\ &= F_x(x_f - x_i) = F_x \Delta x \\ &= \text{force} \times \text{distance} \end{aligned}$$

giving us back our original idea. The reason why we have an integral is in case the force depends on distance. The reason we have the scalar product $\vec{F} \cdot d\vec{r}$ is if the force and distance are at some angle, such as a tall person pulling a toy wagon along with a rope inclined at some angle.

By the way, the units of work must be Newton × meter which is given a special name, Joule. Thus

$$\text{Joule} \equiv \text{Newton meter}$$

Example If I push a sled with a constant force of 100 N along a 5 m path, how much work do I do ?

Solution The force is constant and in only 1-dimension, so

$$\begin{aligned} W &= F_x \Delta x \\ &= 100 \text{ N} \times 5 \text{ m} \\ &= 500 \text{ Nm} \\ &= 500 \text{ Joule} \end{aligned}$$

6.2 Kinetic Energy

Now we know that $\vec{F} = m\vec{a}$ and so work can be written

$$W = \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r} = m \int_{r_i}^{r_f} \vec{a} \cdot d\vec{r}$$

where m is taken outside the integral because it's a constant. Let's just consider 1-dimension to make things easier. Thus

$$W = \int_{x_i}^{x_f} F\, dx = m \int_{x_i}^{x_f} a\, dx$$

Now use an old trick.

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$

using the chain rule for derivatives. But $v = \frac{dx}{dt}$, giving

$$\begin{aligned} a &= \frac{dv}{dx} v \\ &= v\frac{dv}{dx} \end{aligned}$$

Thus

$$\begin{aligned} W &= m\int_{x_i}^{x_f} a\, dx = m\int_{x_i}^{x_f} v\frac{dv}{dx} dx \\ &= m\int_{v_i}^{v_f} v\, dv \\ &= m\left[\frac{1}{2}v^2\right]_{v_i}^{v_f} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \end{aligned}$$

Notice that we have found that the work is equal to the change in the quantity $\frac{1}{2}mv^2$. We give this a special name and call it *Kinetic Energy*

$$\boxed{K \equiv \frac{1}{2}mv^2}$$

Thus we have found that $W = K_f - K_i$ or

$$\boxed{W = \Delta K}$$

The total work is always equal to the *change* in kinetic energy. Kinetic energy is the energy of motion. If m is large and v small, or m is small and

v large the kinetic energy in both cases will be comparable. Note also that K must have the same units as W, namely Joule.

What happens when we do work on an object? Well if you *lift* up an object, you increase its *Potential* energy (more about that in a moment). If you work on an object you can also increase its kinetic energy. If you push a marble on a table its speed will increase and so you have changed its kinetic energy.

Example A sled of mass m is stationary on some frictionless ice. If I push the sled with a force of F over a distance Δx, what will be the speed of the sled ?

Solution The force is constant and is 1-dimension, so

$$\begin{aligned} W = F\,\Delta x &= \Delta K = K_f - K_i \\ &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \end{aligned}$$

Now $v_i = 0$, giving

$$F\,\Delta x = \frac{1}{2}mv_f^2$$

or

$$v_f = \sqrt{\frac{2F\,\Delta x}{m}}$$

The neat thing is that we can get exactly the same answer with our old methods, as the next example shows.

Example Work out the previous example using the constant acceleration equations.

Solution The acceleration is just

$$a = \frac{F}{m}$$

The constant acceleration equation that helps us is

$$v^2 = v_0^2 + 2a(x - x_0)$$

Now $x - x_0 = \Delta x$ m and $v_0 = 0$ giving

$$\begin{aligned} v &= \sqrt{2a(x - x_0)} \\ &= \sqrt{\frac{2F\,\Delta x}{m}} \end{aligned}$$

which is the same answer as the previous example.

In the previous two examples notice how the equation

$$W = F\Delta x = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

is equivalent to

$$v^2 = v_0^2 + 2a(x - x_0)$$

Modify this to

$$\begin{aligned}
\frac{1}{2}v^2 &= \frac{1}{2}v_0^2 + a(x - x_0) \\
\frac{1}{2}v^2 &= \frac{1}{2}v_0^2 + a\Delta x \\
\frac{1}{2}mv^2 &= \frac{1}{2}mv_0^2 + ma\Delta x \\
&= \frac{1}{2}mv_0^2 + F\Delta x
\end{aligned}$$

or

$$\begin{aligned}
F\Delta x &= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \\
&= \Delta K
\end{aligned}$$

as we have above !

Thus the work-energy formulation provides an alternative approach to mechanics.

6.3 Work-Energy Theorem

Let's review what we have done. Work was *defined* as $W \equiv \int \vec{F} \cdot d\vec{r}$ and by putting in $F = ma$ we found that the total work is *always* ΔK where the kinetic energy is defined as $K \equiv \frac{1}{2}mv^2$. Thus

$$W \equiv \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = \Delta K.$$

So far so good. Note *carefully* what we did to get this result. We put in the right hand side of $F = ma$ to prove $W = \Delta K$. What we actually did was

$$W = \int_{r_1}^{r_2} m\vec{a} \cdot d\vec{r} \equiv \Delta K$$

Now let's *not* put $\vec{F} = m\vec{a}$ but just study the integral $\int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$ by itself.

Before we do that, we must recognize that there are two types of forces called *conservative* and *non-conservative*. *You should carefully read Section 8-2 of Halliday to learn about this.*

Anyway, to put it briefly, conservative forces "bounce back" and non-conservative forces don't. Gravity is a conservative force. If you lift an object against gravity and let it go then the object falls back to where it was. Spring forces are conservative. If you pull a spring and then let it go, it bounces back to where it was. However friction is non-conservative. If you slide an object along the table against friction and let go, then the object just stays there.

With *conservative* forces we always associate a *potential energy*.

Thus any force $\vec{F}$ can be broken up into the conservative piece $\vec{F}_C$ and the non-conservative piece $\vec{F}_{NC}$, as in

$$\begin{aligned} W &\equiv \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r} \\ &= \int_{r_i}^{r_f} \vec{F}_C \cdot d\vec{r} + \int_{r_i}^{r_f} \vec{F}_{NC} \cdot d\vec{r} \\ &\equiv W_C + W_{NC} \end{aligned}$$

and each piece corresponds therefore to conservative work W_C and non-conservative work W_{NC}. Let's *define* the conservative piece as the *negative* of the change in a new quantity called potential energy U. The definition is

$$W_C \equiv -\Delta U$$

where $-\Delta U = -(U_f - U_i) = -U_f + U_i$. Now we found that the *total* work W was always ΔK. Combining all of this we have

$$\begin{aligned} W &= W_C + W_{NC} = \Delta K \\ &= -\Delta U + W_{NC} \end{aligned}$$

or

$$\boxed{\Delta U + \Delta K = W_{NC}}$$

which is the famous Work-Energy theorem.

6.4 Gravitational Potential Energy

We have been doing a lot of formal analysis. Let's backtrack a little and try to understand better what we have done. Let's look at the conservative piece a little more closely and examine potential energy in more detail.

Let's consider the simplest conservative force, namely the weight force where $W = mg$ which is a constant. Let's work out W_C and ΔU in 1-dimension.

The gravitational force due to weight is

$$\vec{F}_C = -mg\,\hat{j}$$

giving

$$\begin{aligned} W_C &\equiv -\Delta U = \int \vec{F}_C \cdot d\vec{r} \\ &= -(U_f - U_i) = -mg \int \hat{j} \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = -mg \int_{y_i}^{y_f} dy \\ &= -U_f + U_i = -mg\,[y]_{y_i}^{y_f} = -mg(y_f - y_i) = -mgy_f + mgy_i \end{aligned}$$

which gives $-U_f = -mgy_f$, i.e. $U_f = mgy_f$ and $U_i = mgy_i$. Thus we can simply write

$$\boxed{U = mgy}$$

which is our expression for gravitational potential energy. If an object is raised to a large height y then it has a large potential energy.

If we do *work* in lifting an object, then we *give* that object potential energy, just as we can give an object kinetic energy by doing work. Similarly if an object has potential energy or kinetic energy then the object can do work by releasing that energy. This is the principle of hydro-electric power generators. A large amount of water is stored in a dam at a large height y with a large potential energy. When the water falls and reduces it potential energy (smaller y) the energy is converted into work to drive electric generators.

6.5 Conservation of Energy

Let's summarize again. The work-energy theorem is $\Delta U + \Delta K = W_{NC}$ where $K \equiv \frac{1}{2}mv^2$ and *for gravity* $U = mgy$. W_{NC} is the non-conservative work, such as friction, heat, sound, etc. It is often zero as in the next example.

Example If you drop an object from a height H, with what speed does it hit the ground? Deduce the answer using the work-energy theorem. Assume $W_{NC} = 0$.

Solution $W_{NC} = 0$ because things such as heat and friction are negligible. Thus the work energy theorem is

$$\Delta U + \Delta K = 0$$

or

$$U_f - U_i + K_f - K_i = 0$$

or

$$\boxed{U_f + K_f = U_i + K_i}$$

That is the total energy

$$\boxed{E \equiv U + K}$$

is constant. This is the famous *conservation of mechanical energy*, i.e. $E_f = E_i$.
We have $K = \frac{1}{2}mv^2$ and $U = mgy$ giving

$$mgy_f + \frac{1}{2}mv_f^2 = mgy_i + \frac{1}{2}mv_i^2$$

but $y_f = 0$ and $y_i = H$ and $v_i = 0$. Thus

$$\frac{1}{2}mv_f^2 = mgH$$

or

$$v_f = \sqrt{2gH}$$

Example Complete the previous example using the constant acceleration equations.

Solution The most convenient equation is

$$v^2 = v_0^2 + 2a(y - y_0)$$

but $v_0 = 0$ and $y - y_0 = 0 - H = -H$ and $a = -g$, giving

$$\begin{aligned} v &= \sqrt{2g(y - y_0)} = \sqrt{2g(0 - -H)} \\ &= \sqrt{2gH} \end{aligned}$$

which is the same answer as before.

Example Prove that a swinging pendulum always rises to the same height. (Neglect friction.)

Solution With friction ignored we have $W_{NC} = 0$ and

$$\frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i$$

I let go of the pendulum with speed $v_i = 0$ and it returns with speed $v_f = 0$. Thus

$$mgy_f = mgy_i$$

or

$$y_f = y_i$$

LECTURE DEMONSTRATION: Bowling Ball Pendulum

6.6 Spring Potential Energy

When you pull a spring you feel a force in the *opposite* direction from which you pull. Also the force increases with distance. This can be expressed as

$$\vec{F}_C = -kx\,\hat{i}$$

in the x direction. Thus

$$\begin{aligned} W_C \equiv -\Delta U &= \int \vec{F}_C \cdot d\vec{r} \\ = -(U_f - U_i) &= -k \int x\,\hat{i} \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \\ = -U_f + U_i &= -k \int_{x_i}^{x_f} x\,dx = -k \left[\frac{1}{2}x^2\right]_{x_i}^{x_f} \\ &= -k\left(\frac{1}{2}x_f^2 - \frac{1}{2}x_i^2\right) \\ &= -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_i^2 \end{aligned}$$

which gives $-U_f = -\frac{1}{2}kx_f^2$, i.e. $U_f = \frac{1}{2}kx_f^2$ and $U_i = \frac{1}{2}kx_i^2$. Thus we can simplify and write

$$\boxed{U = \frac{1}{2}kx^2}$$

which is our expression for spring potential energy

Example A spring with spring constant k has a mass of m on one end. The spring is stretched by a distance d. When released, how fast will the mass be moving when it returns to its original position? (Assume the motion occurs on a horizontal frictionless surface.)

Solution $W_{NC} = 0$ giving

$$U_f + K_f = U_i + K_i$$

$$\frac{1}{2}kx_f^2 + \frac{1}{2}mv_f^2 = \frac{1}{2}kx_i + \frac{1}{2}mv_i^2$$

Now $x_f = 0$, $x_i = d$ m and $v_i = 0$. Thus

$$mv_f^2 = kd^2$$

or

$$v_f = d\sqrt{\frac{k}{m}}$$

IMPORTANT NOTE:

The spring is an example of a *variable* force $F = -kx$ which varies as distance. Thus the acceleration $a = -\frac{kx}{m}$ is *not* constant and the constant acceleration equations *cannot* be used to solve the previous example. Also the *variable* force requires the *integral* definition of work as $W = \int \vec{F} \cdot d\vec{r}$.

HALLIDAY SIMULATION: "A Spring"

6.7 Appendix: alternative method to obtain potential energy

Potential energy is defined through

$$W_c = \int \vec{F}_c \cdot d\vec{r} \equiv -\Delta U$$

Let's just ignore the vectors for the moment and write

$$\int F_c dr = -\Delta U$$

Thus we *must* have

$$\boxed{F_c = -\frac{dU}{dr}}$$

To see this write

$$\begin{aligned}\int_i^f F_c dr &= -\int_i^f \frac{dU}{dr} dr = -\int_{U_i}^{U_f} dU = -\left[U\right]_{U_i}^{U_f} \\ &= -(U_f - U_i) = -\Delta U.\end{aligned}$$

(cf. Fundamental Theorem of Calculus).
For gravity we have $\vec{F} = -mg\hat{j}$ or $F = -mg$ and for a spring we have $\vec{F} = -kx\hat{i}$ or $F = -kx$. Thus instead of working out the integral $\int \vec{F} \cdot d\vec{r}$ to get U, just ask what U will give F according to $F = -\frac{dU}{dr}$.

Example For gravity $F = -mg$, derive U without doing an integral.

Solution For gravity $dr \equiv dy$. The question is what U will give

$$F = -mg = -\frac{dU}{dy}$$

The answer is $U = mgy$. Let's check:

$$-\frac{dU}{dy} = -mg\frac{dy}{dy} = -mg$$

which is the F we started with !

Example For a spring $F = -kx$, derive U without doing an integral.

Solution For a spring $d \equiv dx$. The question is what U will give

$$F = -kx = -\frac{dU}{dx}$$

The answer is $U = \frac{1}{2}kx^2$. Let's check

$$-\frac{dU}{dx} = -\frac{1}{2}k\frac{dx^2}{dx} = -\frac{1}{2}k\,2x = -kx$$

which is the F we started with!

6.8 Problems

1. A block of mass m slides down a rough incline of height H and angle θ to the horizontal. Calculate the speed of the block when it reaches the bottom of the incline, assuming the coefficient of kinetic friction is μ_k.

Chapter 7

SYSTEMS OF PARTICLES

SUGGESTED HOME EXPERIMENT:
Locate the center of mass of an object.

THEMES:
FROM ONE TO MANY.

Almost everything we have done so far has referred to the motion of a *single* body of mass m, and we have always been able to treat that single body as though it were a point. But suppose we wich to study the motion of a *complex* object such as a spinning baseball bat (Fig. 9-1 Halliday) or a dancing ballerina (Fig. 9-8 Halliday) ? A bat and a ballerina can be considered as a collection of a huge number of single particles. We now want to study the motion of such *systems of particles*.

7.1 A Special Point

When we studied say a block sliding down an incline, and replaced it with just a single point and studied the motion of that point, we made a very convenient simplification. This special point is called the *center of mass* of an object and by studying its motion alone we avoid all the extra complications of a body of finite size.

"The center of mass of a body or a system of bodies is the points that moves as though all of the mass were aconcentrated there and all external forces were applied there." [Halliday, 1997]. Notice we have included a system of bodies. For instance the motion of the Earth-Moon system around the Sun is actually governed by the center of mass of the two-body Earth-Moon system.

An easy way to find the center of mass is to just regard it as a *balance point*. For example the center of mass of a ruler is located as the point where you can balance the ruler on your finger without it falling off. Thus we already know the answer for a ruler ! The center of mass is located at the center. We will prove this mathematically in a moment.

7.2 The Center of Mass

Systems of Particles

Now let's come up with a *mathematical* definition for center of mass which is more precise than just saying it's the balance point (although the balance point *always* gives the correct answer). The location of the center of mass is defined as

$$\boxed{\vec{r}_{cm} \equiv \frac{1}{M} \sum_i^n m_i \vec{r}_i} \tag{7.1}$$

where the sum over i running from 1 to n *means* sum over all of the point particles within the body, assuming there are a total of n point particles. M is the total mass of all the individual bodies and can be written

$$M \equiv \sum_{i}^{n} m_i \tag{7.2}$$

We have defined the center of mass. Now let's see if our definition makes sense. First of all it's a vector equation and so what it really means is the usual 3-dimensional decomposition as

$$x_{cm} \equiv \frac{1}{M} \sum_{i}^{n} m_i x_i \tag{7.3}$$

$$y_{cm} \equiv \frac{1}{M} \sum_{i}^{n} m_i y_i \tag{7.4}$$

$$z_{cm} \equiv \frac{1}{M} \sum_{i}^{n} m_i z_i \tag{7.5}$$

Let's just consider the 1-dimensional version for the case of 2 bodies only. Then the total mass M becomes

$$M = m_1 + m_2 \tag{7.6}$$

and (7.2) becomes

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{7.7}$$

Does *this* make sense ? Let's see.

Example Where is the position of the center of mass for a system consisting of two dumbells, each with the same mass m each at the end of a $4ft$ massless rod ?

Solution Now you *know* that the answer to this must be at the center of the rod. After all that is the balancing point. That is our guess is that $x_{cm} = 2ft$. Let's use our definition of center of mass, equation (7.1) and see if it gives this answer.

Now we have a 1-dimensional problem and therefore (7.1) reduces to only (7.3). Furthermore we only have two bodies and this reduces further to (7.7). Choosing the origin of the x-coordinate system to be at the left dumbell gives $x_1 = 0ft$ and $x_2 = 4ft$. Substituting gives

$$x_{cm} = \frac{m \times 0ft + m \times 4ft}{m + m} = 2ft \tag{7.8}$$

which is *exactly* what we expected. *Therefore* we can believe that our definition for center of mass (7.1) makes perfect sense.

Let's look at what happens if we use a different coordinate system.

Example Repeat the previous problem, but with the x-origin located at the center between the two dumbells instead of on the left dumbell.

Solution Well now we would guess that the center of mass would be given by $x_{cm} = 0$. Let's see if our formula works here. With the origin of the x-axis chosen to be at the center of the dumbells we have the position of each dumbell given by $x_1 = -2ft$ and $x_2 = +2ft$ respectively. Subsituting we get

$$x_{cm} = \frac{m \times (-2ft) + m \times (+2ft)}{m + m} = 0 \tag{7.9}$$

which is *exactly* what we expected. *Therefore* again we can believe that our definition for center of mass (7.1) makes perfect sense.

Being able to find the center of mass is actually *useful*, as the following example shows.

Example A baby of mass m_B sits on a see-saw. Mary's mass is m_M. Where should Mary sit in order to balance the see-saw ? Work out a formula and also a numerical answer if $m_B = 10\ kg$ and $m_M = 80\ kg$.

Solution Again our intuition tells us that we can guess that the ratio of the distances should be 1/8. That is the baby should be 8 times as far away from the center of the see-saw as Mary. Let's see if our center of mass definition (7.1) tells us this.

Again this is a 1-dimensional, 2-body problem and so the formula for the center of mass is again

$$x_{cm} = \frac{m_B x_B + m_M x_M}{m_B + m_M}.$$

Now we *want* the center of mass located at the center of the see-saw and we will put the origin of our x-axis there as well. Thus

$$x_{cm} = \frac{m_B x_B + m_M x_M}{m_B + m_M} = 0$$

giving

$$m_B x_B + m_M x_M = 0$$

which means that

$$m_B x_B = -m_M x_M$$

or

$$\frac{x_B}{x_M} = -\frac{m_B}{m_M} = -\frac{m}{M}$$

or

$$x_M = -\frac{M}{m} x_B$$

Putting in numbers we get

$$x_M = -\frac{80\ kg}{10\ kg} x_B = -8\ x_B$$

just as we suspected.

Rigid Bodies

Above we considered finding the center of mass of two bodies. This can easily be extended to 3 or more bodies and some of this will be explored in the homework. That's all well and good, but how do we find the center of mass of systems made up of millions of particles such as a baseball bat. In other words how do we find the center of mass of rigid bodies ? That's what we will look at now.

In physics whenever we want to change our study from a collection of *discrete* particles (desribed by a sum $\sum_i$) to a *continuous* collection of particles, the *sum just changes to an integral*. Hopefully this makes perfect sense from what you have studied in calculus. You all now know that an integral is just the limit of a sequence of sums.

Now each of the millions of particles in a rigid body has a tiny little mass denoted by dm. For a discrete collection of particles we had (7.1) as

$$x_{cm} \equiv \frac{1}{M}\sum_i^n x_i m_i \tag{7.10}$$

but for a continuous distribution of particles we now define

$$x_{cm} \equiv \frac{1}{M}\int x dm \tag{7.11}$$

This is easier to work with if we introduce density ρ as mass / volume or

$$\rho \equiv \frac{mass}{volume} \equiv \frac{dm}{dV} \equiv \frac{M}{V}. \tag{7.12}$$

where dV is the volume occupied by the mass dm. Thus our definition can be written

$$\boxed{x_{cm} \equiv \frac{1}{M}\int x dm \equiv \frac{1}{M}\int x\rho dV}$$

and the same for y and z. **If the density is constant**, then it can be taken outside the integral to give

$$\boxed{x_{cm} = \frac{1}{V}\int x dV}$$

and the same for y and z.

There's one additional catch. Above we defined a 3-dimensional density as mass / volume. But what if we have a dense 1-dimensional object such as a very long and thin pencil. Well then we will want a *linear* mass density.

Instead of ρ, we use the symbol λ for linear mass density and define it as

$$\lambda \equiv \frac{\text{mass}}{\text{length}} \equiv \frac{dm}{dL} \equiv \frac{M}{L}$$

so that now we have

$$x_{\text{cm}} \equiv \frac{1}{M}\int x\,dm = \frac{1}{M}\int x\,\lambda\,dL$$

and for a constant λ,

$$x_{\text{cm}} = \frac{1}{L}\int x\,dL$$

Because this is *linear* mass density we do *not* have any equations for x or y. Similarly we may have mass distributed only in 2 dimensions such as the surface of a table. We use *area* mass density defined as

$$\sigma \equiv \frac{\text{mass}}{\text{area}} \equiv \frac{dm}{dA} \equiv \frac{M}{A}$$

giving

$$x_{\text{cm}} \equiv \frac{1}{M}\int x\,dm = \frac{1}{M}\int x\,\sigma\,dA$$

and for constant σ,

$$x_{\text{cm}} = \frac{1}{A}\int x\,dA$$

and similarly for y; but there is no equation for z. (why?)

