left action
G-space
G-map
equivariant
G-equivalence
G-isomorphism
weakly G-equivariant
weak G-equivalence
weak G-isomorphism
right action
TOP(X)
action!smooth

CHAPTER 1

Transformation Groups

Fri Jun 16 11:30:38 2000

1.1. Introduction

1.1.1. Our spaces X are path-connected, completely regular and Hausdorff so that the various slice theorems are valid. For covering space theory, we need and tacitly assume that our spaces are locally path-connected and semi-simply connected.

1.1.2. A *left action* of a topological group G on a topological space X is a continuous function

$$\varphi : G \times X \longrightarrow X$$

such that

(i) $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ for all $g, h \in G$ and $x \in X$,
(ii) $\varphi(1, x) = x$, for all $x \in X$, where 1 is the identity element of G.

We shall usually write $\varphi(g, x)$ simply as gx, $g(x)$, or sometimes $g \cdot x$. Clearly, each element $g \in G$ can be viewed as a homeomorphism of X onto itself. We may denote this action by (G, X, φ), or more simply suppress the φ and call X a *G-space*. If X and Y are G-spaces, then a *G-map* is a continuous function $f : X \to Y$ which is *equivariant*; i.e., $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. If f is a G-map and a homeomorphism, then f is called a *G-equivalence* or *G-isomorphism* (in the relevant category). A map $f : X \to Y$ is *weakly G-equivariant* if there exists a continuous automorphism α_f of G such that $f(gx) = \alpha_f(g)f(x)$, for all $g \in G$, $x \in X$. If f is a homeomorphism, then f is a *weak G-equivalence* or a *weak G-isomorphism*.

There is an analogous notion of a *right action*,

$$\psi : G \times X \longrightarrow X$$

which we denote by $\psi(x, g) = xg$ or $x \cdot g$. Then $\psi(x, gh) = (xg)h = xgh$. Any right G-action $\psi(x, g)$ can be converted to a left G-action $\varphi(g, x)$ by $\varphi(x, g) = \psi(x, g^{-1})$ and vice versa. Note that, for a reasonably nice space X (e.g., locally compact Hausdorff), the set of self-homeomorphisms of X becomes a topological group TOP(X) (see section 1.2.5), and a left G-action is equivalent to having a group homomorphism $G \to \mathrm{TOP}(X)$. A right G-action becomes an anti-homomorphism. We will always assume we have a left action unless we specify otherwise.

An action (G, X, φ) is called a *smooth action* if G is a Lie group, X is a smooth manifold, and the function φ is smooth.

orbit of G through x
orbit space
orbit map
proper mapping
isotropy subgroup
stabilizer
stability subgroup
fixed point set
orbit type

1.1.3. If X is a G-space and $x \in X$, then

$$Gx = \{y \in X : y = gx \text{ for some } g \in G\}$$

is called the *orbit of G through x*. It can be denoted by Gx, $G \cdot x$ or $G(x)$.

The collection of all orbits of X forms a partition of X into disjoint sets. The collection of orbits with the identification topology (i.e., quotient topology) is called the *orbit space* of the G-action on X, and is denoted by $G\backslash X$. The identification map $\nu : X \to G\backslash X$ is called the *orbit map*. It is an open mapping for if U is open in X, then $\nu^{-1}(\nu(U)) = \bigcup_{g\in G} gU$. Note that each gU is open because g is a homeomorphism.

A map $f : X \to Y$ is called a *proper mapping* if preimage of a compact set is compact. For a general G-space, the orbits may fail to be closed subsets of X, and consequently, $G\backslash X$ would not be T_1 (i.e., points in $G\backslash X$ may not be closed). However, when G is compact and X is Hausdorff, we have:

1.1.4 Theorem. [?, 3.1] *If X is a Hausdorff G-space with G compact, then*

(1) *$G\backslash X$ is Hausdorff,*
(2) *the orbit map $X \to G\backslash X$ is a closed mapping,*
(3) *the orbit map $X \to G\backslash X$ is a proper mapping,*
(4) *X is compact if and only if $G\backslash X$ is compact,*
(5) *X is locally compact if and only if $G\backslash X$ is locally compact.*

These facts are easy to prove. To obtain similar properties when G is a non-compact Lie group, we must impose the notion of proper action (see section 1.2 below).

1.1.5. Let

$$G_x = \{g \in G : gx = x\}.$$

This subgroup of G is a closed subgroup of G if points of X are closed (i.e., X is T_1). It is called the *isotropy subgroup*, *stabilizer* or *stability subgroup* of G at x. The set

$$X^G = \{x \in X : gx = x \text{ for all } g \in G\}$$

is called the *fixed point set* of the action of G on X. It is a *closed subset of X if X is Hausdorff.* We sometimes write X^G as $F(G, X)$ or $\mathrm{Fix}(G, X)$.

Clearly,

$$G_{gx} = gG_xg^{-1}$$

so that, if $y \in Gx$, then $G_y = gG_xg^{-1}$ for any g such that $gx = y$. By (G_x) we mean the conjugates of G_x. This set of conjugates is called the *orbit type* of the orbit Gx.