Example Locate the center of mass of a very thin pencil of length L balanced sideways.

Solution Again using intuition we *know* the answer must be at the center of the pencil. Now the element of length $dL \equiv dx$, and the linear mass density λ of the pencil is constant, so that

$$x_{\text{cm}} = \frac{1}{L}\int_0^L x\,dx$$

We have taken the origin ($x = 0$) to be at one end of the pencil. Thus

$$x_{\text{cm}} = \frac{1}{L}\left[\frac{1}{2}x^2\right]_0^L = \frac{1}{L}\left(\frac{1}{2}L^2 - 0\right) = \frac{1}{2}L$$

which is just the answer we expected! Thus we can *believe* that the formulas given previously really do work.

7.3 Newton's Second Law for a System of Particles

For a single particle of mass m we already have encountered Newton's second law, namely $\sum \vec{F} = m\vec{a}$, and $\sum \vec{F}$ are all the forces acting on the mass m and $\vec{a}$ is the resulting acceleration of the mass m. What happens for a system of particles?

The end result is

$$\boxed{\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}}} \tag{7.13}$$

where $\sum \vec{F}_{\text{ext}}$ is the sum of all *external forces* acting on the body (all the internal forces cancel out to zero), M is the *total mass* of the body and $\vec{a}_{\text{cm}}$ is the acceleration of the center of mass of the body.

Example Prove equation (7.13).

Solution Recall our definition of center of mass, namely

$$\vec{r}_{\text{cm}} \equiv \frac{1}{M}\sum_i m_i \vec{r}_i$$

or

$$M\vec{r}_{\text{cm}} = \sum_i m_i \vec{r}_i$$

Taking the time derivative gives

$$M\vec{v}_{\text{cm}} = \sum_i m_i \vec{v}_i$$

and taking the time derivative again gives

$$\begin{aligned} M\vec{a}_{\text{cm}} &= \sum_i m_i \vec{a}_i \\ &= \sum_i \vec{F}_i \end{aligned}$$

which is just the sum of all the forces acting on each mass m_i. These forces will be both *external* and *internal*. However for a rigid body all the internal forces must cancel because in a *rigid* body the particles don't move relative to each other. Thus $\sum\limits_i \vec{F}_i$ just becomes $\sum \vec{F}_{\text{ext}}$ in agreement with (7.13).

7.4 Linear Momentum of a Point Particle

A more fundamental way of discussing Newton's second law is in terms of a new quantity called *momentum*. It is *defined* as

$$\vec{p} \equiv m\vec{v}$$

and it is important because it is a *conserved* quantity just like energy. The *proper* way to write Newton's second law is

$$\boxed{\sum \vec{F} = \frac{d\vec{p}}{dt}}$$

Now $\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$ *if* the mass is constant. Thus $\frac{d\vec{p}}{dt} = m\vec{a}$ if the mass is constant. (If the mass is not constant then $\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{v} = m\vec{a} + \frac{dm}{dt}\vec{v}$ so that Newton's second law actually reads $\sum \vec{F} = m\vec{a} + \frac{dm}{dt}\vec{v}$).

7.5 Linear Momentum of a System of Particles

The total momentum $\vec{P}$ of a system of particles is just the sum of the momenta of each individual particle, namely

$$\vec{P} = \sum_i \vec{p}_i$$

Now from the previous example we had $M\vec{v}_{\rm cm} = \sum_i m_i\vec{v}_i = \sum_i \vec{p}_i$, giving the total momentum of a system of particles as

$$\boxed{\vec{P} = M\vec{v}_{\rm cm}}$$

which is a very nice handy formula for the total momentum equals total mass multiplied by the velocity of the center of mass. Taking the time derivative gives $\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{\rm cm}}{dt} = M\vec{a}_{\rm cm}$ *assuming that M is constant.* Thus Newton's second law for a system of particles can be written

$$\boxed{\sum \vec{F}_{\rm ext} = \frac{d\vec{P}}{dt}}$$

7.6 Conservation of Linear Momentum

If all the external forces are zero ($\sum \vec{F}_{\rm ext} = 0$) then $\frac{d\vec{P}}{dt} = 0$ which implies that the total momentum

$$\boxed{\vec{P} = \text{constant}} \tag{7.14}$$

Note that this is *only* true if all the external forces are zero. Halliday calls this a *closed, isolated system.*

Another way of stating (7.14) is

$$\boxed{\vec{P}_i = \vec{P}_f}$$

Remembering that $\vec{P}$ is the total momentum of a system of particles ($\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$), the conservation equation is

$$\boxed{\vec{p}_{1_i} + \vec{p}_{2_i} + \vec{p}_{3_i} + \cdots = \vec{p}_{1_f} + \vec{p}_{2_f} + \vec{p}_{3_f} + \cdots}$$

This is a vector equation, so we must always write it out in x, y, or z components.

LECTURE DEMONSTRATION: Explosion: spring release on air track.

Example A rifle of mass m_R fires a bullet of mass m_B which emerges at a speed of v_{B_f}. With what speed does the rifle recoil ?

Solution The bullet-rifle system is a closed, isolated system. When the rifle is held at *rest* the sum of all external forces is zero. Thus momentum is conserved for the bullet (B)–rifle (R) two body system. The total momentum is $\vec{P} = \vec{p}_R + \vec{p}_B$, so that conservation of momentum is

$$\vec{p}_{R_i} + \vec{p}_{B_i} = \vec{p}_{R_f} + \vec{p}_{B_f}$$

Now this is a vector equation, so it must be written in terms of components, namely

$$\begin{aligned} p_{R_{x_i}} + p_{B_{x_i}} &= p_{R_{x_f}} + p_{B_{x_f}} \\ p_{R_{y_i}} + p_{B_{y_i}} &= p_{R_{y_f}} + p_{B_{y_f}} \end{aligned}$$

but there is only motion in the x direction and nothing is happening in the y direction, so let's re-write the x-equation, leaving off the x's as

$$p_{R_i} + p_{B_i} = p_{R_f} + p_{B_f}$$

or

$$m_R v_{R_i} + m_B v_{B_i} = m_R v_{R_f} + m_B v_{B_f}$$

But $v_{R_i} + v_{B_i} = 0$ because before the gun is fired (initial situation) the bullet and gun do not move. After the gun is fired (final situation) they both move. Thus

$$O = m_R v_{R_f} + m_B v_{B_f}$$

$$\Rightarrow \quad v_{R_f} \quad = \quad -\frac{m_B}{m_R} v_{B_f}$$

where the minus sign indicates that the rifle moves in a direction opposite to the bullet.

7.7 Problems

1. A particle of mass m is located on the x axis at the position $x = 1$ and a particle of mass $2m$ is located on the y axis at position $y = 1$ and a third particle of mass m is located off-axis at the position $(x, y) = (1, 1)$. What is the location of the center of mass?

2. Consider a square flat table-top. Prove that the center of mass lies at the center of the table-top, assuming a constant mass density.

3. A child of mass m_c is riding a sled of mass m_s moving freely along an icy frictionless surface at speed v_0. If the child falls off the sled, derive a formula for the change in speed of the sled. (Note: energy is not conserved !) WRONG WRONG WRONG ??????????????
   speed of sled remains same - person keeps moving when fall off ???????

Chapter 8

COLLISIONS

SUGGESTED HOME EXPERIMENT:
Design a simple experiment illustrating momentum conservation.

THEMES:
COLLISIONS.

8.1 What is a Collision?

Read

8.2 Impulse and Linear Momentum

Leave out

8.3 Elastic Collisions in 1-dimension

Recall our work energy theorem for a single particle,

$$\Delta U + \Delta K = W_{NC}$$

or

$$U_f - U_i + K_f - K_i = W_{NC}$$

or

$$U_f + K_f = U_i + K_i + W_{NC}$$

If $W_{NC} \neq 0$ then energy will not be conserved. For a two-body collision process, then an *inelastic* collision is one in which energy is not conserved (i.e. $W_{NC} \neq 0$), but an *elastic* collision is one in which energy is conserved ($W_{NC} = 0$).

Now if you think of a collision of two billiard balls on a horizontal pool table then $U_f = mgy_f$ and $U_i = mgy_i$, but $y_f = y_i$ and thus $U_f = U_i$ or $\Delta U = 0$. Thus the above work-energy theorem would be

$$K_f = K_i + W_{NC}$$

Thus for collisions where $U_i = U_f$, we often say more simply that an elastic collision is when the *kinetic energy* alone is conserved and an inelastic collision is when it is not conserved.

In this section we first will deal only with *elastic* collisions in 1-dimension.

Example A billiard ball of mass m_1 and initial speed v_{1i} hits a stationary ball of mass m_2. All the motion occurs in a straight line. Calculate the final speeds of both balls in terms of m_1, m_2, v_{1i}, assuming the collison is elastic (Is this a good assumption?).

Solution All the motion is in 1-dimension and so conservation of momentum (with $v_{2i} = 0$) is just

$$m_1 v_{1i} + 0 = m_1 v_{1f} + m_2 v_{2f}$$

and conservation of kinetic energy is

$$\frac{1}{2} m_1 v_{1i}^2 + 0 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Here we have two equations with the two unknowns v_{1f} and v_{2f}. Thus the rest of the problem is simply doing some algebra. Let's solve for v_{1f} in the first equation and then substitute into the second equation to get v_{2f}. Thus

$$v_{1f} = v_{1i} - \frac{m_2}{m_1} v_{2f}$$

or

$$v_{1f}^2 = v_{1i}^2 - 2\frac{m_2}{m_1} v_{2f} v_{1i} + \left(\frac{m_2}{m_1}\right)^2 v_{2f}^2$$

Substituting this into the conservation of kinetic energy equation gives

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1i}^2 - m_2 v_{2f} v_{1i} + \frac{1}{2}\frac{m_2^2}{m_1} v_{2f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

which simplifies to

$$0 = -2 m_2 v_{1i} + v_{2f}\left(\frac{m_2^2}{m_1} + m_2\right)$$

giving

$$v_{2f} = \frac{2m_2}{\frac{m_2^2}{m_1} + m_2} v_{1i}$$

which is finally

$$\boxed{v_{2_f} = \frac{2m_1}{m_1 + m_2} v_{1i}}$$

Substituting this back into the conservation of momentum equation gives

$$m_1 v_{1i} = m_1 v_{1f} + \frac{2m_1 m_2}{m_1 + m_2} v_{1i}$$

which gives

$$v_{1f} = v_{1i}\left(1 - \frac{2m_2}{m_1 + m_2}\right) = v_{1i}\frac{m_1 + m_2 - 2m_2}{m_1 + m_2}$$

or

$$\boxed{v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}}$$

There are some interesting special situations to consider.

1) Equal masses ($m_1 = m_2$). This implies that $v_{1f} = 0$ and $v_{2f} = v_{1i}$. That is the projectile billiard ball *stops* and transfers all of its speed to the target ball. (This is also true if the target is moving.)

2) Massive target ($m_2 \gg m_1$). In this case we get $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx \frac{2m_1}{m_2} v_{1i} \approx 0$ which means the projectile bounces off at the same speed and the target remains stationary.

3) Massive projectile ($m_1 \gg m_2$). Now we get $v_{2f} \approx 2v_{1i}$ and $v_{1f} \approx v_{1i}$ meaning that the projectile keeps charging ahead at about the same speed and the target moves off at double the speed of the projectile.

COMPUTER SIMULATIONS

LECTURE DEMONSTRATION: colliding pendula (Sample Problem 10-3)

All students should carefully study the Moving Target discussion on Pg. 220 of Halliday.

8.4 Inelastic Collisions in 1-dimension

A *completely inelastic* collision is defined as one in which the two particles *stick together* after the collision.

Example Repeat the previous example for a completely inelastic collision.

Solution If the particles stick to each other after the collision then their final speeds are the same; let's call it V,

$$v_{1f} = v_{2f} \equiv V$$

And writing $v_{1i} \equiv v$ we have from conservation of momentum

$$m_1 v + 0 = m_1 V + m_2 V$$

or

$$V = \frac{m_1}{m_1 + m_2} v$$

Let's look again at the special situations.

1) Equal masses ($m_1 = m_2$). This gives

$$V = \frac{1}{2} v$$

2) Massive target ($m_2 \gg m_1$). This gives

$$V \approx \frac{m_1}{m_2} v \approx 0$$

3) Massive projectile ($m_1 \gg m_2$). This gives

$$V \approx v$$

8.5 Collisions in 2-dimensions

Glancing collisions (i.e. *not* head-on) are more complicated to analyze. Figure 10.1 shows a typical configuration.

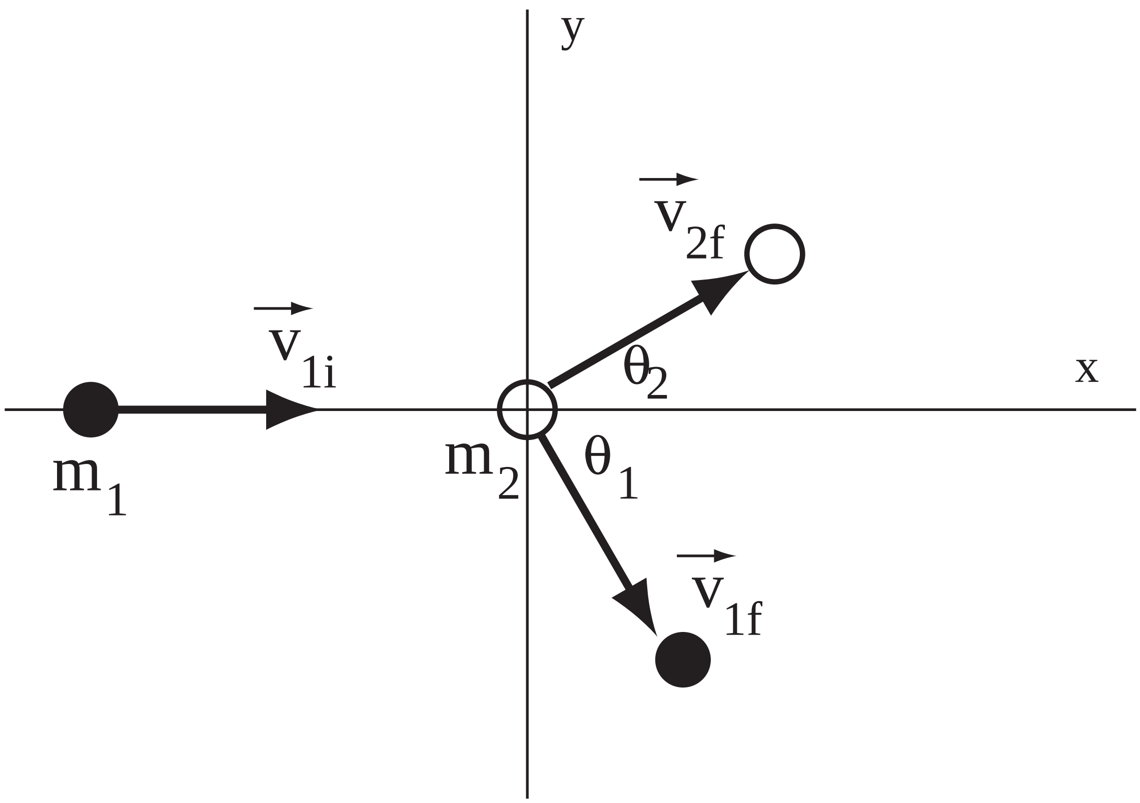


FIGURE 10.1 Glancing collision.

Example Write down the conservation of energy and momentum equations for the glancing collision depicted in Fig. 10.1 where the target ball is initially at rest.

Solution Conservation of momentum is

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

but $\vec{p}_{2i} = 0$. In x and y components these are

$$\begin{aligned} m_1 v_{1ix} &= m_1 v_{1fx} + m_2 v_{2fx} \\ m_1 v_{1iy} &= m_1 v_{1fy} + m_2 v_{2fy} \end{aligned}$$

or

$$\begin{aligned} m_1 v_{1i} &= m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2 \\ 0 &= -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2 \end{aligned}$$

If the collision is elastic we also have conservation of kinetic energy,

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

These three equations must then be solved for the quantities of interest.

Students should carefully study sample Problems 10-7, 10-8 [Halliday].

Example A ball of mass m_1 and speed v_{1i} collides with a stationary target ball of mass m_2, as shown in Fig. 10.1. If the target is scattered at an angle of θ_2 what is the scattering angle θ_1 of the projectile in terms of m_1, m_2, v_{1i}, θ_2 and v_{2f} where v_{2f} is the final speed of the target ?

Solution Conservation of momentum gives

$$\Sigma\vec{p}_i = \Sigma\vec{p}_f$$

or

$$\Sigma p_{ix} = \Sigma p_{fx} \quad \text{and} \quad \Sigma p_{iy} = \Sigma p_{fy}$$

The x direction gives

$$m_1 v_{1i} + 0 = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$$

$$0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$$

We want to find θ_1. Solve the first and second equations for θ_1 giving

$$\cos\theta_1 = \frac{m_1 v_{1i} - m_2 v_{2f} \cos\theta_2}{m_1 v_{1f}}$$

and

$$\sin\theta_1 = \frac{m_2 v_{2f} \sin\theta_2}{m_1 v_{1f}}$$

giving

$$\tan\theta_1 = \frac{m_2 v_{2f} \sin\theta_2}{m_1 v_{1i} - m_2 v_{2f} \cos\theta_2}$$

(Notice that this result is valid for both elastic and inelastic collisions. We did not use conservation of energy.)

8.6 Reactions and Decay Processes

Leave out.

Center of Mass Reference Frame

Remember that the total momentum $\vec{P}$ of a system of particles was given by $\vec{P} = M\vec{v}_{cm}$ or

$$\vec{P} = M\vec{v}_{cm} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i$$

Up to now we have been measuring velocities with respect to the "Lab" reference frame, which is the name for the reference frame associated with a stationary target. The Lab does not move, or in other words $\vec{v}_{\text{Lab}} = 0$.

We can also measure velocities with respect to the center of mass frame where $\vec{v}_{cm} = 0$. This is also often called the center of momentum frame because if $\vec{v}_{cm} = 0$ then $\sum_i \vec{p}_i = 0$. If $\vec{v} =$ velocity in Lab frame and $\vec{u} =$ velocity in cm frame then

$$\boxed{\vec{u} = \vec{v} - \vec{v}_{cm}}$$

Example A red billiard ball of mass m_R moving at a speed v_R collides head on with a black billiard ball of mass m_B at rest. A) What is the speed of the center of mass ? B) What is the speed of both balls in the cm frame ?

Solution

$$v_B = 0$$

$$\begin{aligned} v_{\rm cm} &= \frac{m_R\vec{v}_R + m_B\vec{v}_B}{m_R + m_B} \\ &= \frac{m_R \times v_R + 0}{m_R + m_B} \\ &= \frac{m_R}{m_R + m_B}\, v_R \end{aligned}$$

which is the speed of the center of mass. Now get the speed of the red ball via

$$\begin{aligned} u_R &= v_R - v_{\rm cm} = v_R - \frac{m_R}{m_R + m_B}\, v_R \\ &= v_R\left(1 - \frac{m_R}{m_R + m_B}\right) = v_R\left(\frac{m_R + m_B - m_R}{m_R + m_B}\right) \\ &= \frac{m_B}{m_R + m_B}\, v_R \end{aligned}$$

and the speed of the black ball is

$$\begin{aligned} u_B &= v_B - v_{\rm cm} \\ &= 0 - \frac{m_R}{m_R + m_B}\, v_R \\ &= -\frac{m_R}{m_R + m_B}\, v_R \end{aligned}$$

8.7 Problems

1. In a game of billiards, the player wishes to hit a stationary target ball with the moving projectile ball. After the collision, show that the sum of the scattering angles is 90^o. Ignore friction and rolling motion and assume the collision is elastic. Also both balls have the same mass.

Chapter 9

ROTATION

<u>SUGGESTED HOME EXPERIMENT</u>:
Calculate the speed of an ant at the edge of the minute hand on your kitchen clock.

<u>THEMES</u>:
SPIN.

9.1 Translation and Rotation

We have studied how point particles and systems of particles (rigid bodies) move as a whole. The next thing to consider is *rotational* motion, as opposed to the translational motion studied previously.

When studying rotational motion it is very convenient and instructive to develop the whole theory in analogy to translational motion. I have therefore written the Master Table that we shall refer to often.

9.2 The Rotational Variables

Previously we denoted translational position in 1-dimension with the symbol x. If a particle is located on the rim of a circle we often use s instead of x to locate its position around the circumference of the circle. Thus s and x are equivalent translational variables

$$s \equiv x$$

Now the *angular* position is described by angle which is *defined* as

$$\boxed{\theta \equiv \frac{s}{r}}$$

where s (or x) is the translation position and r is the radius of the circle. Notice that *angle has no units* because s and r both have units of m. The angle defined above is measured in *radian*, but of course this is *not* a unit. One complete revolution is 2π radian often also called $360°$. (All students should carefully read Pg. 240 of Halliday for a clear distinction between radian and degrees.)

Translational position is given by x (or s) and translation displacement was $\Delta x \equiv x_2 - x_1$ (or $\Delta s \equiv s_2 - s_1$). Similarly *angular displacement* is

$$\Delta\theta \equiv \theta_2 - \theta_1$$

and because $\theta \equiv \frac{s}{r}$ then it is related to translation displacement by

$$\boxed{\Delta\theta = \frac{\Delta s}{r} = \frac{\Delta x}{r}}$$

This is the first entry in the Master Table.

Secondly we defined translational average velocity as $\bar{v} \equiv \frac{\Delta x}{\Delta t} \equiv \frac{\Delta s}{\Delta t}$ and instantaneous velocity as $v \equiv \frac{dx}{dt} = \frac{ds}{dt}$. Similarly we define average angular velocity as

$$\bar{\omega} \equiv \frac{\Delta \theta}{\Delta t}$$

and instantaneous velocity as

$$\boxed{\omega \equiv \frac{d\omega}{dt}}$$

Now because we have $\Delta\theta = \frac{\Delta x}{r}$ we must also have $\frac{\Delta\theta}{\Delta t} = \frac{\Delta x}{r\Delta t}$ or $\bar{\omega} = \frac{\bar{v}}{r}$ as relating average velocity and average angular velocity. Similarly

$$\boxed{\omega = \frac{v}{r}}$$

This is the second entry in the Master Table.

Finally the angular acceleration α is defined as

$$\alpha \equiv \frac{d\omega}{dt}$$

and

$$\alpha = \frac{a}{r}$$

relating angular acceleration α to translational acceleration a_t. (Notice that a is *not* the centripetal acceleration. For *uniform* circular motion $\alpha = 0$ and $a_t = 0$ because the particle moves in a circle at constant speed v and the centripetal acceleration is $a_r = \frac{v^2}{r}$. For *non-uniform* circular motion, where the speed keeps *increasing* (or decreasing) then $\alpha \neq 0$ and $a \neq 0$.) See the third entry in the Master Table.

9.3 Are Angular Quantities Vectors?

Yes. Read Halliday.

9.4 Rotation with Constant Angular Acceleration

The equations for constant angular acceleration are obtained in identical fashion to the translational constant acceleration equations. They are listed in the Master Table.

Example A flywheel is spinning at 100 revolutions per second and is stopped by a brake in 10 seconds. What is the angular acceleration of the flywheel ?

Solution The initial angular velocity is

$$\omega_0 = 100 \times 2\pi \,\text{sec}^{-1}$$

and the final angular velocity is $\omega = 0$. Using $\omega = \omega_0 + \alpha t$ gives

$$\begin{aligned} \alpha &= \frac{\omega - \omega_0}{t} = \frac{0 - 100 \times 2\pi \,\text{sec}^{-1}}{10 \,\text{sec}} \\ &= -62.8 \,\text{sec}^{-2} \end{aligned}$$

Study Sample Problems 11-3, 11-4, 11-5.

9.5 Relating the Linear and Angular Variables

We have already discussed this. Read Halliday. Especially read about a_t and a_r on Pg. 246 Halliday.

9.6 Kinetic Energy of Rotation

To calculate the kinetic energy of a rotating object we add up all of the kinetic energies of the individual particles making up the object, namely

$$K = \sum_i \frac{1}{2} m_i v_i^2$$

The speeds are $v_i = \omega r_i$. Note we do not write $v_i = \omega_i r_i$ because the rotational velocity of all particles is the same value ω. That is $\omega_1 = \omega_2 = \omega_3 = \cdots \equiv \omega$. Substituting gives

$$K = \sum_i \frac{1}{2} m_i \omega^2 r_i^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

Define *rotational inertia* or *rotational mass* as

$$\boxed{I \equiv \sum_i m_i r_i^2}$$

and we get

$$\boxed{K = \frac{1}{2} I \omega^2}$$

which looks exactly like $K = \frac{1}{2} m v^2$ where instead of velocity v we have ω and instead of mass (or inertia) m we have rotational mass (or rotational inertia) I. Recall that mass, or inertia, tells us how difficult it is to *move* an object. Similarly the rotational mass, or rotational inertia, tells us how difficult it is to *rotate* an object. (Carefully read Pg. 248, Halliday.) See Master Table.

9.7 Calculating the Rotational Inertia

For a continuous distribution of mass the rotational inertia has the sum replaced by an integral, namely

$$\boxed{\begin{gathered} I \equiv \sum_i r_i^2 m_i = \int r^2 dm \\ = \int r^2 \rho dV = \int r^2 \sigma dA = \int r^2 \lambda dL \end{gathered}}$$

where dm has been replaced by ρdV or σdA or λdL depending on whether the rigid body is 3-dimensional, 2-dimensional or 1-dimensional.

Now when you spin an object, you always spin it about some axis. Take your physics book for example. It is *easy* to spin about an axis through the center (i.e. center of mass) but more difficult to spin about an axis through the edge of the book.

Remember that the rotational inertia I tells us how difficult it is to get something rotating, or spinning, just as ordinary inertia m tells us how difficult it is to get something moving. Thus I is small for the spin axis through the center of the book, but large for an axis through the edge of the book. In the formula for $I = \sum_i r_i^2 m_i = \int r^2 dm$ then r will always be measured from the rotation axis.

A very handy formula which helps a lot in calculating I is the famous parallel axis theorem,

$$I = I_{\rm cm} + Mh^2$$

where I is the rotational inertia about an axis located a distance h from the center of mass and *parallel* to a line through the center of mass. M is the *total mass* of the whole rigid body. This theorem is proved on Pg. 250 of Halliday.

Let's now look at some examples of how to calculate I. Many results are listed on Pg. 249 of Halliday.

Example A rod of length L and negligible mass has a dumbbell of mass m located at each end. Calculate the rotational inertia about an axis through the center of mass (and perpendicular to the rod).

Solution See Fig. 11-13(a) in Halliday, Pg. 250. Each dumbbell is a discrete mass and so we use

$$\begin{aligned} I &= \sum_i r_i^2 m_i \\ &= r_1^2 m + r_2^2 m \end{aligned}$$

where there are only two terms because there are only two dumbbells, and also $m_1 = m_2 \equiv m$. Now $r_1 = \frac{1}{2}L$ and $r_2 = -\frac{1}{2}L$ giving

$$\begin{aligned} I &= \left(\frac{1}{2}L\right)^2 m + \left(-\frac{1}{2}L\right)^2 m \\ &= \frac{1}{2}mL^2 \end{aligned}$$

Example Repeat the previous example for an axis through one of the dumbbells (but still perpendicular to the rod).

Solution See Fig. 11-13(b) in Halliday, Pg. 250. Now we have $r_1 = 0$ and $r_2 = L$ giving

$$\begin{aligned} I &= r_1^2 m + r_2^2 m \\ &= 0 + L^2 m \\ &= mL^2 \end{aligned}$$

Example Repeat the previous example using the parallel axis theorem.

Solution The parallel axis theorem is $I = I_{\rm cm} + Mh^2$ where the total mass if $M = 2m$ and h is the distance from the center of mass to the rotation axis. Thus $h = L/2$ giving

$$\begin{aligned} I &= \frac{1}{2}mL^2 + (2m)\left(\frac{L}{2}\right)^2 \\ &= mL^2 \end{aligned}$$

This is the *same* as before and so we have good reason to believe that the parallel axis theorem is true.

Example Calculate the rotational inertia of a thin uniform rod of mass M and length L about an axis through the center of the rod (and perpendicular to its length).

Solution See the figure in Table 11-2 of Halliday, Pg. 249. Let the linear mass density of the rod be $\lambda \equiv \frac{M}{L}$. Then (with $dr = dL$)

$$I = \int r^2 dm = \int_{-L/2}^{L/2} r^2\,\lambda\,dr$$

where the integration limits are $-L/2$ to $L/2$ because the axis is through the center of the rod. The rod is uniform which means λ is constant and can be taken outside the integral to give

$$\begin{aligned} I &= \lambda \int_{-L/2}^{L/2} r^2 dr = \lambda\left[\frac{1}{3}r^3\right]_{-L/2}^{L/2} \\ &= \lambda\left[\frac{1}{3}\left(\frac{L}{2}\right)^3 - \frac{1}{3}\left(-\frac{L}{2}\right)^3\right] = \lambda\frac{L^3}{12} \\ &= \frac{M}{L}\frac{L^3}{12} = \frac{1}{12}ML^2 \end{aligned}$$

Example Repeat the previous example for an axis through one end of the rod.

Solution Now we have

$$\begin{aligned} I &= \lambda \int_0^L r^2 dr = \lambda \left[\frac{1}{3} r^2\right]_0^2 \\ &= \lambda \left(\frac{1}{3} L^3 - 0\right) = \lambda \frac{1}{3} L^3 \\ &= \frac{M}{L} \frac{1}{3} L^3 = \frac{1}{3} M L^2 \end{aligned}$$

Example Repeat the previous example using the parallel axis theorem.

Solution

$$\begin{aligned} I &= I_{\rm cm} + M h^2 \\ &= \frac{1}{12} M L^2 + M \left(\frac{L}{2}\right)^2 \\ &= \frac{1}{3} M L^2 \end{aligned}$$

9.8 Torque

We now want to determine the rotational equivalent of force. Rotational force is called *torque*. It is a vector defined as the *cross product* of $\vec{r}$ and $\vec{F}$,

$$\boxed{\vec{\tau} \equiv \vec{r} \times \vec{F}}$$

Its magnitude is

$$\vec{\tau} = rF \sin \phi$$

where ϕ is the angle between $\vec{r}$ and $\vec{F}$. Now $r \sin \phi$ is just a perpendicular distance $r_\perp = r \sin \phi$, so that

$$\tau = r_\perp F$$

Carefully read Halliday Pg. 252-253 for a detailed discussion of the meaning of torque. See the Master Table.

9.9 Newton's Second Law for Rotation

Now just as we have $\sum \vec{F} = m\vec{a}$ for translational dynamics we would guess that

$$\boxed{\sum \vec{\tau} = I \vec{\alpha}}$$

would be Newton's second law for rotation. This is exactly right!

Example Do Sample Problem 11-11 in class (Pg. 253 Halliday).

9.10 Work and Rotational Kinetic Energy

We have seen that in 1-dimension, work is $W = \int F\,dx$. Similarly for rotations we have

$$W \equiv \int \tau\, d\theta$$

See Master Table.

MASTER TABLE

Translational Motion	Rotational Motion	Relation
Displacement $\Delta x \equiv \Delta s$	Angular Displ. $\Delta\theta$	$\Delta x = \Delta s = r\Delta\theta$
Velocity $v \equiv \frac{dx}{dt}$	Angular Vel. $\omega \equiv \frac{d\theta}{dt}$	$v = r\omega$
Acceleration $a_t \equiv \frac{dv}{dt}$	Angular Accel. $\alpha \equiv \frac{d\omega}{dt}$	$a_t = r\alpha$
Constant Accel. Eqns:	Constant Angular Accel:	
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$	
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	
$x - x_0 = \frac{v+v_0}{2}t$	$\theta - \theta_0 = \frac{\omega+\omega_0}{2}t$	
$= v_0 t + \frac{1}{2}at^2$	$= \omega_0 t + \frac{1}{2}\alpha t^2$	
$= vt - \frac{1}{2}at^2$	$= \omega t - \frac{1}{2}\alpha t^2$	
$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$	$I \equiv \sum_i r_i^2 m_i = \int r^2 dm$
$\sum \vec{F} = m\vec{a}$	$\sum \vec{\tau} = I\vec{\alpha}$	$\vec{\tau} \equiv \vec{r} \times \vec{F}$
$W = \int F\, dx$	$W = \int \tau d\theta$	

9.11 Problems

1. Show that the ratio of the angular speeds of a pair of coupled gear wheels is in the inverse ratio of their respective radii. [WS 13-9]

2. Show that the magnitude of the total linear acceleration of a point moving in a circle of radius r with angular velocity ω and angular acceleration α is given by $a = r\sqrt{\omega^4 + \alpha^2}$ [WS 13-8]

3. The turntable of a record player rotates initially at a rate of 33 revolutions per minute and takes 20 seconds to come to rest. How many rotations does the turntable make before coming to rest, assuming constant angular deceleration ?

4. A cylindrical shell of mass M and radius R rolls down an incline of height H. With what speed does the cylinder reach the bottom of the incline ? How does this answer compare to just dropping an object from a height H ?

5. Four point masses are fastened to the corners of a frame of negligible mass lying in the xy plane. Two of the masses lie along the x axis at positions $x = +a$ and $x = -a$ and are both of the same mass M. The other two masses lie along the y axis at positions $y = +b$ and $y = -b$ and are both of the same mass m.

A) If the rotation of the system occurs about the y axis with an angular velocity ω, find the moment of inertia about the y axis and the rotational kinetic energy about this axis.

B) Now suppose the system rotates in the xy plane about an axis through the origin (the z axis) with angular velocity ω. Calculate the moment of inertia about the z axis and the rotational kinetic energy about this axis. [Serway, 3rd ed., pg. 151]

6. A uniform object with rotational inertia $I = \alpha m R^2$ rolls without slipping down an incline of height H and inclination angle θ. With what speed does the object reach the bottom of the incline? What is the speed for a hollow cylinder ($I = mR^2$) and a solid cylinder ($I = \frac{1}{2}MR^2$)? Compare to the result obtained when an object is simply dropped from a height H.

7. A pencil of length L, with the pencil point at one end and an eraser at the other end, is initially standing vertically on a table with the pencil point on the table. The pencil is let go and falls over. Derive a formula for the speed with which the eraser strikes the table, assuming that the pencil point does not move. [WS 324]

Chapter 10

ROLLING, TORQUE & ANGULAR MOMENTUM

SUGGESTED HOME EXPERIMENT:
Design a simple experiment showing conservation of angular momentum.

THEMES:
SPIN.

10.1 Rolling

All students should read this whole section in Halliday carefully. Note that when a wheel rolls *without* slipping, then *static* friction is involved. When the wheel slips then *kinetic* friction is involved. This is discussed in Halliday. I shall now discuss an important example. (See also Sample Problems 12-1, 12-2, 12-3.)

Example Calculate the rotational inertia of a hollow cylinder and a solid cylinder, about the long axis through the center of the cylinder as shown in Fig. 12.1.

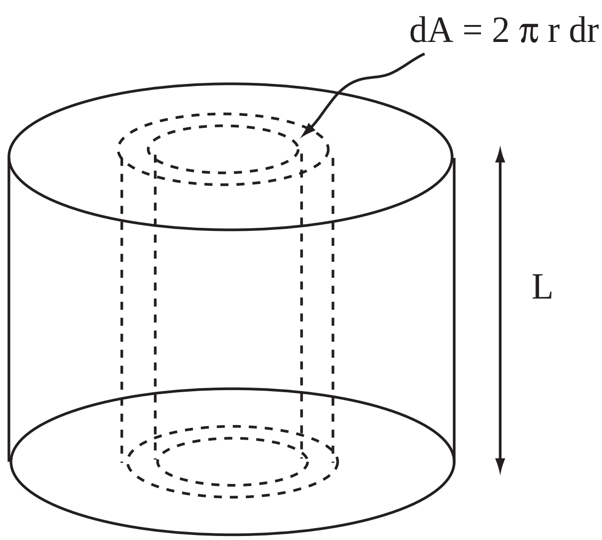


FIGURE 12.1 Solid Cylinder.

Solution The rotational inertia of a hollow cylinder is simply

$$I = MR^2$$

To calculate the rotational inertia of the solid cylinder, refer to Fig. 12.1. The small element of area indicated is $dA = 2\pi r dr$ corresponding to a small element of volume $dV = dA\ L = 2\pi r dr L$. Thus the rotational indertia (with $\rho = \frac{M}{L\pi R^2}$ being the density of the cylinder) is

$$I = \int r^2 dm$$

$$
\begin{aligned}
&= \int r^2 \rho dV \\
&= \rho 2\pi L \int_0^R r^3 dr \\
&= \rho 2\pi L \frac{1}{4} R^4 \\
&= \frac{M}{L\pi R^2} 2\pi L \frac{1}{4} R^4 \\
&= \frac{1}{2} M R^2
\end{aligned}
$$

Example If a solid cylinder and a hollow cylinder with the same mass and radius roll down an incline, which reaches the bottom first?

Solution The kinetic energy of a rolling object now consists of *two* terms; one rotational and one translational, i.e.

$$K = \frac{1}{2} I_{\rm cm} \omega^2 + \frac{1}{2} M v_{\rm cm}^2 \equiv K_{\rm rotation} + K_{\rm translation}$$

where $I_{\rm cm}$ is the rotational inertia about the center of mass and $v_{\rm cm}$ is the translational speed of the center of mass. The rotational inertias of Hoop, Disk and Sphere are

$$
\begin{aligned}
I_{\rm Hollow\ cylinder} &= MR^2 \\
I_{\rm Solid\ cylinder} &= \frac{1}{2} MR^2
\end{aligned}
$$

The hollow cylinder has the larger moment of inertia and therefore more kinetic energy will go into rotation, and thus less into translation. Therefore the solid cylinder reaches the bottom first.

LECTURE DEMONSTRATION: show the above example.

10.2 Yo-Yo

Read

10.3 Torque Revisited

Read carefully; *Review of Cross Product*

10.4 Angular Momentum

10.5 Newton's Second Law in Angular Form

We have previously defined torque (or angular force) as $\vec{\tau} \equiv \vec{r} \times \vec{F}$. Now Newton's Second Law is $\sum \vec{F} = \frac{d\vec{p}}{dt}$ where $\vec{p} \equiv m\vec{v}$ is the momentum. We therefore expect an *angular* version of Newton's Second Law involving angular force or torque and angular momentum $\vec{l}$. Thus we expect

$$\boxed{\sum \vec{\tau} = \frac{d\vec{l}}{dt}}$$

But we haven't said what $\vec{l}$ is. We can figure it out.

Consider the following quantity,

$$\begin{aligned} \frac{d}{dt}(\vec{r} \times \vec{p}) &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\ &= \vec{v} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt} \\ &= m(\vec{v} \times \vec{v} + \vec{r} \times \vec{a}) \end{aligned}$$

but $\vec{v} \times \vec{v} = 0$ giving

$$\begin{aligned} \frac{d}{dt}(\vec{r} \times \vec{p}) &= m\vec{r} \times \vec{a} \\ &= \vec{r} \times \vec{F} \\ &= \vec{\tau} \end{aligned}$$

Thus the unknown $\vec{l}$ *must* be

$$\boxed{\vec{l} \equiv \vec{r} \times \vec{p}}$$

10.6 Angular Momentum of a System of Particles

Let's call the angular momentum of a system of particles $\vec{L}$. In terms of the angular momentum $\vec{l}_i$ of each particle, it is

$$\vec{L} = \sum_i \vec{l}_i$$

and Newton's Second Law for a system of particles becomes

$$\boxed{\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}}$$

as we would expect, based on analogy with $\sum \vec{F}_{\text{ext}} = \frac{d\vec{p}}{dt}$ where $\vec{p}$ was the total momentum.

10.7 Angular Momentum of a Rigid Body Rotating About a Fixed Axis

In a rigid body, all particles rotate at the same speed. Halliday (Pg. 281) shows that

$$\vec{L} = I\vec{\omega}$$

which is exactly analogous to $\vec{p} = m\vec{v}$.

10.8 Conservation of Angular Momentum

For translational motion we had $\sum \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$ and for $\sum \vec{F}_{\text{ext}} = 0$ we had $\vec{P} =$ constant, i.e. conservation of momentum. Similarly from $\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$, then if there are no external torques $\sum \vec{\tau}_{\text{ext}} = 0$ then the total angular momentum is conserved, namely

$$\vec{L} = \text{constant}$$

LECTURE DEMONSTRATION: example below

Example A student is spinning on a stool and holding two heavy weights with outstretched hands. If the student brings the weights closer inward, show that the spin rate increases.

Solution For a rigid body spinning about a fixed axis we had $\vec{L} = I\vec{\omega}$. Angular momentum is conserved, thus

$$\vec{L}_i = \vec{L}_f$$

or

$$I_i\omega_i = I_f\omega_f$$

The moment of inertia of the two weights is $I = 2Mr^2$ where r is the length of the student's arm. The rotational inertia of the student remains the same. Thus

$$2Mr_i^2\omega_i = 2Mr_f^2\omega_f$$

giving

$$\omega_f = \left(\frac{r_i}{r_f}\right)^2 \omega_i$$

And $r_i > r_f$ giving $\omega_f > \omega_i$.

MASTER TABLE 2

Translational Motion	Rotational Motion	Relation
$\sum \vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$	$\sum \vec{\tau} = I\vec{\alpha} = \frac{d\vec{l}}{dt}$	$\vec{l} = \vec{r} \times \vec{p}$
$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$	

10.9 Problems

1. A bullet of mass m travelling with a speed v is shot into the rim of a solid circular cylinder of radius R and mass M as shown in the figure. The cylinder has a fixed horizontal axis of rotation, and is originally at rest. Derive a formula for the angular speed of the cylinder after the bullet has become imbedded in it. (Hint: The rotational inertia of a solid cylinder about the center axis is $I = \frac{1}{2}MR^2$). [WS354-355]

Chapter 11

GRAVITATION

SUGGESTED HOME EXPERIMENT:
Design some observations so that you can detect the retrograde motion of a planet. (Obviously you won't be able to actually carry out these observations this week. Why ?)

THEMES:
The Solar System.

The study of gravitation has been one of the core areas of physics research for the last 500 years. Indeed it was the study of gravity that revolutionized much of our thinking of our place in the universe, for one of the key results in the last 500 years was the realization that Earth is NOT the center of the universe. This has had profound and dramatic consequences for all of humankind. (I personally believe that an equally profound effect will take place if extraterrestrial intelligent life is found.)

We shall approach our study of gravitation a little different from the way Halliday discusses it. I wish to emphasize the historical approach to the subject because it is interesting and helps us understand the physics much better. A wonderful book that tells the whole story in nice detail is by R. Kolb, "Blind Watchers of the Sky" (Helix Books, Addison-Wesley, New York, 1996). This would be great reading between semesters! Some of the key historical figure are the following:

Claudius Ptolemy (l40 A.D.)
Nicolaus Copernicus (1473-1543)
Tycho Brahe (1546-1601)
Galileo Galilei (1564-1642)
Johannes Kepler (1571-1630)
Isaac Newton (1642-1727)
Albert Einstein (1879-1955)

I would now like to just briefly describe the contributions of each of these figures. We shall elaborate on the mathematical details afterwards.

In the system of Ptolemy (l40 A.D.), the Earth was believed to be at the center of the universe and the Sun, Moon, stars and planets all revolved around the Earth, as seems to be indicated by simple observation. However, upon closer inspection it can be seen that the *planets* (Greek word meaning wanderer) actually do not move in smooth circles about the Earth but rather do a kind of wandering motion. Actually they undergo a retrograde motion with respect to an observer on Earth. This retrograde motion was very puzzling to the ancients, and ran afoul of the idea that all heavenly bodies moved in pure circles. In order to save the theoretical notion of pure circles and yet to explain the observational fact of retrograde motion for the planets, Ptolemy introduced the idea of *epicycles*. Figure 14.1 shows that instead of a planet moving in a great circle about the Earth, as do the Sun and Stars, Ptolemy's idea was that another circle called an epicycle moves ina great circle around the Earth and the planets move around on the epicycles. This 'explains' the observations of retrograde motion. But Ptolemy's system leaves unanswered the question of where the epicycle comes from. However this system of epicylces enjoyed great success for over a thousand years.

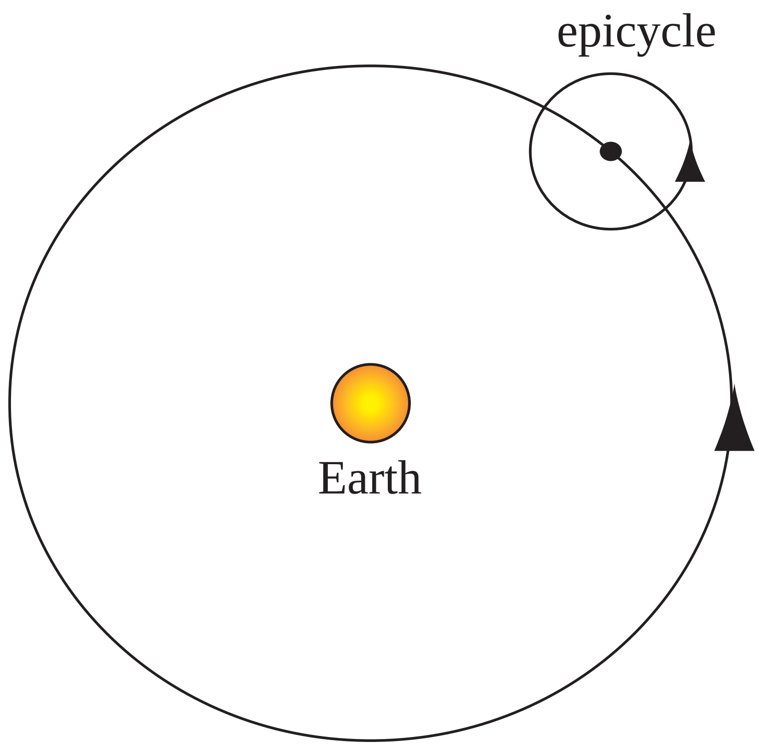


FIGURE 14.1 Ptolemaic epicycle.

However later on came Copernicus (1473-1543), a Polish monk, who suggested that the Earth is *not* at the center of the universe. From a psychological point of view, this is probably the most important scientific idea in history. Copernicus thought instead that the Sun was at the center of the universe and that all the planets, including Earth, revolved around it. This provided an alternative explanation for the retrograde motion of the planets, for if the planets move at different speeds around the Sun, then from the point of view of an observer on Earth, the planets will appear to move forward and then backward depending upon the relative orientation.

Tycho Brahe (1546-1601) was one of the greates observational astronomers in history. Of course the telescope had not yet been invented and all of Tycho's observations were with some geometric instruments and the naked eye. He mounted an intensive campaign to accurately record the motion of all the planets. After Tycho died, Johannes Kepler (1571-1630) obtained access to Tycho's precision data and was able to use it to figure out the exact motion of the planets to a high degree of precision. In particular Kepler discovered that the motion of the planets was not the perfect circle after all, but rather the motions were elliptical. From analyzing Tycho's data Kepler discovered 3 important facts about the planets. These are usually called Kepler's laws of planetary motion. They are

1) All planets move in elliptical orbits with the Sun at one focus.

2) The line joining any planet to the Sun sweeps out equal areas in equal times.

3) The period squared is proportional to the mean distance cubed, i.e. $T^2 \propto R^3$. (The period T is the time it takes for a planet to complete one orbit of the Sun. For Earth this is 365 days. The mean distance R is the average distance from the Sun to the planet in question.)

Meanwhile, Galileo Galilei (1564-1642) used the newly invented telescope to view the heavens for the first time. Among his many great discoveries, were observations of the moons of Jupiter clearly showing orbits around the planet itself. This was the first direct observation of bodies which did not orbit Earth.

One important point to note about Kepler's laws is that they were 'mere' empirical facts. No one understood why they were true. In fact Kepler spent the rest of his life trying to explain then. It was not until Isaac Newton (1642-1727) invented a *theory of gravity* that Kepler's laws were finally understood on a theoretical basis. Newton had been thinking deeply about what holds the moon in orbit around Earth and what holds the planets in orbit around the Sun. The story goes that Newton was sitting under an apple tree watching the apples fall off the tree onto the ground. It suddenly occured to Newton that the force causing the apples to fall to the ground is the *same* force that keeps the moon in orbit about Earth and the planets in orbit about the Sun. What a great leap of imagination ! Newton hypothesized that the gravitational force between any two objects was given by an inverse square law of the form

$$F = G\frac{m_1 m_2}{r^2} \tag{11.1}$$

where m_1 and m_2 are the masses of the bodies and r is the distance between their centers. G is a constant. Note that this says that if the distance between two bodies is doubled the force drops by a factor of 4. The great triumph of Newton's gravitational theory was that he could *derive* Kepler's laws. We shall go through this derivation in a moment.

The story of gravity is not complete without mentioning Einstein's General Theory of Relativity which was another theory of *gravity* completely at odds with Newton's theory. In Einstein's theory there is no mention of any forces at all. Rather, gravity is seen to be due to a curvature of space and time. The concept of force is more of an illusion. Einstein's theory was also able to explain Kepler's laws, but its advantage over Newton'e theory was that it explained additional facts about the planets such as the precession of the orbit of mercury and the deflection of starlight by the Sun.

Actually even today the story of gravity is not complete. In fact of the 4 forces that we have identified in nature (gravity, electromagnetism, strong force, weak force), it is gravity that *still* remains poorly understood. The theory of quantum mechanics was invented early this century to describe the motion of tiny particles such as atoms. The great problem with gravity is that no one has succeeded in making it consistent with quantum mechanics. A recent theory, called Superstring theory, may be the answer but we will have to wait and see. By the way, the physics department at the University of Wisconsin-Milwaukee is one of the leading centers in the nation for the modern study of gravity.

11.1 The World and the Gravitational Force

Read.

11.2 Newton's Law of Gravitation

We already know about Newton's three laws of motion, the second of which is $\sum \vec{F} = m\vec{a}$. These three laws describe motion *in general*. They never refer to a specific force. Newton however did also study in detail a specific force, namely gravity. He conjectured that the gravitational force between two bodies of mass m_1 and m_2 whose *centers* are separated by a distance of r has a magnitude of

$$F = -G\frac{m_1 m_2}{r^2}$$

The minus sign tells us that the force points inwards. The value of G was determined later in 1798 by Cavendish. It's value measured today is

$$G = 6.67 \times 10^{-11} \text{Nm}^2 \text{ kg}^{-2}$$

However, it is interesting that today the gravitational constant is the least accurately known of all the fundamental constants. For instance, its most accurately known value is actually $G = (6.67259 \pm 0.00085) \times 10^{-11}$ Nm^{2}kg^{-2} [see Particle Properties Data Booklet, 1996] whereas for example the charge of the electron is $(1.60217733 \pm 0.00000049) \times 10^{-19}$ Coulomb or the speed of light is 299 792 458 m sec^{-1} which are known much more accurately than G. Another example is the strength of the electrical force, called the fine structure constant, $\alpha^{-1} = 137.0359895 \pm 0.0000061$.

Note: Halliday (Pg. 323) writes $F = G\frac{m_1 m_2}{r^2}$ (i.e. with a plus sign) but then writes $F = -G\frac{m_1 m_2}{r^2}$ on Pages 329 and 331. The equation should always be written with a *minus* sign to indicate an atractive inwards force.

Now the *vector* form of Newton's Law is

$$\boxed{\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}}$$

where $\hat{r}$ is a unit vector point from one mass *out* to the other. The gravitational force is an *inward* force and that's why the minus sign appears.

11.3 Gravitation and Principle of Superposition

Read carefully.

11.4 Gravitation Near Earth's Surface

Newton's formula $F = G\frac{m_1 m_2}{r^2}$ is often called the *law of Universal Gravitation* because it applies to all bodies in the universe. How does it fit in with our concept of *Weight* which we defined to be the gravitational force at the *surface* of the Earth, namely

$$W \equiv mg$$

where $g = 9.8$ m sec^{-1} is the acceleration due to gravity at the surface of the Earth? Well, if $F = G\frac{m_1 m_2}{r^2}$ is *universal* then it should *predict* the Weight force. Let's see how this comes about.

Example Show that $F = G\frac{m_1 m_2}{r^2}$ gives the same result as $W = mg$ near the surface of Earth.

Solution Let $m_1 \equiv M$ be the mass of Earth, which is $m_1 = M = 5.98 \times 10^{24}$ kg. Let $m_2 \equiv m$ be the mass of a person of weight $W = mg$. The distance between the *centers* of the masses is just the radius of Earth, i.e. $r = 6370$ km (which is about 4000 miles, only slightly larger than the width of the United States or Australia). Thus the gravitational force between the two masses is

$$\begin{aligned} F &= G\frac{mM}{r^2} \\ &= 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times \frac{m \times 5.98 \times 10^{24}\text{ kg}}{(6.37 \times 10^6\, m)^2} \\ &= m \times 9.8 \text{ m sec}^{-2} \end{aligned}$$

which is the *same* as $W = mg$. In other words we have *predicted* the value of g from the mass and radius of Earth. You could now do the same for the other planets.

Example Explain how to measure the mass of Earth.

Solution In the previous example, we found

$$g = G\frac{M}{r^2}$$

where M is the mass of Earth and r is the radius of Earth. Thus by *measuring* g (which you do in the lab) and by *measuring* r (which the ancient Greeks knew how to do by comparing the depth of a shadow in a well at two different locations at the same time) then M is given by

$$M = \frac{gr^2}{G}$$

and G was measured in the famous Cavendish experiment (look this up).

11.5 Gravitation Inside Earth

If you go down a deep mine shaft then there will be Earth below you and Earth above you. It is interesting to figure out that the Earth above you won't have any overall gravitational effect. The easiest way to see this is to suppose you were located exactly at the center of Earth. Then the gravitational pull of all the Earth surrounding you above will all cancel out and you will fee zero net force. Now consider Figure 14.2 where a person is located at point P inside the Earth, at a distance r from the center of Earth.

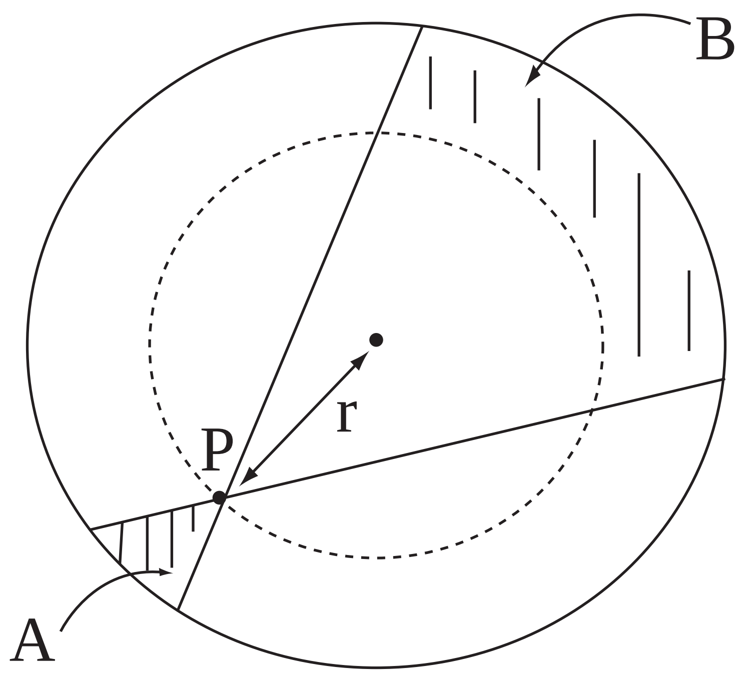


FIGURE 14.2 A person is located as point P inside the Earth, as a distance r from the center of Earth.

I have drawn a dotted circle of radius r intersecting point P. We all agree that the total mass located *inside* the dotted circle produces a net gravitational force on the person. However the mass *outside* the dotted circle produces *no* net gravitational force. This can be seen by considering the shaded regions A and B. Region A contains a small amount of mass which will pull the person at P outwards. However the mass contained in B will pull in the opposite direction. Now there is *more* mass in B, but it is *further* away and so the gravitational effects of the mass in A and in B cancel out. Thus we can *ignore* all of the mass located outside of the dotted circle.

Example (See also Sample Problem 14-5): A hole is drilled from the United States to China through the center of Earth. Ignoring the rotation of Earth, show that a particle dropped into the hole experiences a gravitational force like Hooke's law, and therefore will undergo oscillation in the hole.

Solution Newton's law is $\vec{F} = -G\frac{Mm}{r^2}\hat{r}$ where M is the mass contained *within* the dotted circle (Figure 14.2) and r is the radius of the dotted circle. Now when the particle falls through the hole, M keeps getting *smaller* because r gets smaller as the particle falls towards the center of Earth. The density of material in Earth is

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi r^3}$$

giving

$$M = \frac{4}{3}\pi r^3 \rho$$

where ρ is constant. Thus

$$\begin{aligned}\vec{F} &= -G\frac{\frac{4}{3}\pi r^3 \rho m}{r^2}\hat{r} \\ &= -\frac{4\pi G}{3}\rho m\, r\hat{r} \\ &= -K\vec{r}\end{aligned}$$

where $K \equiv \frac{4\pi G}{3}\rho m$ and $\vec{r} = r\hat{r}$. This is exactly Hooke's law, i.e. the same as for a spring. Thus the particle will oscillate.

Example When you go down a mine shaft, do you weigh more or less than you did at the surface of the Earth ?

Solution We found in the previous example that

$$\vec{F} \;=\; -\frac{4\pi G}{3}\rho m\, r\hat{r}$$

Now ρ is constant and thus $\vec{F}$ is bigger when r is big. Thus when r gets small, $\vec{F}$ gets small and your weight therefore decreases. In fact $F = W = mg = \frac{4\pi G}{3}\rho m r$ giving $g = \frac{4\pi G}{3}\rho r$ indicating that g gets smaller as r gets smaller.

11.6 Gravitational Potential Energy

Let's briefly recall our ideas about work and energy. The *total* work was defined as $W \equiv \int \vec{F} \cdot d\vec{r}$. By substituting $\vec{F} = m\vec{a}$ we found the work was *always* equal to the change in kinetic energy, i.e.

$$W \equiv \int \vec{F} \cdot d\vec{r} = \Delta K$$

The total work consisted of two parts namely, conservative W_C and non-conservative W_{NC}. We *defined* potential energy U via

$$W_C = \int \vec{F}_C \cdot d\vec{r} \equiv -\Delta U$$

giving

$$W = W_C + W_{NC} = -\Delta U + W_{NC} = \Delta K$$

or

$$\Delta U + \Delta K = W_{NC}$$

which we called the work-energy theorem. Now K is *always* given by $K = \frac{1}{2}mv^2$ (which came from $\int m\vec{a} \cdot d\vec{r} = \Delta K$) but U is *different* for different forces (because $-\Delta U = \int \vec{F} \cdot d\vec{r}$).

For a spring force $\vec{F} = -kx\hat{i}$ we found $U = \frac{1}{2}kx^2$. For gravity near the surface of Earth, $\vec{F} = -mg\hat{j}$ we found $U = mgy$. For universal gravitation $\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$ we will find that the gravitational potential energy is

$$\boxed{U = -G\frac{m_1 m_2}{r}}$$

Example For gravity near the surface of Earth, prove that $U = mgy$.

Solution This was already done in Chapter 8 (these notes). Let's do it again.

$$W_C \equiv \int \vec{F}_C \cdot d\vec{r} \equiv -\Delta U$$

Now $\vec{F} = -mg\hat{j}$ and $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$. Thus

$$\vec{F} \cdot d\vec{r} = -mg\,dy$$

giving

$$\begin{aligned} W_C &= -mg \int_{y_i}^{y_f} dy \equiv -\Delta U \\ &= -mg(y_f - y_i) = -(U_f - U_i) \\ &= -mgy_f + mgy_i = -U_f + U_i \end{aligned}$$

giving

$$\begin{aligned} U_f &= mgy_f \\ U_i &= mgy_i \end{aligned}$$

or just

$$U = mgy$$

Example For universal gravitation, prove that $U = -G\frac{m_1 m_2}{r}$.

Solution

$$W_C = \int \vec{F}_C \cdot d\vec{r} \equiv -\Delta U$$
$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r} \text{ and } d\vec{s} \equiv d\vec{r} = \hat{r}\,dr$$
$$\vec{F} \cdot d\vec{r} = -G\frac{m_1 m_2}{r^2} dr\,\hat{r} \cdot \hat{r} = -G\frac{m_1 m_2}{r^2} dr$$

giving

$$\begin{aligned}
W_C &= -Gm_1 m_2 \int_{r_i}^{r_f} \frac{1}{r^2} dr = -\Delta U \\
&= -Gm_1 m_2 \left[-\frac{1}{r}\right]_{r_i}^{r_f} = -(U_f - U_i) \\
&= -Gm_1 m_2 \left(-\frac{1}{r_f} - -\frac{1}{r_i}\right) \\
&= -Gm_1 m_2 \left(-\frac{1}{r_f} + \frac{1}{r_i}\right) \\
&= +G\frac{m_1 m_2}{r_f} - G\frac{m_1 m_2}{r_i} = -U_f + U_i
\end{aligned}$$

giving

$$\begin{aligned}
U_f &= -G\frac{m_1 m_2}{r_f} \\
U_i &= -G\frac{m_1 m_2}{r_i}
\end{aligned}$$

or just

$$U = -G\frac{m_1 m_2}{r}$$

Recall that we also had an alternative way of finding U without having to work out the integral $\int \vec{F}_C \cdot d\vec{r}$. We had $W_C = \int \vec{F}_C \cdot d\vec{r} \equiv -\Delta U$. Ignoring the vectors we write

$$\int F_C dr = -\Delta U$$

meaning that we *must* have

$$F_C = -\frac{dU}{dr}$$

This occurs because

$$\begin{aligned}\int_i^f F_C dr &= -\int_i^f \frac{dU}{dr} dr = -\int_{U_i}^{U_f} dU = -[U]_{U_i}^{U_f} \\ &= -(U_f - U_i) = -\Delta U\end{aligned}$$

Example For universal gravitation $F = -G\frac{m_1 m_2}{r^2}$, derive U without doing an integral.

Solution For universal gravitation, the question is what U will give

$$F = -G\frac{m_1 m_2}{r^2} = -\frac{dU}{dr}$$

The answer is $U = -G\frac{m_1 m_2}{r^2}$. Let's check:

$$-\frac{dU}{dr} = +Gm_1 m_2 \frac{d}{dr}\left(\frac{1}{r^2}\right) = -\frac{Gm_1 m_2}{r}$$

which is the F we started with!

Escape Speed

If you throw a ball up in the air it always comes back down. If you throw it faster it goes higher before returning. There is a speed, called the *escape speed*, such that the ball will not return at all. Let's find out what this is.

Example Calculate the speed with which a ball must be thrown, so that it never returns to the ground.

Solution The ball usually returns to the ground because of its gravitational potential energy $U = -G\frac{m_1 m_2}{r}$. However if we can throw the ball to an infinite distance, $r = \infty$, then U will be zero and the ball will not return. We want to throw the ball so that it *just barely* escapes to infinity, that is its speed, when it gets to infinity, has dropped off to zero. Using conservation of energy we have

$$K_i + U_i = K_f + U_f$$

or

$$\frac{1}{2}mv_i^2 - G\frac{Mm}{R} = O + O$$

where M is the mass of Earth, m is the mass of the ball and R is the radius of Earth, because we throw the ball from the surface of Earth. v_i is the *escape speed* that we are looking for. Thus

$$\frac{1}{2}mv_i^2 = G\frac{Mm}{R}$$

and m cancels out giving

$$\boxed{v_i = \sqrt{\frac{2GM}{R}}}$$

for the escape speed. Now the mass and radius of Earth are $M = 6 \times 10^{24}$ kg and $R = 6370$ km, giving

$$\begin{aligned} v_i &= \sqrt{\frac{2 \times 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6 \times 10^{24}\text{kg}}{6.37 \times 10^6\text{m}}} \\ &= 40,353 \text{ km hour}^{-1} \\ &\approx 25,000 \text{ miles per hour} \end{aligned}$$

Now you can see that if M is very large or R is very small then the escape speed gets very big. The speed of light is $c = 3 \times 10^8$ m/sec. You can imagine an object so massive *or* so small that the escape speed is *bigger* than the speed of light. Then light itself cannot escape. Such an object is called a *Black Hole*.

Example To what size would we need to squeeze Earth to turn it into a Black Hole ?

Solution Let's set the escape speed equal to the speed of light $c = 3 \times 10^8$ m/sec. Thus

$$c = \sqrt{\frac{2GM}{R}}$$

$$c^2 = \frac{2GM}{R}$$

giving

$$\begin{aligned} R &= \frac{2GM}{c^2} \\ &= \frac{2 \times 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6 \times 10^{24}\text{kg}}{(3 \times 10^8 \text{m sec}^{-1})^2} \\ &= 4.4 \text{ mm} \end{aligned}$$

(where M = mass of Earth = 6×10^{24} kg). Thus if we could squeeze the Earth to only 4 mm it would be a black hole!

Example The size of the universe is about 10 billion light years and its total mass is about 10^{53} kg. Calculate the escape speed for the universe.

Solution A light year is the distance that light travels in one year. Thus

$$\begin{aligned} \text{light year} &= c \times 1 \text{ year} \\ &= 3 \times 10^8 \frac{\text{m}}{\text{sec}} \times 365 \times 24 \times 60 \times 60 \text{ sec} \\ &= 10^{16}\text{m} \end{aligned}$$

Thus

$$\begin{aligned} v &= \sqrt{\frac{2GM}{r}} \\ &= \sqrt{\frac{2 \times 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 10\text{kg}}{10 \times 10^9 \times 10^{16}\,\text{m}}} \\ &= 3.7 \times 10^8 \text{ m/sec} \\ &= 1.2c \end{aligned}$$

which is 1.2 times the speed of light. Thus is our universe really a black hole? Do we actually live inside a black hole?

11.7 Kepler's Laws

Let's now use Newton's law of gravitation to prove some of Kepler's laws of planetary motion.

Kepler's first law is that the planets move in *elliptical* orbits with the Sun at one focus. This is somewhat difficult to prove and we will leave it to a more advanced physics course. A picture is shown in Figure 14.3 with the Sun at the focus of an ellipse.

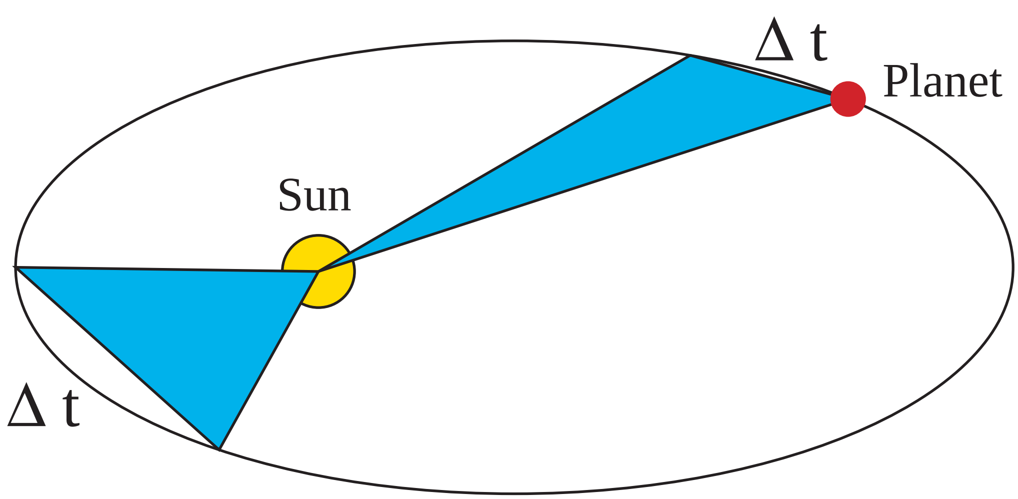


FIGURE 14.3 Planets sweep out equal areas in equal times.

Kepler's second law states that the line joining a planet to the Sun sweeps out equal areas in equal times. This is shown in Fig. 14.3. In the upper part of the figure there are two shaded regions with the same area. The planet takes the same time Δt to sweep out this area. Thus the planets move quickly when close to the Sun and move slowly when farther away.

Example Prove that Kepler's second law can be derived from Newton's law of universal gravitation.

Solution Figure 14.4 shows the radius vector $\vec{r}$ and the displacement $\vec{v}\,dt$ for the planet of mass m.

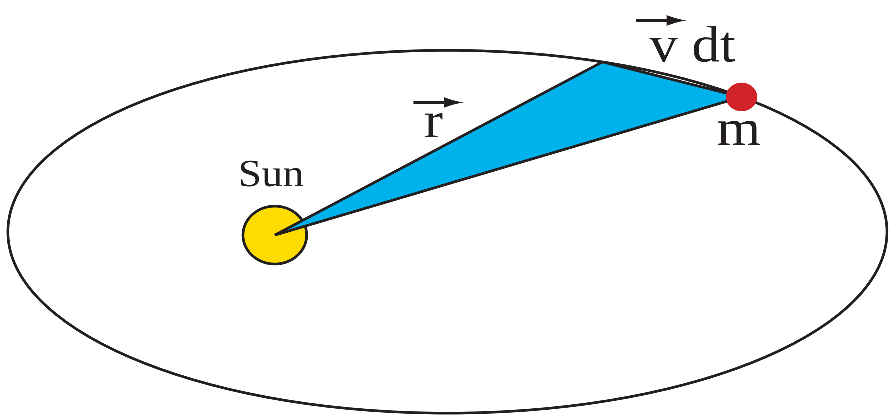


FIGURE 14.4 Area swept out by planet.

The shaded portion is the area swept out and has the shape of a triangle of area

$$dA = \frac{1}{2} r\,v\,dt$$

The rate of change of area is

$$\frac{dA}{dt} = \frac{1}{2} r\,v = \frac{1}{2m} mr\,v = \frac{l}{2m}$$

where l is the angular momentum of the planet. But angular momentum is *constant*, therefore

$$\frac{dA}{dt} = \text{constant}$$

meaning that equal areas are swept out in equal times!

Kepler's third law is that the period squared is proportional to the average distance cubed ($T^2 \propto r^3$) for a planetary orbit. This is difficult to prove for elliptical orbits, which is done in a more advanced physics course. We will prove it for a circular orbit only.

Actually the essenticity of the elliptical orbits are typically very small. In other words the elliptical orbits are *very close* to *circular* orbits with the Sun at the center. We shall prove Kepler's other two laws with the assumption that the orbits are circles. Thus we immediately know that the right hand side of $F = ma$ is $\frac{mv^2}{r}$ because *all uniform* circular motion has the centripetal acceleration given by $a = \frac{v^2}{r}$.

Example Prove that Kepler's third law can be derived from Newton's law of universal gravitation. (Assume circular orbits only)

Solution
$$F = ma$$
gives
$$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$
Now the period T is the time to complete *one* orbit. Thus
$$v = \frac{2\pi r}{T}$$
or
$$G\frac{M}{r^2} = \frac{1}{r}\frac{4\pi^2 r^2}{T^2} = \frac{4\pi^2 r}{T^2}$$
giving
$$\boxed{T^2 = \frac{4\pi^2}{GM}r^3}$$
or
$$T^2 \propto r^3$$

Example The distance between Earth and the Sun is about 93 million miles and can easily be determined using parallax and trigonometry. How can the mass of the Sun be subsequently determined ?

Solution Kepler's law is $T^2 = \frac{4\pi^2}{GM} r^3$ giving

$$M = \frac{4\pi^2}{G} \frac{r^3}{T^2}$$

Now r = 93,000,000 miles = 150,000,000 km and the period of Earth is 1 year or

$$T = 365 \times 24 \times 60 \times 60 \text{ sec}$$

Thus the mass of the Sun is

$$\begin{aligned} M &= \frac{4\pi^2}{6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}} \times \frac{(150,000,000 \times 10^3 \text{ m})^3}{365 \times 24 \times 60 \times 60 \text{ sec})^2} \\ &= 2 \times 10^{30} \text{ kg} \end{aligned}$$

Notice that the mass of Earth did not enter. Thus if we observe two bodies in orbit and know the distance between them we can get the mass of the other body. this is how astronomers determine the mass of double star systems. (More than half of the stars in the sky are actually double stars.)

11.8 Problems

Chapter 12

OSCILLATIONS

<u>**SUGGESTED HOME EXPERIMENT**</u>:
Measure g from the period of a pendulum.

<u>**THEMES**</u>:
Clocks.

12.1 Oscillations

Much of the motion that we have considered, such as motion of a car in a straight line or projectile motion, has started and then finished, i.e. it does not repeat. However a great deal of motion in nature is repetitive or oscillatory, such as a satellite undergoing circular motion, or an object suspended on a spring or a buoy bobbing up and down in the water. We would now like to study oscillations in detail. This will later lead to the study of *wave motion* which is also oscillatory in nature.

Oscillations are of great technological importance, especially in regard to time keeping.

(Note: Mechanical Universe tapes very good – especially discussion of clocks and navigation.)

12.2 Simple Harmonic Motion

An important property of oscillatory motion is the *frequency* f which is the number of oscillations completed each second. The units are $\sec^{-1}$ or Hertz, often abbreviated as Hz. Thus

$$\begin{aligned} 1 \text{ Hertz} &= 1 \text{ Hz} = 1 \text{ oscillation per second} \\ &= 1 \sec^{-1}. \end{aligned}$$

Another related quantity is the *period* T which is the time taken to complete 1 full oscillation. Now

$$f = \frac{\text{number of oscillations}}{\text{time}}$$

and if the time is simply T then 1 oscillation is completed. Thus

$$\boxed{f = \frac{1}{T}}$$

In circular motion, which is a type of oscillatory motion, we introduced the angular speed ω defined as

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Clearly if $\Delta\theta = 2\pi$ then $\Delta t = T$ giving $\omega = \frac{2\pi}{T}$. Thus angular velocity and frequency are related by

$$\boxed{\omega = 2\pi f}$$

In oscillations ω is often called *angular frequency.*

Any motion that repeats itself at regular intevals is called oscillatory motion or harmonic motion. Now of *all* the mathematical functions that you have ever come across, there is one famous function that displays oscillations and that is $\cos\theta$, which is plotted in Figure 16.1.

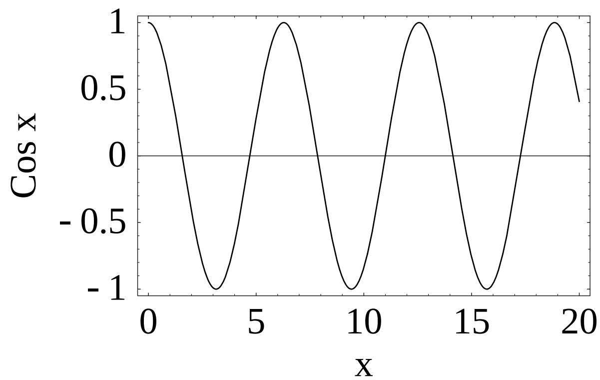


FIGURE 16.1 Plot of $\cos\theta$.

Thus the displacement x for oscillatory motion can be written

$$x = x_m \cos\theta$$

but $\omega = \frac{\theta}{t}$, giving

$$\boxed{x = x_m \cos\omega t}$$

We can also introduce a phase angle ϕ if we want and instead write

$$x = x_m \cos(\omega t + \phi)$$

This is discussed on Pg. 374 of Halliday. Here x_m refers to the maximum value of the displacement x. And x_m is often called the *amplitude* of the motion.

LECTURE DEMONSTRATION: Spring and Pendulum

Any motion that obeys the above equation $x = x_m \cos\omega t$ is called Simple Harmonic Motion (SHM).

The velocity of SHM is easy to figure out. First recall that if $y = \cos kx$ then $\frac{dy}{dx} = -k\sin kx$. Now the velocity is

$$v = \frac{dx}{dt} = -\omega x_m \sin\omega t$$

Also recall if $y = \sin kx$ when $\frac{dy}{dx} = k \cos kx$. Now the acceleration is

$$a = \frac{dv}{dt} = -\omega^2 x_m \cos \omega t$$

from which it follows that

$$a = -\omega^2 x$$

In Figure 16-4 of Haliday, there is a plot of x, v, a. Notice that when x and a are at a maximum, then v is a minimum and vice-versa.

LECTURE DEMONSTRATION: Show this for Spring

12.3 Force Law for SHM

Now consider Newton's law for a Spring where the force is given by $F = -kx$ (Hooke's law), where k is called the spring constant. Substituting into

$$\begin{aligned} F &= ma \\ -kx &= ma \end{aligned}$$

but we found that $a = -\omega^2 x$ giving

$$-kx = -m\omega^2 x$$

or

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

which is the angular frequency for an oscillating spring. The period is obtained from $\omega = 2\pi f = \frac{2\pi}{T}$ or

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

Notice an amazing thing. *The period does not depend on the amplitude of oscillation* x_m! When a spring is oscillating, the oscillations tend to die down in amplitude x_m but the period of oscillation remains the same! *This is crucial to the operation of clocks.* I can "wind" my spring clock by just pulling on it a bit and still the period is the same.

LECTURE DEMONSTRATION: Show this for Spring and Pendulum. Also show $T \propto \sqrt{m}$ and $T \propto \frac{1}{\sqrt{k}}$.

Navigation and Clocks

NNN - FIX For a pendulum, this independence of the period on the amplitude was first noticed by Galileo and led to the development of clocks which was very important for navigation. The reason was that it enabled one to determine *longitude* on Earth. (*Latitude* was easy to determine just by measuring the height of the Sun in the sky at noon.) By dragging knotted ropes behind a ship it was easy to measure the speed of a ship. If one knew how long one had been travelling (i.e. measure the time of travel, say with a pendulum or spring clock) then one knew the *distance* from the port from which one had set sail. Knowing longitude and latitude gives one's position on the Earth. Thus the invention of accurate clocks (based on the independence of period and amplitude) enabled accurate estimates of longitude and thus revolutionized navigation.

Example $F = ma$ is really a *differential equation*, that is an *equation* involving *derivatives*. For the spring, it becomes $-kx = ma = m\ddot{x}$ where $\ddot{x} = \frac{d^2x}{dt^2}$. Thus the differential equation is

$$m\ddot{x} + kx = 0$$

In mathematics there are special techniques for solving differential equations, which you will learn about in a special differential equations course. Using these special techniques one can *prove* that $x = x_m \cos\omega t$ is a *solution* to the above differential equation. (Just like the solution to the *algebraic equation* $x^2 - 5 = 4$ is $x = \pm 3$. We *verify* this solution by sustituting, $(\pm 3)^2 - 5 = 9 - 5 = 4$). Many students will not have yet learned how to solve differential equations, but we can *verify* that the solution given is correct.

Verify that $x = x_m \cos\omega t$ is a solution to the differential equation $m\ddot{x} + kx = 0$.

Solution

$$\begin{aligned} x &= x_m \cos\omega t \\ \dot{x} &= \frac{dx}{dt} = -\omega x_m \sin\omega t \\ \ddot{x} &= \frac{d\dot{x}}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x_m \cos\omega t \end{aligned}$$

Substitute into

$$m\ddot{x} + kx = 0$$

giving

$$-m\omega^2 x_m \cos\omega t + kx_m \cos\omega t = 0$$

or

$$-m\omega^2 + k = 0$$

Thus *if*

$$\omega = \sqrt{\frac{k}{m}}$$

then $x = x_m \cos\omega t$ is a solution.

Example When a mass is suspended from the end of a massless spring, the spring stretches by a distance x. If the spring and mass are then put into oscillation, what is the period ?

Solution We saw that the period is given by $T = 2\pi\sqrt{\frac{m}{k}}$. We don't know m or k ! We can get k from Hooke's law $F = -kx$. The weight $W = mg$ stretches the spring, thus $mg = kx$ or $k = \frac{mg}{x}$. Thus

$$T = 2\pi\sqrt{\frac{mx}{mg}}$$

and fortunately m cancels out giving

$$T = 2\pi\sqrt{\frac{x}{g}}$$

12.4 Energy in SHM

We found before that the potential energy stored in a spring is $U = \frac{1}{2}kx^2$ and the kinetic energy is $K = \frac{1}{2}mv^2$. The conservation of mechanical energy says that

$$E_i = E_f$$

where the total energy is

$$E \equiv K + U$$

That is

$$K_i + U_i = K_f + U_f$$

Thus E is constant. However for a spring x and v are always changing. Can we be sure that E is always constant ?

Example For SHM, show that the total energy is always constant even though K and U always change.

Solution Recall that for SHM we have $x = x_m \cos \omega t$ and $v = -\omega x_m \sin \omega t$. Thus

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2 \omega t$$

and

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2 \omega t.$$

Thus U and K always change. Let's add them.

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}m\omega^2 x_m^2 \sin^2 \omega t + \frac{1}{2}kx_m^2 \cos^2 \omega t \end{aligned}$$

but we previously found that $\omega = \sqrt{\frac{k}{m}}$ giving

$$\begin{aligned} E &= \frac{1}{2}m\frac{k}{m}x_m^2 \sin^2 \omega t + \frac{1}{2}kx_m^2 \cos^2 \omega t \\ &= \frac{1}{2}kx_m^2(\sin^2 \omega t + \cos^2 \omega t) \end{aligned}$$

$$\boxed{E = \frac{1}{2}kx_m^2}$$

which is always constant because the amplitude x_m is constant!

12.5 An Angular Simple Harmonic Oscillator

leave out

12.6 Pendulum

The Simple Pendulum

A pendulum is a very important type of oscillating motion and a very important clock (e.g. "Grandfather Clock"). The forces on a pendulum are shown in Fig. 16-10 of Halliday. Let's analyze the forces and show that the *period is independent of amplitude.*

Example Prove that the period of a pendulum undergoing small oscillations is given by $T = 2\pi\sqrt{\frac{L}{g}}$ where L is the length of the pendulum.

Solution From Figure 16-10 (Halliday) we have

$$\sum F_k = ma_x$$

where we take the x direction to be perpendicular to the string. Thus

$$-mg\sin\theta = m\alpha L$$

where α is the angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$. Now for *small oscillations*, $\sin\theta \approx \theta$, so that

$$-g\theta = m\frac{d^2\theta}{dt^2}L$$

Now compare this to our spring equation which was

$$\begin{aligned} -kx &= ma \\ -kx &= m\frac{d^2x}{dt^2} \end{aligned}$$

which had period $T = 2\pi\sqrt{\frac{m}{k}}$. Thus for the pendulum we must have

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

LECTURE DEMONSTRATION: Show $T \propto \sqrt{L}$

Example A Physical Pendulum consists of a solid piece of matter undergoing oscillations as shown in Fig. 16.11 (Halliday). Prove that the period of oscillation is $T = 2\pi\sqrt{\frac{I}{mgh}}$, where I is the rotational inertia, m is the total mass and h is the distance from the rotation axis to the center of mass. (See Haliday, Pg. 382) Assume *small* oscillations.

Solution The torque is

$$\tau = -(mg\sin\theta)h$$

where the minus sign indicates that when θ increases the torque acts in the opposite direction. For small oscillations $\sin\theta \approx \theta$ giving

$$\tau \approx -mg\theta h$$

Substitute into Newton's second law

$$\sum \tau = I\alpha$$

gives

$$\begin{aligned} -mg\theta h &= I\ddot{\theta} \\ &= I\frac{d^2\theta}{dt^2} \end{aligned}$$

Now compare this to our spring equation which was

$$\begin{aligned} -kx &= ma \\ -kx &= m\frac{d^2x}{dt^2} \end{aligned}$$

which had period $T = 2\pi\sqrt{\frac{m}{k}}$. Thus for the physical pendulum we must have

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

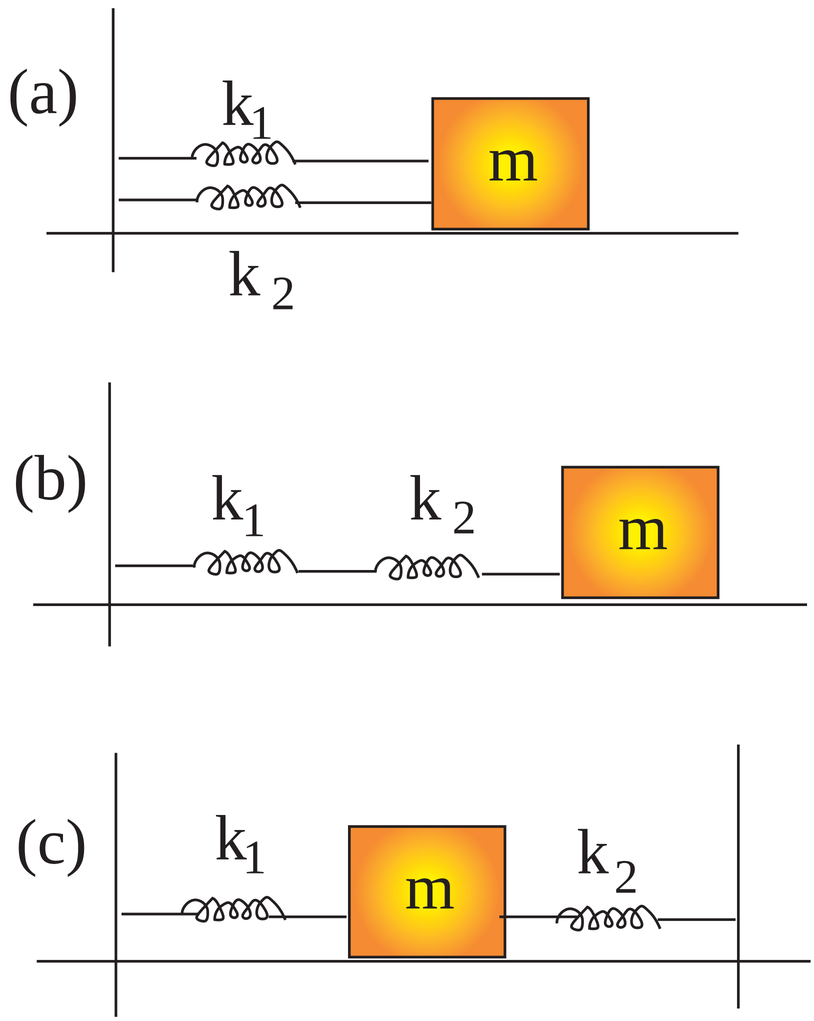


FIGURE 16.2 Block sliding on frictionless surface with various spring combinations.

Example Two springs, with spring constants k_1 and k_2, are connected in parallel to a mass m sliding on a frictionless surface, as shown in Fig. 16.2a. What is the effective spring constant K? (i.e. If the two springs were replaced by a single spring with constant K, what is K in terms of k_1 and k_2?) Assume both springs have zero mass.

Solution If m moves by an amount x then it feels two forces $-k_1x$ and $-k_2x$, giving

$$\sum F = ma$$

$$-k_1x - k_2x = m\ddot{x}$$

$$-(k_1 + k_2)x = m\ddot{x}$$

giving

$$K = k_1 + k_2$$

Example The two springs of the previous example are connected in series, as shown in Fig. 16.2b. What is the effective spring constant K ?

Solution If spring 1 moves a distance x_1 and spring 2 moves a distance x_2 then the mass moves a distance $x_1 + x_2$. The force the mass feels is

$$F = -K(x_1 + x_2)$$

Now consider the motion of the *mass plus spring 2* system. The force it feels is

$$f = -k_1 x_1$$

but we must have $F = f$ because ma is *same* for mass m and *mass plus spring 2* system because spring 2 has zero mass. Thus

$$K = \frac{k_1 x_1}{x_1 + x_2}$$

but

$$k_1 x_1 = k_2 x_2$$

(the ratio of stretching $\frac{x_1}{x_2} = \frac{k_2}{k_1}$ is inversely proportional to spring strength.) Thus $K = \frac{k_1 x_1}{x_1 + \frac{k_1}{k_2} x_1}$ giving

$$K = \frac{k_1 k_2}{k_1 + k_2}$$

or

$$\frac{1}{K} = \frac{1}{k_1} + \frac{1}{k_2}$$

Example The two springs of the previous example are connected as shown in Fig.16.2c. What is the effective spring constant K ?

Solution If spring 1 is *compressed* by x then spring 2 is *stretched* by $-x$. Thus

$$\sum F = ma$$

$$-k_1 x + k_2(-x) = m\ddot{x}$$

$$-(k_1 + k_2)x = m\ddot{x}$$

giving

$$K = k_1 + k_2$$

12.7 Problems

1. An object of mass m oscillates on the end of a spring with spring constant k. Derive a formula for the time it takes the spring to stretch from its equilibrium position to the point of maximum extension. Check that your answer has the correct units.

2. An object of mass m oscillates at the end of a spring with spring constant k and amplitude A. Derive a formula for the speed of the object when it is at a distance d from the equilibrium position. Check that your answer has the correct units.

3. A block of mass m is connected to a spring with spring constant k, and oscillates on a horizontal, frictionless surface. The other end of the spring is fixed to a wall. If the amplitude of oscillation is A, derive a formula for the speed of the block as a function of x, the displacement from equilibrium. (Assume the mass of the spring is negligible.)

4. A particle that hangs from a spring oscillates with an angular frequency ω. The spring-particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car), as the car descends at a constant speed v. The car then stops suddenly. Derive a formula for the amplitude with which the particle oscillates. (Assume the mass of the spring is negligible.) [Serway, 5th ed., pg. 415, Problem 14]

5. A large block, with a second block sitting on top, is connected to a spring and executes horizontal simple harmonic motion as it slides across a frictionless surface with an angular frequency ω. The coefficient of static friction between the two blocks is μ_s. Derive a formula for the maximum amplitude of oscillation that the system can have if the upper block is not to slip. (Assume that the mass of the spring is negligible.) [Serway, 5th ed., pg. 418, Problem 54]

6. A simple pendulum consists of a ball of mass M hanging from a uniform string of mass m, with $m \ll M$ (m is much smaller than M). If the period of oscillation for the pendulum is T, derive a formula for the speed of a transverse wave in the string when the pendulum hangs at rest. [Serway, 5th ed., pg. 513, Problem 16]

Chapter 13

WAVES - I

SUGGESTED HOME EXPERIMENT:
Pluck some strings and verify the frequency equation for strings.

THEMES:
Violin and Guitar.

So far we have studied the motion of single particles and systems of particles. However the motion of *waves* requires a different type of approach, although we will use extensively some of our results from harmonic motion.

Waves are an important phenomenon in nature. There are water waves, sound waves by which we hear, light waves by which we see, and radio waves by which we communicate. Thus in today's modern society it is important to understand wave motion.

13.1 Waves and Particles

Read.

13.2 Types of Waves

Read.

13.3 Transverse and Longitudinal Waves

There are *two* different types of waves. *Transverse waves* are the ones you are most familiar with, such as water waves or waves on a string. Transverse waves have the property that the wave displacement is *perpendicular* to the velocity of the wave, as shown in Fig. 17-1 (Halliday). Sound waves are an example of *longitudinal waves* in which the wave displacement is *parallel* to the wave velocity, as shown in Fig. 17-2 (Halliday). When you hear a sound wave, the wave travels to your ear and vibrates your ear drum in the same direction as travel.

LECTURE DEMONSTRATION: Slinky showing tranverse and longitudinal waves.

13.4 Wavelength and Frequency

There are 3 important variables for a wave, namely, i) the height y of the wave, ii) the distance x that the wave travels and iii) the time t that the wave travels. When *visualizing* a wave we usually think of a $y-x$ plot or a $y-t$ plot as shown in Fig. 17-4 (Halliday).

The $y-x$ plot represents an *instant of time* t and is similar to a photograph or *snapshot* of a water wave that we would take at the beach. The distance between wave crests (that we could measure from our snapshot) is called the *wavelength* λ.

The $y-t$ plot represents a *single location* x and is similar to a *movie* of a buoy bobbing up and down in the water as a wave passes through. The buoy is anchored to the ocean floor at a fixed distance x. The *time* it takes the buoy to bob up and down *once* is called the *period* T of the wave.

Thus, to summarize, λ is determined from the $y-x$ graph (instant of time t) whereas T is determined from the $y-t$ graph (fixed distance x). Carefully study Fig. 17-4 (Halliday). Thus y is a function of both x and t, written as $y(x,t)$. Now the $y-x$ graph can be written

$$y(x,0) = y_m \sin kx$$

where we have taken the instant of time to be $t = 0$. The reason we have written $\sin kx$ and not just $\sin x$ is because the domain of the sine function is an angle. We can *only* ever have $\sin\theta$ where θ is an angle. Thus we *cannot* write $\sin x$ because x is not an angle. Actually x is a distance with units of m. However we *want* to use x as a plotting variable. To do this we have to multiply it by something called k, so that the quantity kx is an *angle*, i.e. $\theta \equiv kx$. Now what is k? Well if kx is an angle then after one complete wave cycle, the angle kx must be 2π. Now after one complete cycle the distance the wave moves is $x = \lambda$. Thus we *must* have

$$\theta = kx$$

or

$$2\pi = k\lambda$$

giving

$$\boxed{k = \frac{\lambda}{2\pi}}$$

which is called the *wave number*. Similarly, the $y-t$ graph can be written

$$y(0,t) = y_m \sin \omega t$$

where we have taken the fixed distance to be $x = 0$. We did not write $\sin t$ because t is not an angle, whereas ωt is an angle. ω is the angular speed that we have discussed before. Again after one complete wave cycle ωt must be 2π and after one cycle the time t will just be one period T. Thus we *must* have

$$\theta = \omega t$$

or

$$2\pi = \omega T$$

giving

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi f}$$

which is often called the angular frequency ω. We previously defined $f \equiv \frac{1}{T}$ in Chapter 16. A general wave can be written

$$\boxed{y(x,t) = y_m \sin(kx + \omega t)}$$

Does this agree with what we had before? Yes. We can see that $y(x, 0) = y_m \sin kx$ and $y(0, t) = y_m \sin \omega t$.

13.5 Speed of a Travelling Wave

A handy formula for wave speed is *easy* to get! In one complete cycle the wave travels a distance $x = \lambda$ and takes a time $t = T$ to do it. Thus the wave *speed* must be

$$v = \frac{\text{distance}}{\text{time}} = \frac{x}{t} = \frac{\lambda}{T}$$

Simple algebra also gives

$$\boxed{v = \frac{\lambda}{T} = f\lambda = \frac{\omega}{k}}$$

Example What is the amplitude, wavelength, frequency and speed of the wave described by

$$y(x,t) = 5\sin(3x + 2t)$$

with all quantities in SI units (i.e. 5 m, 3 m^{-1} and 2 sec^{-1}).

Solution The general wave is

$$y(x,t) = y_m \sin(kx + \omega t)$$

Thus the amplitude is

$$y_m = 5 \text{ m}$$

the wave number is

$$k = 3 \text{ m}^{-1}$$

and angular frequency is

$$\omega = 2 \text{ sec}^{-1}$$

Now $k = \frac{2\pi}{k} = 3 \text{ m}^{-1}$ giving

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{3 \text{ m}^{-1}} = 2.1 \text{ m}$$

and $\omega = 2\pi f = 2 \text{ sec}^{-1}$ giving

$$f = \frac{\omega}{2\pi} = \frac{2 \text{ sec}^{-1}}{2\pi} = 0.32 \text{ sec}^{-1}$$

and the speed is

$$v = f\lambda = 0.32 \text{ sec}^{-1} \times 2.1 \text{ m} = 0.67 \text{ m/sec}$$

13.6 Wave Speed on a String

LECTURE DEMONSTRATION: Wave speed depends on tension.

When a wave travels on a string, the wave speed depends on both the string *tension* τ and the mass per unit length μ, or *linear mass density.* What must the exact formula be? (τ is now tension, not torque) Well the units of v are m sec^{-1} and units of τ are $N \equiv$ kg m sec^{-1} and units of μ are kg m^{-1}. To get m sec^{-1} from kg m sec^{-2} and kg m^{-1} can *only* be obtained with

$$\begin{aligned} \text{m sec}^{-1} &= \sqrt{\frac{\text{kg m sec}^{-2}}{\text{kg m}^{-1}}} \\ &= \sqrt{\text{m}^2\text{sec}^{-2}} = \text{m sec}^{-1} \end{aligned}$$

Thus we must have

$$\boxed{v = \sqrt{\frac{\tau}{\mu}}}$$

And we can combine with our previous formula, so that the wave speed on a string is $v = f\lambda = \sqrt{\frac{\tau}{\mu}}$.

13.7 Energy and Power of a Travelling String Wave

Leave out.

13.8 Principle of Superposition

Read carefully.

13.9 Interference of Waves

Read carefully.

LECTURE DEMONSTRATION: Show wave interference using slinky.

13.10 Phasors

Leave out.

13.11 Standing Waves

Read carefully.

13.12 Standing Waves and Resonance

When waves travel down a string they can *reflect* back from the other end and *interfere* with the other waves.

LECTURE DEMONSTRATION: Standing waves on slinky.

In this way standing waves of different wavelength can be produced. The wave of lowest frequency (longest wavelength) is called the *fundamental harmonic*. Higher frequencies are called higher harmonics. The various allowed harmonics are shown in Fig. 17-18 (Haliday). The relations between the wavelength λ and the length of the string L for the various harmonics are

$$\begin{aligned} L &= \frac{\lambda}{2} \\ L &= \lambda = \frac{2\lambda}{2} \\ L &= \frac{3\lambda}{2} \end{aligned}$$

etc. These can be written in general as

$$L = n\frac{\lambda}{2} \quad \text{with} \quad n = 1, 2, 3, \cdots$$

Now the wave speed is $v = f\lambda = \sqrt{\frac{\tau}{\mu}}$ and writing $\lambda = \frac{2L}{n}$ gives $f\frac{2L}{n} = \sqrt{\frac{\tau}{\mu}}$ or

$$\boxed{f = \frac{n}{2L}\sqrt{\frac{\tau}{\mu}}}$$

This is an extremely important formula for the design of muscial instruments.

LECTURE DEMONSTRATION: Show how frequency of Sound from Violin depends on length L, tension τ and mass density μ, thus verifying the above formula.

Example Middle C has a frequency of 262 Hz. What tension do we need to apply to a violin string to get this frequency for the fundamental harmonic? (Assume the string has a mass of about 10 gram and a length of 1/4 m.)

Solution The mass per unit length μ is

$$\mu = \frac{10 \text{ gram}}{1/4 \text{ m}} = \frac{0.01 \text{ kg}}{.25 \text{ m}} = 0.04 \text{ kg m}^{-1}$$

The frequency is given by $f = \frac{n}{2L}\sqrt{\frac{\tau}{\mu}}$. The fundamental harmonic corresponds to $n = 1$, giving

$$\begin{aligned}
\tau &= \mu(2Lf)^2 \\
&= 0.04 \text{ kg m}^{-1} \, (2 \times 0.25 \text{ m} \times 262 \text{ sec}^{-1})^2 \\
&= 686 \text{ kg m}^{-1}\text{m}^2\text{sec}^{-2} \\
&= 686 \text{ kg m sec}^{-2} \\
&= 686 \text{ N}
\end{aligned}$$

13.13 Problems

Chapter 14

WAVES - II

SUGGESTED HOME EXPERIMENT:
Blow in some pipes and verify the frequency equation for pipes.

THEMES:
Flute and Recorder.

14.1 Sound Waves

This chapter is mostly devoted to the study of *sound* waves, although much of what we have to say can also be applied to light waves. By the way, sound waves are longitudinal whereas light waves are transverse.

14.2 Speed of Sound

The speed of sound in any medium is given by

$$v = \sqrt{\frac{B}{\rho}}$$

where ρ is the density of the medium and B is the Bulk Modulus defined as

$$B \equiv -\frac{\Delta p}{\Delta V/V}$$

where a change in pressure Δp causes a change in the volume ΔV of a medium. Students should read Halliday (Pg. 426-427) for a careful discussion of these concepts.

In *air* the speed of sound is

$$343 \text{ m/sec} = 1125 \text{ ft/sec} = 767 \text{ mph}$$

The speed of sound was exceeded in an *airplane* many years ago. However the sound barrier was broken by an *automobile* only for the first time in October 1997!

14.3 Travelling Sound Waves

Leave out.

14.4 Interference

Read carefully.

14.5 Intensity and Sound Level

Read Halliday carefully. Understand the formula for sound level

$$\beta \equiv 10\text{dB} \log \frac{I}{I_o}$$

14.6 Sources of Musical Sound

All students should carefully read Pg. 435-436 Halliday. There it is explained how standing sound waves occur in pipes filled with air. See Fig. 18-14 (Halliday). The various maxima and minima locations of the standing waves correspond to maximum and minimum *pressures* in the pipe as shown in Fig. 18-13 (Halliday).

LECTURE DEMONSTRATION: Standing Sound Waves & Water Column

Example For a pipe open at both ends, determine the relationship between the length of the pipe L and the frequencies of the various harmonics.

Solution The pipe open at both ends is shown in Figs. 18-13, 18-14a (Halliday). There is a pressure *node* at the closed end and an anitnode at the open end. The relations between the wavelength λ and the pipe length L for the various harmonics is *Note the* **first** *harmonic is actually Fig. 18-13 (Halliday) and the higher harmonics are in Fig. 18-14a*

$$\begin{aligned} L &= \frac{\lambda}{2} = \frac{1}{2}\lambda \\ L &= \lambda = \frac{2}{2}\lambda \\ L &= \frac{3}{2}\lambda \\ L &= 2\lambda = \frac{4}{2}\lambda \end{aligned}$$

etc. These can be written in general as

$$L = \frac{n\lambda}{2} \quad \text{with} \quad n = 1, 2, 3 \cdots$$

Example Repeat the previous example for a pipe open at only one end.

Solution This is shown in Fig. 18-14b (Halliday). Obviously

$$L = \frac{\lambda}{4} = \frac{1\lambda}{4}$$
$$L = \frac{3\lambda}{4}$$
$$L = \frac{5\lambda}{4}$$

etc. These can be written in general as

$$L = \frac{n\lambda}{4} \quad \text{with} \quad n = 1, 3, 5 \cdots$$

Now recall that $v = f\lambda = \sqrt{\frac{B}{\rho}}$. Thus for the pipe open at *both* ends

$$f = \frac{n}{2L}\sqrt{\frac{B}{\rho}} \quad \text{with} \quad n = 1, 2, 3, \cdots$$

and for the pipe open at *one* end,

$$f = \frac{n}{4L}\sqrt{\frac{B}{\rho}} \quad \text{with} \quad n = 1, 3, 5, \cdots$$

These are very important formulas for the design of wind musical instruments, such as a flute or recorder.

Note that a longer instrument (larger L) will give a lower frequency.

LECTURE DEMONSTRATION: Two recorders.

Also note that the frequency depends on the *density* of air.

LECTURE DEMONSTRATION: Talking with Helium gas.

14.7 Beats

Read carefully.

14.8 Doppler Effect

Everyone has noticed the pitch of the sound of a train varies when the train passes. You can also easily hear this just listening to cars drive down the road. This change in frequency of a moving sound source is called the Doppler effect.

LECTURE DEMONSTRATION: Moving Microphone (twirl on a string)

The same Doppler effect is also observed when the listener is moving and the source is stationary.

We have previously seen that for a stationary observer and source, then

$$f = \frac{v}{\lambda}$$

where v is the wave speed and λ is the wavelength.

Example An observer *moves toward* a stationary source of sound waves at a speed v_D (detector speed). Derive a formula for the observed frequency f' in terms of the stationary frequency f.

Solution This situation is shown in Fig. 18-18 (Halliday). The detector will sense a *higher frequency* as in

$$f' = \frac{v + v_D}{\lambda}$$

Now

$$\frac{f'}{f} = \frac{\frac{v+v_D}{\lambda}}{\frac{v}{\lambda}} = \frac{v + v_D}{v}$$

or

$$f' = f\frac{v + v_D}{v}$$

Note: if the observer was moving away, the result would be

$$f' = f\frac{v - v_D}{v}$$

Example A sound wave *moves toward* a stationary observer at a speed v_s. Derive a formula for the observed frequency f' in terms of the stationary frequency f.

Solution This situation is shown in Fig. 18-21 (Halliday). This time it is the *wavelength* which changes and it will be smaller as in

$$\lambda' = \lambda - \frac{v_s}{f}$$

f' is now (due to change in λ')

$$\begin{aligned} f' &= \frac{v}{\lambda'} \\ &= \frac{v}{\lambda - \frac{v_s}{f}} = \frac{vf}{\lambda f - v_s} \\ &= \frac{vf}{v - v_s} \end{aligned}$$

or

$$f' = f\frac{v}{v - v_s}$$

Note: if the source was smoving *away*, the result would be

$$f' = f\frac{v}{v + v_s}$$

All of the previous results can be combined into a single formula,

$$\boxed{f' = f\frac{v \pm v_D}{v \mp v_s}}$$

If $v_s = 0$ we get $f' = f\frac{v \pm v_D}{v}$ as before and if $v_D = 0$ we get $f' = f\frac{v}{v \mp v_s}$ as before. An easy way to remember the signs is that if detector and source are moving *toward* each other the frequency *increases*. If they are moving *away* from each other the frequency*decreases*.

The Austrian physicist, *Johann Christian Doppler* proposed the effect in 1842. In 1845 it was tested experimentally by *Buys Ballot* using a locomotive drawing an open train car with trumpeters playing.

Example Middle C has a frequency of 264 Hz. The D note has a fequency of 300 Hz. If a trumpeter is playing the C note on a train, how fast would the train need to travel for a stationary person (with perfect pitch) on the ground to hear a D note ?

Solution Here $v_D = 0$ and we want to find v_s. The frequency increases and we have

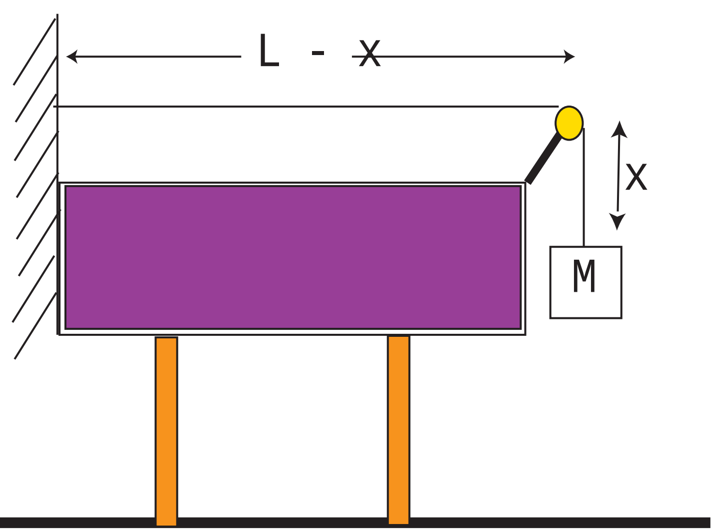
$$\begin{aligned} f' &= f\frac{v}{v - v_s} \\ \Rightarrow \frac{1}{f'} &= \frac{v - v_s}{fv} \\ \Rightarrow v - v_s &= \frac{fv}{f'} \\ \Rightarrow v_s &= v(1 - \frac{f}{f'}) \\ &= 767\ mph\ (1 - \frac{264Hz}{300Hz}) \\ &= 92\ mph \end{aligned}$$

14.9 Problems

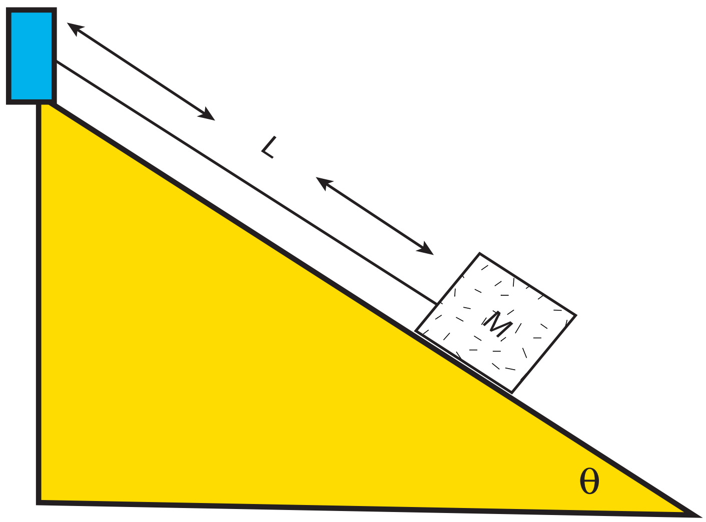
1. A uniform rope of mass m and length L is suspended vertically. Derive a formula for the time it takes a transverse wave pulse to travel the length of the rope.

   (Hint: First find an expression for the wave speed at any point a distance x from the lower end by considering the tension in the rope as resulting from the weight of the segment below that point.) [Serway, 5th ed., p. 517, Problem 59]

2. A uniform cord has a mass m and a length L. The cord passes over a pulley and supports an object of mass M as shown in the figure. Derive a formula for the speed of a wave pulse travelling along the cord. [Serway, 5 ed., p. 501]



3. A block of mass M, supported by a string, rests on an incline making an angle θ with the horizontal. The string's length is L and its mass is $m \ll M$ (i.e. m is negligible compared to M). Derive a formula for the time it takes a transverse wave to travel from one end of the string to the other. [Serway, 5th ed., p. 516, Problem 53]



4. A stationary train emits a whistle at a frequency f. The whistle sounds higher or lower in pitch depending on whether the moving train is approaching or receding. Derive a formula for the *difference* in frequency Δf, between the approaching and receding train whistle in terms of u, the speed of the train, and v, the speed of sound. [Serway, 5th ed., p. 541, Problem 54]

Chapter 15

TEMPERATURE, HEAT & 1ST LAW OF THERMODYNAMICS

SUGGESTED HOME EXPERIMENT:
Put a block of ice into an insulated container of water and measure the temperature change. Is it what you expect ?

THEMES:
Heating and Cooling.

15.1 Thermodynamics

We now leave our study of *mechanics* and begin our study of *thermodynamics*. The most important system that we will study is an *ideal gas* and how the temperature, pressure and volume are related. (Actually, however, thermodynamic quantities are related to our study of mechanics. This is the study of the kinetic theory of gases, i.e. a microscopic approach to thermodynamics.)

One of the most important properties of a macroscopic system, such as a liquid or gas is the *temperature*, or *thermal energy*.

15.2 Zeroth Law of Thermodynamics

If two bodies have the same temperature then they are said to be in thermal equilibrium. Thus the zeroth law of thermodynamics simply states that:

"If two bodies are in thermal equilibrium with a third body, then they are in thermal equilibrium with each other."

15.3 Measuring Temperature

Read.

15.4 Celsius, Farenheit and Kelvin Temperature Scales

The antiquated Farenheit temperature scale is only still used in a few countries (including the United States). Water freezes at 32°F and boils at 212°F. A much more natural temperature scale, called Celsius or Centigrade, rates the freezing and boiling point of water at 0°C and 100°C respectively. To *convert* between the two scales use

$$\boxed{F = \frac{9}{5}C + 32}$$

where F is the temperature in Farenheit and C is the temperature in Centigrade.

Example If you set your house thermostate to 70°F what is the temperature in Centigrade ?

Solution

$$F = \frac{9}{5}C + 32$$

$$F - 32 = \frac{9}{5}C$$

$$\begin{aligned} C &= \frac{5}{9}(F - 32) \\ &= \frac{5}{9}(70 - 32) \\ &= 23^{\circ}C \end{aligned}$$

Example At what temperature are the Farenheit and Centigrade scales equal ?

Solution When they are equal the $F = C = x$ giving

$$x = \frac{9}{5}x + 32$$

$$x\left(1 - \frac{9}{5}\right) = 32$$

$$-\frac{4}{5}x = 32$$

$$x = -40^{\circ}$$

i.e.

$$-40^{\circ}F = -40^{\circ}C$$

From a *microscopic* point of view (see Chapter 20), the *temperature* of a substance is related to the *speed* of the individual molecules which also give rise to *pressure*. Thus a gas which has fast moving molecules will have a high temperature and pressure. What happens if we slow all the molecules to zero speed? Well then the gas pressure will be *zero*. The temperature at which this happens is $-273.15°C$.

LECTURE DEMONSTRATION: Show this.

This leads to a *third* type of tmperature scale called *Absolute* temperature or *Kelvin* temperature. The Kelvin temperature at which a gas has zero pressure is defined to be $0°K$. Thus

$$\boxed{C = K - 273.15}$$

where C is the temperature in Centigrade and K is the temperature in Kelvin.

Example What is the relationship between Farenheit and Kelvin ?

Solution

$$C = K - 273$$

and

$$C = \frac{5}{9}(F - 32)$$

giving

$$\frac{5}{9}(F - 32) = K - 273$$

or

$$\begin{aligned} F &= \frac{9}{5}(K - 273) + 32 \\ &= \frac{9}{5}K - 459.4 \end{aligned}$$

15.5 Thermal Expansion

Read.

15.6 Temperature and Heat

Read *very* carefully.

15.7 The Absorption of Heat by Solids and Liquids

Heat Capacity

If you put a certain amount of *energy* or *heat* into a block of wood then the temperature will increase by a certain amount. If you do the same thing to a lump of steel (of the same mass) its temperature increase will be larger than for the wood. *Heat capacity* tells us how much the temperature of an object will increase for a given amount of energy or heat input. It is *defined* as

$$\boxed{C \equiv \frac{Q}{\Delta T}}$$

where C is the heat capacity, Q is the heat and ΔT is the temperature change, or

$$Q = C(T_f - T_i)$$

Example Which has the largest heat capacity; wood or steel ?

Solution For a given Q then ΔT will be larger for steel. From $C = \frac{Q}{\Delta T}$ it means that C is small for steel and large for wood.

Specific Heat

If we put a certain amount of heat into a small block of steel compared to a large block then the small block will change its temperature the most. Thus we also need to include the *mass* of the block in determining temperature change. Thus we define specific heat (with a lower case c) as

$$\boxed{c \equiv \frac{Q}{m\Delta T}}$$

or

$$Q = cm(T_f - T_i)$$

In other words the specific heat is just the heat capacity per unit mass or

$$c = \frac{C}{m}$$

Molar Specific Heat

Instead of defining specific heat with the mass of the object, we could define it according to the total number of molecules in the object. But if we write down the total number of molecules we will be writing down *huge* numbers. Now we always use other words for huge numbers. Instead of saying "one hundred tens" we say "thousand", i.e.

$$\text{thousand} \equiv 1000$$

or instead of saying "one thousand thousands" we say "million", i.e.

$$\text{million} \equiv 1,000,000$$

Now even million, billion and trillion are too small for the number of molecules in an object. Thus define

$$\boxed{\text{mole} \equiv 6.02 \times 10^{23}}$$

(This number arose because in 12 grams of ^{12}C there is 1 mole of atoms.) Thus molar specific heat is defined as

$$c_m \equiv \frac{Q}{N\Delta T}$$

where N is the number of moles of molecules in the substance.

Table 19-3 in Halliday has a list of specific heats and molar specific heats for various substances.

Example How much heat is required to increase the temperature of 2 kg of water from $20°C$ to $30°C$?

Solution From Table 19-3 of Halliday, the specific heat of water is 1.00 cal $g^{-1}K^{-1}$. Thus the temperature should be in $°K$. Now $\Delta T = 30°C - 20°C = 20°C$ or

$$\Delta T = -243°K - -253°K = 10°K$$

giving

$$\begin{aligned} Q &= mc\Delta T \\ &= 2\text{kg} \times 1 \text{ cal g}^{-1}K^{-1} \times 10\ K \\ &= 2000 \text{ g} \times 1 \text{ cal g}^{-1}K^{-1} \times 10\ K \\ &= 20{,}000 \text{ cal} = 83{,}720 \text{ J} \\ &= 20 \text{ kcal} \end{aligned}$$

where we have used 1 cal $\equiv$ 4.186 J.

Heats of Transformation

When you put heat or energy into an object the temperature does *not* always change! For example, if you put heat into a block of ice at $0°C$ it may just melt to a pool of water *still* at $0°C$. Thus heat can cause a *change of phase*. Putting heat into water at $100°C$ may just vaporize the water to steam at $100°C$. The *heat of transformation* L is defined via

$$\boxed{Q \equiv Lm}$$

where Q is the heat and m is the mass. If melting is involved L is called a heat of fusion L_f or for vaporizing L is called a heat of vaporization L_v.

Exercise The latent heat of fusion for water is $L_f = 333\,kJ/kg$ and the latent heat of vaporization is $L_v = 2256\,kJ/kg$. Does it take more heat to melt ice or vaporize water (of the same mass)?

Example How much heat is required to melt 2 kg of ice at $0°C$ to water at $0°C$?

Solution The latent heat of fusion is L_f = 79.5 cal g^{-1} giving

$$\begin{aligned} Q &= Lm \\ &= 79.5 \text{ cal g}^{-1} \times 2000 \text{ g} \\ &= 159{,}000 \text{ cal} \\ &= 159 \text{ kcal} \end{aligned}$$

Example Sample Problem 19-6 (Halliday). (done in class)

Example Sample Problem 19-7 (Halliday). (done in class)

15.8 A Closer Look at Heat and Work

When discussing work and energy for thermodynamic systems it is useful to think about compressing the gas in a piston, as shown in Fig. 19.1.

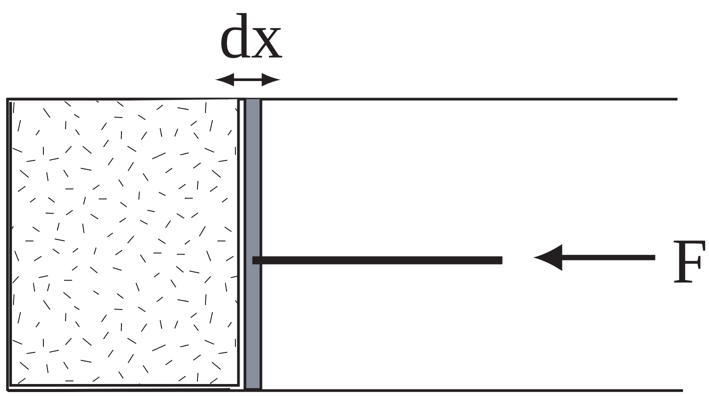


FIGURE 19.1 Piston.

By pushing on the piston the gas is compressed, or if the gas is heated the piston expands. Such pistons are crucial to the operation of automobile engines. The gas consists of a mixture of gasoline which is compressed by the piston. Sitting inside the chamber is a spark plug which ignites the gas and pushes the piston out. The piston is connected to a crankshaft connecting the auto engine to the wheels of the automobile.

Another such piston system is the simple bicycle pump. Recall our definition of Work as

$$W \equiv \int \vec{F} \cdot d\vec{s}$$

For the piston, all the motion occurs in 1-dimension so that

$$W = \int F\, dx$$

(or equivalently $\vec{F} \cdot d\vec{s} = F\, dx \cos 0° = F\, dx$). The *pressure* of a gas is defined as *force* divided by area (of the piston compressing the gas) or

$$p \equiv \frac{F}{A}$$

giving $dW = pAdx = pdV$ where the volume is just area times distance or $dV = Adx$. That is when we compress the piston by a distance dx, the volume of the gas changes by $dV = Adx$ where A is the cross-sectional area of the piston. Writing $W = \int dW$ gives

$$\boxed{W = \int_{V_i}^{V_f} p\,dV}$$

which is the work done *by* a gas of pressure p changing its volume from V_i to V_f (or the work done *on* the gas).

15.9 The First Law of Thermodynamics

We have already studied this! The first law of thermodynamics is nothing more than a re-statement of the work energy theorem, which was

$$\Delta U + \Delta K = W_{NC}$$

Recall that the total work W was *always* $W = \Delta K$. Identify heat Q as $Q \equiv W_{NC}$ and internal energy (such as energy *stored* in a gas, which is just potential energy) is $E_{\rm int} \equiv U$ and we have

$$\Delta E_{\rm int} + W = Q$$

or

$$\boxed{\Delta E_{\rm int} = Q - W}$$

which is the first law of thermodynamics. The *meaning* of this law is that the internal energy of a system can be changed by adding heat or doing work. Often the first law is written for *tiny* changes as

$$dE_{\rm int} = dQ - dW$$

15.10 Special Cases of 1st Law of Thermodynamics

1. Adiabatic Processes

Adiabatic processes are those that occur so rapidly that there is no transfer of heat between the system and its environment. Thus $Q = 0$ and

$$\Delta E_{\text{int}} = -W$$

For example if we push in the piston very quickly then our work will increase the internal energy of the gas. It will store potential energy ($\Delta U = \Delta E_{\text{int}}$) like a spring and make the piston bounce back when we let it go.

2. Constant-volume Processes

If we glue the piston so that it won't move then obviously the volume is constant, and $W = \int pdV = 0$, because the piston can't *move*. Thus

$$\Delta E_{\text{int}} = Q$$

which means the only way to increase the internal energy of the gas is by adding heat Q.

3. Cyclical Processes

Recall the motion of a *spring*. It is a cyclical process in which the spring oscillates back and forth. After *one complete cycle* the potential energy U of the spring has not changed, thus $\Delta U = 0$. Similarly we can push in the piston, then let it go and it will push back to where it started, similar to the spring. Thus $\Delta E_{\text{int}} = 0$ and

$$Q = W$$

meaning that work done equals heat gained.

4. Free Expansion

Another way to get $\Delta E_{\text{int}} = 0$ is for

$$Q = W = 0$$

Free expansion is illustrated in Fig. 19-15 [Halliday].

Example Sample Problem 19-8 (Halliday). (done in class)

15.11 Heat Transfer Mechanisms

There are three basic processes by which heat is always *transferred* from one body to another. These are

1) Convection

2) Conduction

3) Radiation

Students should *carefully* read Section 19.11 of Halliday.

15.12 Problems

1. The coldest that any object can ever get is 0 K (or -273 C). It is rare for physical quantities to have an upper or lower possible limit. Explain why temperature has this lower limit.

2. Suppose it takes an amount of heat Q to make a cup of coffee. If you make 3 cups of coffee how much heat is required?

3. How much heat is required to make a cup of coffee? Assume the mass of water is 0.1 kg and the water is initially at 0°C. We want the water to reach boiling point.
   Give your answer in Joule and calorie and Calorie.

   (1 cal = 4.186 J; 1 Calorie = 1000 calorie.

   For water: $c = 1\frac{\text{cal}}{\text{gC}} = 4186\frac{\text{J}}{\text{kg C}}$; $L_v = 2.26 \times 10^6\frac{\text{J}}{\text{kg}}$; $L_f = 3.33 \times 10^5\frac{\text{J}}{\text{kg}}$)

4. How much heat is required to change a 1 kg block of ice at -10°C to steam at 110°C ?
   Give your answer in Joule and calorie and Calorie.

   (1 cal = 4.186 J; 1 Calorie = 1000 calorie.
   $\text{c}_{\text{water}} = 4186\frac{\text{J}}{\text{kg C}}$; $\text{c}_{\text{ice}} = 2090\frac{\text{J}}{\text{kg C}}$; $\text{c}_{\text{steam}} = 2010\frac{\text{J}}{\text{kg C}}$
   For water, $L_v = 2.26 \times 10^6\frac{\text{J}}{\text{kg}}$; $L_f = 3.33 \times 10^5\frac{\text{J}}{\text{kg}}$)

Chapter 16

KINETIC THEORY OF GASES

SUGGESTED HOME EXPERIMENT:
Put a block of ice into an insulated container of water and measure the temperature change. Is it what you expect ?

THEMES:
Behavior of a Gas.

16.1 A New Way to Look at Gases

The subject of classical thermodynamics, studied in the last chapter, was developed in the 18th and 19th centuries *before* we knew about molecules and atoms. The *kinetic theory* of gases attempts to explain all of the concepts of classical thermodynamics, such as temperature and pressure, in terms of an underlying *microscopic theory* based on atoms and molecules. For example, we shall see that the *temperature* of a gas is related to the *average kinetic energy* of all molecules in the gas.

16.2 Avagadro's Number

One **mole** *is the number of atoms in a 12 gram sample of* ^{12}C, and this number is determine from experiment to be 6.02×10^{23}. This is often called *Avagadro's number*. The number of molecules must be the number of moles times the number of molecules *per* mole. Thus we write Avagadro's number as

$$N_A = 6.02 \times 10^{23} \text{ mole}^{-1}$$

and

$$N = nN_A$$

where N is the number of molecules and n is the number of moles.

16.3 Ideal Gases

One of the most fundamental properties of any macroscopic system is the so-called *equation of state*. This is the equation that specifies the exact relation between pressure p, volume V, and temperature T for a substance. The equation of state for a gas is very different to the equation of state of a liquid. Actually there is a giant accelerator, called the *Relativistic Heavy Ion Collider* (RHIC) currently under construction at Brookhaven National Laboratory on Long Island. This accelerator will collide heavy nuclei into each other at extremely high energies. One of the main aims is to determine the nuclear matter equation of state at very high temperatures and densities, simulating the early universe.

Now it turns out that *most gases* obey a simple equation of state called the *ideal gas law*

$$\boxed{pV = nRT}$$

where p is the pressure, V is the volume, T is the temperature (in $^\circ K$), n is the number of moles of the gas and R is the so called gas constant with the value

$$R = 8.31 \text{ J mol}^{-1} \text{ K}^{-1}$$

Recall that the number of molecules is given by $N = nN_A$ where n is the number of moles. Thus $pV = nRT = \frac{N}{N_A}RT$ and define Boltzmann's constant

$$\begin{aligned} k &\equiv \frac{R}{N_A} = \frac{8.31\text{J mole}^{-1}\text{K}^{-1}}{6.02 \times 10^{23} \text{ mole}^{-1}} \\ &= 1.38 \times 10^{-23} \text{ JK}^{-1} \\ &= 8.62 \times 10^{-5} \text{ eV K}^{-1} \end{aligned}$$

where an electron volt is defined as

$$\text{eV} \equiv 1.6 \times 10^{-19} \text{ J}$$

Thus the ideal gas law is also often written as

$$pV = NkT$$

where N is the total number of molecules.

The ideal gas law embodies exactly the properties we expect of a gas:

1) If the volume V is held constant, then the pressure p increases as temperature T increases.

2) If the pressure p is held constant, then as T increases, p increases.

3) If the temperature T is held constant, then as p increases, V decreases.

LECTURE DEMONSTRATIONS: Show this

Work Done by an Ideal Gas

The equation of state can be represented on a graph of pressure vs. volume, often called a pV diagram. Remember an equation of state is an equation relating the three variables p, V, T. A pV diagram takes care of two variables. The third variable T represents different lines on the pV diagram. These difference lines are called *isotherms* (meaning same temperature). An example is given in Fig.20-1 [Halliday]. For fixed T (say 310 K) the pressure is inversely proportional to volume as specified in the ideal gas law. Fig. 20-1 [Halliday] would look *different* for an equation of state different from the ideal gas law.

Example What is the work done by *any* gas (ideal or not) at constant volume (isometric) ?

Solution If $V_i = V_f$ then

$$W = \int_{V_i}^{V_f} pdV = 0$$

which is obvious when we think of the piston in the previous chapter. If the volume does not change then the piston doesn't move and the work is zero.

Example Derive a formula for the work done by *any* gas (ideal or not) which expands isobarically (i.e. at constant pressure).

Solution If p is a constant it can be taken outside the integral, giving

$$\begin{aligned} W &= \int_{V_i}^{V_f} pdV \\ &= p\int_{V_i}^{V_f} dV \\ &= p\,[V]_{V_i}^{V_f} \\ &= p(V_f - V_i) \\ &= p\Delta V \end{aligned}$$

Example Derive a formula for the work done by a gas when it expands isothermally (i.e. at constant temperature).

Solution The work done by an expanding gas is given by

$$W = \int_{V_i}^{V_f} p dV$$

But this time the pressure *changes*. For an ideal gas we have $p = \frac{nRT}{V}$ giving

$$\begin{aligned} W &= nRT \int_{V_i}^{V_f} \frac{1}{V} dV \\ &= nRT \left[\ln V\right]_{V_i}^{V_f} \\ &= nRT(\ln V_f - \ln V_i) \\ &= nRT \ln\left(\frac{V_f}{V_i}\right) \end{aligned}$$

Carefully study Sample Problems 20-1, 20-2 in Halliday.

16.4 Pressure, Temperature and RMS Speed

Carefully study Section 20.4 in Halliday.

Now consider our first kinetic theory problem. Imagine a gas, consisting of n moles being confined to a cubical box of volume V. "*What is the connection between the pressure p exerted by the gas on the walls and the speeds of the molecules*?" (Halliday Pg. 487) Pressure is *defined* as Force divided by Area or $p \equiv \frac{F}{A}$ where $F = \frac{dp}{dt}$. Using Newtonian Mechanics, Halliday (Pg. 488) shows that

$$\boxed{p = \frac{nMv_{RMS}^2}{3V}}$$

where n is the number of moles, M is the mass of 1 mole of the gas (so that nM is the total mass of the gas), v_{RMS} is the average speed of the molecules and V is the volume of the gas. The above equation is derived purely from applying Newtonian mechanics to the individual molecules. All students should study the derivation in Halliday (Pg. 488) carefully.

Now by comparing to the ideal gas law $pV = nRT$ or $p = \frac{nRT}{V}$ we *must* have $\frac{nMv_{RMS}^2}{3} = nRT$ or

$$\boxed{v_{RMS} = \sqrt{\frac{3RT}{M}}}$$

which shows that the *temperature T is related to the speed of molecules*!

As shown in Table 20-1 (Halliday) the speed of molecules at room temperature is very large; about 500 m/sec for air (about 1000 mph).

16.5 Translational Kinetic Energy

For a single molecule its average kinetic energy is

$$\bar{K} = \frac{1}{2}mv_{RMS}^2$$

and using $v_{RMS} = \sqrt{\frac{3RT}{M}}$ gives $\bar{K} = \frac{1}{2}m\frac{3RT}{M}$.

Remember that M is the molar mass, which is the mass of 1 mole of gas and m is the mass of the molecule. Thus $\frac{M}{m} = 1$ mole $= 6.02 \times 10^{23} = N_A$, Avagadro's number. Thus $\bar{K} = \frac{3RT}{2N_A}$ or

$$\boxed{\bar{K} = \frac{3}{2}kT}$$

This is a very interesting result. For a given temperature T, all gas molecules, no matter what their mass, have the same average translational kinetic energy.

Example In the center of the Sun the particles are bare hydrogen nuclei (protons). Calculate their average kinetic energy.

Solution The center of the Sun is at a temperature of about 20,000,000°K. Thus

$$\begin{aligned}\bar{K} &= \frac{3}{2}kT \\ &= \frac{3}{2} \times 8.62 \times 10^{-8}\frac{\text{eV}}{K} \times 20 \times 10^6 K \\ &= 2586 \text{ eV} \\ &\approx 3 \text{ MeV}\end{aligned}$$

16.6 Mean Free Path

Even though room temperature air molecules have a large RMS speed $v_{RMS} \approx 500$ m/sec, that does not mean that they move across a room in a fraction of a second. If you open a bottle of perfume at one end of a room, it takes a while for you to notice the smell at the other end of the room. This is because the molecules undergo an enormous number of collisions on their way across the room, as shown very nicely in Fig. 20-4 (Halliday).

The *mean free path* λ is the average distance that a molecule travels in between collisions. It is given by

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 N/V}$$

where d is the average diameter of a molecule, and N/V is the average number of molecules per unit volume. This formula is discussed on Pages 490-491 (Halliday).

16.7 Distribution of Molecular Speeds

Not all molecules travel at the speed v_{RMS}. this is just the *average* molecular speed. We would like to know how many molecules travel above or below this speed. This was worked out by Maxwell. The *probbility* of a given speed is

$$P(v) = 4\pi \left(\frac{M}{2\pi RT}\right)^{3/2} v^2 \, e^{-\frac{Mv^2}{2RT}}$$

where M is the molar mass of the gas. This probability distribution is plotted in Fig. 20-7 (Halliday).

16.8 Problems

1.

A) If the number of molecules in an ideal gas is doubled, by how much does the pressure change if the volume and temperature are held constant?

B) If the volume of an ideal gas is halved, by how much does the pressure change if the temperature and number of molecules is constant?

C) If the temperature of an ideal gas changes from 200 K to 400 K, by how much does the volume change if the pressure and number of molecules is constant.

D) Repeat part C) if the temperature changes from 200 C to 400 C.

2. If the number of molecules in an ideal gas is doubled and the volume is doubled, by how much does the pressure change if the temperature is held constant ?

3. If the number of molecules in an ideal gas is doubled, and the absolute temperature is doubled and the pressure is halved, by how much does the volume change ?
(Absolute temperature is simply the temperature measured in Kelvin.)

Chapter 17

Review of Calculus

17.1 Derivative Equals Slope

17.1.1 Slope of a Straight Line

All students will be familiar with the equation for a straight line

$$y(x) = mx + c \tag{17.1}$$

where c is the intercept on the y axis and m is the slope of the line. To *prove* to ourselves that m really is the slope, we need a good definition of slope. Let's define

$$Slope \equiv \frac{\Delta y}{\Delta x} \equiv \frac{y_f - y_i}{x_f - x_i} \tag{17.2}$$

where Δy is the difference between final and initial values y_f and y_i. In Fig. 22.1 the graph of $y(x) = 2x+1$ is plotted and the slope has been determined by measuring Δy and Δx.

Rather than always having to verify the slope graphically, let's do it analytically for *all* lines. Take $x_i = x$ as the initial x value and $x_f = x + \Delta x$ as the final value. Obviously $x_f - x_i = \Delta x$. The initial value of y is

$$\begin{aligned} y_i \equiv y(x_i) &= mx_i + c \\ &= mx + c \end{aligned} \tag{17.3}$$

and the final value is

$$\begin{aligned} y_f &\equiv y(x_f) = mx_f + c \\ &= y(x + \Delta x) = m(x + \Delta x) + c \end{aligned} \tag{17.4}$$

Thus $\Delta y = y_f - y_i = m(x + \Delta x) + c - mx - c = m\Delta x$. Therefore the slope becomes

$$\frac{\Delta y}{\Delta x} = \frac{m\Delta x}{\Delta x} = m \tag{17.5}$$

which is a *proof* that $y = mx + c$ has a slope of m.

From above we can re-write our formula (17.2) using $y_f = y(x + \Delta x)$ and $y_i = y(x)$, so that

$$Slope \equiv \frac{\Delta y}{\Delta x} = \frac{y_f - y_i}{x_f - x_i} = \frac{y(x + \Delta x) - y(x)}{\Delta x} \tag{17.6}$$

17.1.2 Slope of a Curve

A straight line always has constant slope m. That's why it's called *straight*. The parabola $y(x) = x^2 + 1$ is plotted in Fig. 22.2 and obviously the slope changes. In fact the concept of the *slope of a parabola doesn't make any sense* because the parabola continuously *curves*. However we might think about little *pieces* of the parabola. If you look at any *tiny* little piece it looks straight. These tiny little pieces are all tiny little line segments, each with their own slope. Notice that the slope of the tiny little line segments keeps *changing*. At $x = 0$ the slope is 0 (the tiny little line is flat) whereas around $x = 1$ the slope is larger.

One of the most important ideas in calculus is the concept of the *derivative*, which is nothing more than

$$Derivative = Slope\ of\ tiny\ little\ line\ segment.$$

In Fig. 22.1 we got the slope from Δy and Δx on the large triangle in the top right hand corner. But we would get the same answer if we had used the tiny triangles in the bottom left hand corner. What characterizes these tiny triangles is that Δx and Δy are both tiny (but their ratio, $\frac{\Delta y}{\Delta x} = 2$ always). Another way of saying that Δx is tiny is to say

$$Tiny = \lim_{\Delta x \to 0}$$

That is the *limit* as Δx goes to zero is another way of saying Δx is tiny.

Examples

1) $\lim_{\Delta x \to 0} [\Delta x + 3] = 3$

2) $\lim_{\Delta x \to 0} \Delta x = 0$

3) $\lim_{\Delta x \to 0}[(\Delta x)^2 + 4] = 4$

4) $\lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 4\Delta x}{\Delta x} = \lim_{\Delta x \to 0}(\Delta x + 4) = 4$

5) $\lim_{\Delta x \to 0} 3 = 3$

For a *curve* like the parabola we can't draw a big triangle, as in Fig. 22.1, because the *hypotenuse would be curved*. But we can get the *slope at a point* by drawing a tiny triangle at that point. Thus let's define the

$$\begin{matrix}\text{Slope of} \\ \text{curve at} \\ \text{a point}\end{matrix} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \begin{matrix}\text{Slope of tiny} \\ \text{little line} \\ \text{segment}\end{matrix} \equiv \text{Derivative}$$

So it's the same definition as before in (17.6) except $\lim_{\Delta x \to 0}$ is an *instruction* to use a tiny triangle. Now $\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x)-y(x)}{\Delta x}$ from (17.6) and the derivative is given a fancy new symbol $\frac{dy}{dx}$ so that

$$\boxed{\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}} \tag{17.7}$$

The symbol dy simply means

$$dy \equiv \textit{tiny } \Delta y$$

That is, usually Δy can be big or small. If we are talking about a *tiny* Δy we write dy instead. Similarly for Δx.

Example Calculate the derivative of the straight line $y(x) = 3x$

Solution $y(x) = 3x$

$y(x + \Delta x) = 3(x + \Delta x)$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{3(x + \Delta x) - 3x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3x + 3\Delta x - 3x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 3 = 3 \end{aligned}$$

Thus the derivative is the slope.

Example Calculate the derivative of the straight line $y(x) = 4$
Solution $y(x) = 4$
$y(x + \Delta x) = 4$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{4 - 4}{\Delta x} = 0$$

The line $y(x) = 4$ has 0 slope and therefore 0 derivative.

(do Problem 1)

The derivative was defined to give us the slope of a *curve* at a point. The two examples above show that it also works for a straight line (A straight line is a special case of a curve). Now do some examples for *real* curves.

Example Calculate the derivative of the parabola $y(x) = x^2$
Solution $y(x) = x^2$

$$\begin{aligned} y(x + \Delta x) &= (x + \Delta x)^2 \\ &= x^2 + 2x\Delta x + (\Delta x)^2 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= 2x \end{aligned}$$

Example Calculate the slope of the parabola $y(x) = x^2$ at the points $x = -2$, $x = 0$, $x = 3$.
Solution We already have $\frac{dy}{dx} = 2x$. Thus

$$\left.\frac{dy}{dx}\right|_{x=-2} = -4$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0$$

$$\left.\frac{dy}{dx}\right|_{x=3} = 6$$

which shows how the slope of a tiny little line segment varies as we move along the parabola.

Example Calculate the slope of the curve $y(x) = x^2 + 1$ (see Fig. 22.2) at the points $x = -2$, $x = 0$, $x = 3$

Solution $y(x) = x^2 + 1$

$$\begin{aligned} y(x + \Delta x) &= (x + \Delta x)^2 + 1 \\ &= x^2 + 2x\Delta x + (\Delta x)^2 + 1 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - (x^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= 2x \end{aligned}$$

Thus the slopes are the same as in the previous example.

(do Problem 2)

17.1.3 Some Common Derivatives

In a previous example we saw that the derivative of $y(x) = 4$ was $\frac{dy}{dx} = 0$, which make sense because a graph of $y(x) = 4$ reveals that the slope is always 0. This is true for any constant c. Thus

$$\boxed{\frac{dc}{dx} = 0} \tag{17.8}$$

We also saw in a previous example that $\frac{d}{dx}x^2 = 2x$. In general we have

$$\boxed{\frac{dx^n}{dx} = nx^{n-1}} \tag{17.9}$$

This is a very important result. We have already verified it for $n = 2$. Let's verify it for $n = 3$.

Example Check that (17.9) is correct for $n = 3$.

Solution Formula (17.9) gives

$$\frac{dx^3}{dx} = 3x^{3-1} = 3x^2$$

We wish to verify this. Take $y(x) = x^3$.

$$\begin{aligned} y(x + \Delta x) &= (x + \Delta x)^3 \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 \\ &= 3x^2 \text{ in agreement with our result above.} \end{aligned}$$

(do Problem 3)

A list of very useful results for derivatives is given below. You will prove most of these results in your calculus course. I will just make some comments about them.

Table A-4 Properties of Derivatives and Derivatives of Particular Functions [Tipler, pg. AP-16, 1991].

Multiplicative constant rule

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1. The derivative of a constant times a function equals the constant times the derivative of the function:

$$\frac{d}{dx}[Cy(x)] = C\frac{dy(x)}{dx}$$

Addition rule

---

2. The derivative of a sum of functions equals the sum of the derivatives of the functions:

$$\frac{d}{dx}[y(x) + z(x)] = \frac{dy(x)}{dx} + \frac{dz(x)}{dx}$$

Chain rule

---

3. If y is a function of x and x is in turn a function of t, the derivative of y with respect to t equals the product of the derivative of y with respect to z and the derivative of z with respect to x:

$$\frac{d}{dx}y(x) = \frac{dy}{dz}\frac{dz}{dx}$$

Derivative of a product

---

4. The derivative of a product of functions $y(x)z(x)$ equals the first function times the derivative of the second plus the second function times the derivative of the first:

$$\frac{d}{dx}[y(x)z(x)] = y(x)\frac{dz(x)}{dx} + \frac{dy(x)}{dx}z(x)$$

Reciprocal derivative

5. The derivative of y with respect to x is the reciprocal of the derivative of x with respect to y, assuming that neither derivative is zero:

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} \quad \text{if} \quad \frac{dx}{dy} \neq 0$$

Derivatives of particular functions

6. $\frac{dC}{dx} = 0$ where C is a constant	10. $\frac{d}{dx} \tan \omega x = \omega \sec^2 \omega x$
7. $\frac{d(x^n)}{dx} = nx^{n-1}$	11. $\frac{d}{dx} e^{bx} = be^{bx}$
8. $\frac{d}{dx} \sin \omega x = \omega \cos \omega x$	12. $\frac{d}{dx} \ln bx = \frac{1}{x}$
9. $\frac{d}{dx} \cos \omega x = -\omega \sin \omega x$	

Multiplicative constant rule Example $\quad \frac{d}{dx}[Cy(x)] = C\frac{dy(x)}{dx}$.

This just means, for example, that

$$\frac{d}{dx}(3x^2) = 3\frac{dx^2}{dx} = 3 \times 2x = 6x$$

(do Problem 4).

Addition rule Example $\quad \frac{d}{dx}[y(x) + z(x)] = \frac{dy(x)}{dx} + \frac{dz(x)}{dx}$

Take for example $y(x) = x$ and $z(x) = x^2$. This rule just means

$$\frac{d}{dx}(x + x^2) = \frac{dx}{dx} + \frac{dx^2}{dx} = 1 + 2x$$

(do Problem 5)

Chain Rule $\quad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$

(A rough "proof" of this is to just note that the dz cancels in the numerator and denominator.) The use of the chain rule is best seen in the following example, where y is *not* given as a function of x.

Example Verify the chain rule for $y = z^3$ and $z = x^2$.

Solution We have $y(z) = z^3$ and $z(x) = x^2$. Thus $y(x) = x^6$.

$$\begin{aligned} \therefore \frac{dy}{dx} &= 6x^5 \\ \frac{dy}{dz} &= 3z^2 \\ \frac{dz}{dx} &= 2x \end{aligned}$$

Now $\frac{dy}{dz}\frac{dz}{dx} = (3z^2)(2x) = (3x^4)(2x) = 6x^5$. Thus we see that $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$.

Product Rule $\quad \frac{d}{dx}[y(x)z(x)] = y(x)\frac{dz(x)}{dx} + \frac{dy(x)}{dx}z(x)$

The use of this arises when multiplying two functions together as illustrated in the next example.

Example If $y(x) = x^3$ and $z(x) = x^2$, verify the product rule.

Solution $\quad y(x)z(x) = x^5$

$$\Rightarrow \frac{d}{dx}[y(x)z(x)] = \frac{dx^5}{dx} = 5x^4$$

Now let's show that the product rule gives the same answer.

$$y(x)\frac{dz(x)}{dx} = x^3\frac{dx^2}{dx} = x^3 2x = 2x^4$$

$$\frac{dy(x)}{dx}z(x) = \frac{dx^3}{dx}x^2 = 3x^2x^2 = 3x^4$$
$$y(x)\frac{dz(x)}{dx} + \frac{dy(x)}{dx}z(x) = 2x^4 + 3x^4 = 5x^4$$

in agreement with our answer above.

(do Problem 6)

17.1.4 Extremum Value of a Function

A final important use of the derivative is that it can be used to tell us when a function attains a maximum or minimum value. This occurs when the derivative or slope of the function is zero.

Example What are the (x, y) coordinates of the place where the parabola $y(x) = x^2 + 3$ has its minimum value?

Solution The minimum value occurs where the slope is 0. Thus

$$0 = \frac{dy}{dx} = \frac{d}{dx}(x^2 + 3) = 2x$$

$$\therefore\ x = 0$$
$$y = x^2 + 3 \qquad \therefore\ y = 3$$

Thus the minimum is at $(x, y) = (0, 3)$. You can verify this by plotting a graph.

(do Problem 7)

17.2 Integral

17.2.1 Integral Equals Antiderivative

The derivative of $y(x) = 3x$ is $\frac{dy}{dx} = 3$. The derivative of $y(x) = x^2$ is $\frac{dy}{dx} = 2x$. The derivative of $y(x) = 5x^3$ is $\frac{dy}{dx} = 15x^2$.

Let's play a game. I tell you the answer and you tell me the question. Or I tell you the derivative $\frac{dy}{dx}$ and you tell me the original function $y(x)$ that it came from. Ready?

$$\begin{aligned} &\text{If} \quad \frac{dy}{dx} = 3 && \text{then} \quad y(x) = 3x \\ &\text{If} \quad \frac{dy}{dx} = 2x && \text{then} \quad y(x) = x^2 \\ &\text{If} \quad \frac{dy}{dx} = 15x^2 && \text{then} \quad y(x) = 5x^3 \end{aligned}$$

We can generalize this to a rule.

$$\text{If} \quad \frac{dy}{dx} = x^n \quad \text{then} \quad y(x) = \frac{1}{n+1}x^{n+1}$$

Actually I have cheated. Let's look at the following functions

$$\begin{aligned} y(x) &= 3x + 2 \\ y(x) &= 3x + 7 \\ y(x) &= 3x + 12 \\ y(x) &= 3x + C \quad (C \text{ is an arbitrary constant}) \\ y(x) &= 3x \end{aligned}$$

All of them have the *same* derivative $\frac{dy}{dx} = 3$. Thus in our little game of re-constructing the original function $y(x)$ from the derivative $\frac{dy}{dx}$ there is always an ambiguity in that $y(x)$ could always have some *constant* added to it.

Thus the *correct* answers in our game are

$$\text{If} \quad \frac{dy}{dx} = 3 \quad \text{then} \quad y(x) = 3x + \text{ constant}$$

(Actually instead of always writing `constant`, let me just write C)

If $\frac{dy}{dx} = 2x$ then $y(x) = x^2 + C$

If $\frac{dy}{dx} = 15x^2$ then $y(x) = 5x^3 + C$

If $\frac{dy}{dx} = x^n$ then $y(x) = \frac{1}{n+1}x^{n+1} + C$.

This original function $y(x)$ that we are trying to get is given a special name called the *antiderivative* or *integral*, but it's nothing more than the original function.

17.2.2 Integral Equals Area Under Curve

Let's see how to extract the integral from our original definition of derivative.

The slope of a curve is $\frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$ when the Δ increments are tiny. Notice that $y(x)$ is a function of x but so also is $\frac{dy}{dx}$. Let's call it

$$f(x) \equiv \frac{dy}{dx} = \frac{\Delta y}{\Delta x} \tag{17.10}$$

Thus if $f(x) = \frac{dy}{dx} = 2x$ then $y(x) = x^2 + C$, and similarly for the other examples.

In equation (17.10) I have written $\frac{\Delta y}{\Delta x}$ also because $\frac{dy}{dx}$ is just a tiny version of $\frac{\Delta y}{\Delta x}$.

Obviously then

$$\Delta y = f\,\Delta x \tag{17.11}$$

or

$$dy = f\,dx \tag{17.12}$$

What happens if I add up *many* Δy's. For instance suppose you are aged 18. Then if I add up many age increments in your life, such as

$$\begin{aligned} \text{Age} &= \Delta\text{Age}_1 + \Delta\text{Age}_2 + \Delta\text{Age}_3 + \Delta\text{Age}_4 \cdots \\ &1 \text{ year} + 3 \text{ years} + 0.5 \text{ year} + 5 \text{ years} + 0.5 \text{ year} + 5 \text{ years} + 3 \text{ years} \\ &= 18 \text{ years} \end{aligned}$$

I get your complete age. Thus if I add up all possible increments of Δy I get back y. That is

$$y = \Delta y_1 + \Delta y_2 + \Delta y_3 + \Delta y_4 + \cdots$$

or symbolically

$$y = \sum_i \Delta y_i \tag{17.13}$$

where

$$\Delta y_i = f_i \, \Delta x_i \tag{17.14}$$

Now looking at Fig. 22.3 we can see that the area of the shaded section is just $f_i \Delta x_i$. Thus Δy_i *is an area* of a little shaded region. Add them all up and we have the total area under the curve. Thus

$$\begin{array}{l}\text{Area under}\\ \text{curve } f(x)\end{array} = \sum_i f_i \Delta x_i = \sum_i \frac{\Delta y_i}{\Delta x_i} \Delta x_i = \sum_i \Delta y_i = y \tag{17.15}$$

Let's now make the little intervals Δy_i and Δx_i very tiny. Call them dy and dx. If I am using *tiny* intervals in my sum $\sum$ I am going to use a new symbol $\int$. Thus

$$\text{Area} = \int f dx = \int \frac{dy}{dx} dx = \int dy = y \tag{17.16}$$

which is just the tiny version of (17.15). Notice that the dx "cancels".

In formula (17.16) recall the following. The derivative is $f(x) \equiv \frac{dy}{dx}$ and y is my original function which we called the integral or antiderivative. *We now see that the integral or antiderivative or original function can be interpreted as the area under the derivative curve* $f(x) \equiv \frac{dy}{dx}$.

By the way $\int f \, dx$ reads "integral of f with respect to x."

$$\text{Summary: if } f = \frac{dy}{dx} \Rightarrow y = \int f \, dx$$

Summary of 1.2.1 and 1.2.2

$$\begin{array}{ll} y(x) = x^2 & \dfrac{dy}{dx} = 2x \equiv f(x) \\ y(x) = x^2 + 4 & \dfrac{dy}{dx} = 2x \equiv f(x) \end{array}$$

$$\Rightarrow \text{ if } f(x) \equiv \frac{dy}{dx} = 2x \Rightarrow y(x) = x^2 + c$$

$$\begin{aligned} f(x) &= \frac{dy}{dx} = \frac{\Delta y}{\Delta x} \\ \Delta y &= f \Delta x \qquad dy = f \, dx \\ y &= \sum_i \Delta y_i = \int dy \end{aligned}$$

$$
\begin{aligned}
&= \sum_i f_i \Delta x_i = \int f\,dx \\
&= \text{Area under curve } f(x) \\
&= \text{Antiderivative}
\end{aligned}
$$

$$\boxed{y = \int f\,dx}$$

E.g.

$$\int 2x\,dx = x^2 + c$$

do a few more examples.

Example What is $\int x\,dx$?

Solution The derivative function is $f(x) = \frac{dy}{dx} = x$. Therefore the original function must be $\frac{1}{2}x^2 + c$. Thus

$$\int x\,dx = \frac{1}{2}x^2 + c$$

(do Problem 8)

17.2.3 Definite and Indefinite Integrals

The integral $\int x\,dx$ is supposed to give us the area under the curve x, but our answer in the above example ($\frac{1}{2}x^2 + c$) doesn't look much like an area. We would expect the area to be a number.

Example What is the area under the curve $f(x) = 4$ between $x_1 = 1$ and $x_2 = 6$?

Solution This is easy because $f(x) = 4$ is just a horizontal straight line as shown in Fig. 22.4. The area is obviously $4 \times 5 = 20$.

Consider $\int 4dx = 4x + c$. This is called an *indefinite integral* or *antiderivative*. The integral which gives us the *area* is actually the *definite* *integral* written

$$\int_{x_1}^{x_2} 4dx \equiv [4x+c]_{x_1}^{x_2} \equiv (4x_2+c)-(4x_1+c)$$
$$= [4x]_{x_1}^{x_2} = 4x_2 - 4x_1 \qquad (17.17)$$

Let's explain this. The *formula* $4x+c$ by itself does not give the area directly. For an area we must always specify x_1 and x_2 (see Fig. 22.4) so that we know what area we are talking about. In the previous example we got $4\times 5 = 20$ from $4x_2 - 4x_1 = (4\times 6)-(4\times 1) = 24-4 = 20$, which is the same as (17.17). Thus (17.17) *must* be the correct formula for area. Notice here that it doesn't matter whether we include the c because it cancels out.

Thus $\int 4dx = 4x + c$ is the antiderivative or indefinite integral and it gives a *general formula* for the area but not the *value* of the area itself. To evaluate the *value* of the area we need to specify the *edges* x_1 and x_2 of the area under consideration as we did in (17.17). Using (17.17) to work out the previous example we would write

$$\int_1^6 4dx = [4x+c]_1^6 = [(4\times 6)+c]-[(4\times 1)+c]$$
$$= 24+c-4-c$$
$$= 24-4 = 20 \qquad (17.18)$$

Example Evaluate the area under the curve $f(x) = 3x^2$ between $x_1 = 3$ and $x_2 = 5$.

Solution

$$\int_3^5 3x^2dx = [x^3+c]_3^5$$
$$= (125+c)-(27+c) = 98$$

(do Problem 9)

Figure 22.1 Plot of the graph $y(x) = 2x + 1$. The slope $\frac{\Delta y}{\Delta x} = 2$.

Figure 22.2 Plot of $y(x) = x^2 + 1$. Some *tiny* little pieces are indicated, which look straight.

Figure 22.3 A general function $f(x)$. The area under the shaded rectangle is approximately $f_i \Delta x_i$. The total area under the curve is therefore $\sum_i f_i \Delta x_i$. If the Δx_i are tiny then write $\Delta x_i = dx$ and write $\sum_i = \int$. The area is then $\int f(x) dx$.

Figure 22.4 Plot of $f(x) = 4$. The area under the curve between $x_1 = 1$ and $x_2 = 6$ is obviously $4 \times 5 = 20$.

17.3 Problems

1. Calculate the derivative of $y(x) = 5x + 2$.

2. Calculate the slope of the curve $y(x) = 3x^2 + 1$ at the points $x = -1$, $x = 0$ and $x = 2$.

3. Calculate the derivative of x^4 using the formula $\frac{dx^n}{dx} = nx^{n-1}$. Verify your answer by calculating the derivative from $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x)-y(x)}{\Delta x}$.

4. Prove that $\frac{d}{dx}(3x^2) = 3\frac{dx^2}{dx}$.

5. Prove that $\frac{d}{dx}(x + x^2) = \frac{dx}{dx} + \frac{dx^2}{dx}$.

6. Verify the chain rule and product rule using some examples of your own.

7. Where do the extremum values of $y(x) = x^2 - 4$ occur? Verify your answer by plotting a graph.

8. Evaluate $\int x^2 dx$ and $\int 3x^3 dx$.

9. What is the area under the curve $f(x) = x$ between $x_1 = 0$ and $x_2 = 3$? Work out your answer i) graphically and ii) with the integral.

Bibliography

[1] D. Halliday, R. Resnick and J. Walker, *Fundamentals of Physics* (Wiley, New York, 1997).