If K is a normal subgroup of G and X is a G-space, then there is induced a natural action of G/K on $K\backslash X$, and the orbit mapping $\nu : X \to G\backslash X$ factors through $X \xrightarrow{K\backslash} K\backslash X \xrightarrow{G/K\backslash} G\backslash X$. The induced natural action

$$G/K \times K\backslash X \longrightarrow K\backslash X$$

is given by $gK \cdot (Kx) = g(Kx)$. (Strictly speaking, if K is not a closed subgroup of G, then the topological group G/K is not a Hausdorff topological group. It is often assumed that a topological group is Hausdorff which will automatically imply that it is completely regular. Of course, our main concern is with Lie groups and their closed subgroups).

action!ineffective
action!effective
action!ineffective part
action!transitive
action!simply transitive
action!free

If there exists a subgroup K of G, $K \neq 1$, such that $X^K = X$, we say that G acts *ineffectively*; otherwise, G acts *effectively*. Therefore, the action is effective if and only if $\bigcap_{x\in X} G_x = 1$. We call the largest subgroup K of G that fixes all of X, the *ineffective part* of the action. It is a closed (assuming X is T_1) normal subgroup and G/K acts effectively on X.

1.1.6. An action of G on X is *transitive* if the orbit through some point of X consists of all of X. That is to say, that given any x and y in X, there exists $g \in G$, such that $gx = y$. Such an action is *simply transitive* if such g is unique. If G acts transitively on X, then the function $G/G_x \to G(x) \hookrightarrow X$ given by $gG_x \to gx$ is *onto*. Clearly the map is one-one. Moreover, the map is continuous. This is an immediate consequence of the universal properties of quotient mappings. If G/G_x is compact, and X is Hausdorff, then the mapping is a homeomorphism. In general, however, the inverse mapping may fail to be continuous. See section 1.2.3 for a condition guaranteeing continuity of the inverse.

1.1.7 Example. Let $G(\cong \mathbb{R})$ be a linear subspace of $\mathbb{R}^2$ which consists of the points on a line through the origin with irrational slope. Reducing the coordinates in $\mathbb{R}^2$ modulo 1 induces the standard covering projection,

$$p : \mathbb{R}^2 \longrightarrow \mathbb{Z}^2\backslash\mathbb{R}^2 = T^2,$$

of $\mathbb{R}^2$ onto the 2-torus T^2. This is a homomorphism of the additive group $\mathbb{R}^2$ onto T^2 with kernel the standard integral lattice subgroup $\mathbb{Z}^2$. Since $G \cap \mathbb{Z}^2 = \{0\}$ is the trivial group, G descends to T^2 as a one-one continuous homomorphism onto its image. Let $X = p(G)$ with the relative topology of T^2 Then G acts on G by left translations and this descends to a transitive G-action on X, with the stabilizer $G_0 = 0$. Now note that this topology on X is strictly weaker than the topology of the original G.

1.1.8 Exercise. (a) Show that the two $\mathbb{Z}_5 = \langle\lambda\rangle$ actions on the unit disk given by

$$\lambda \times z \mapsto e^{\frac{2\pi i}{5}} z, \quad \lambda \times z \mapsto e^{\frac{4\pi i}{5}} z$$

are not G-equivariant but, are weakly G-equivariant.

(b) Show that if $f : (G, X) \to (G, Y)$ is a weak G-equivalence, then f induces a homeomorphism $G\backslash X \to G\backslash Y$ which sends orbits of type (H) to orbits of type $(\alpha_f(H))$.

1.1.9 Definition. G acts *freely* on X if $G_x = 1$ for all $x \in X$.

Examples of free actions are groups of covering transformations, and the left translations in a principal G-bundle. However, free actions are more general than the left translations in a principal G-bundle. See section 1.3 for a definition. In Example 1.1.7, $\mathbb{R}$ acts freely on $p(G)$ but $p(G) \to \mathbb{R}\backslash p(G)$ (a point), is not a principal

action!locally proper
Cartan G-space
action!proper

$\mathbb{R}$-bundle projection. The reason is that the action fails to be *locally proper* (see the next section).

1.2. (Locally) Proper G-spaces

1.2.1. We shall be largely dealing with actions of Lie groups on spaces. Because our group may not be compact, we need to recall the notion of a *proper action* of a locally compact topological group on a topological space. Compact Lie group actions tend to leave important geometrical structure of spaces invariant whereas non-compact Lie groups often do not.

Properness is the concept that enables properties of the actions of non-compact Lie groups to resemble those of compact groups. There are good sources for properties of proper actions; e.g., R. Palais[**?**], R. Kulkarni[**?**]. However, we caution the reader that there is no uniformity in terminology for this concept. We shall, for our convenience, recall what we shall need and refer to Palais for some of the proofs.

OK?

In the section 1.2 through 1.6,♣ *G is a locally compact topological group and X is a completely regular Hausdorff space and neighborhoods will be open sets unless specified or commented differently.* The point of complete regularity is to help ensure that $G\backslash X$ will have nice separation properties and this coupled with the notion of (local) *properness* enables us to have a *slice theorem*, (cf. section 1.5.1).

1.2.2 Definition. An action of G on X is called *locally proper* if for each $x \in X$, there exists a neighborhood U of x such that

$$\{g \in G : gU \cap U \neq \emptyset\}$$

has compact closure. In particular, G_x, being a closed subset of the above set (since X is T_1), is compact . If G is discrete, the above set is finite. (In Palais[**?**], a locally proper G-space is called a *Cartan G-space*). The action is called *proper* if for each x, there exists a neighborhood U of x such that, for each $y \in X$, there exists a neighborhood V of y for which the closure of

$$\{g \in G : gV \cap U \neq \emptyset\}$$

is compact. Observe that if closure of $\{g \in G : gV \cap U \neq \emptyset\}$ is compact, then closure of $\{g \in G : V \cap gU \neq \emptyset\}$ is compact: For, if C is compact, then C^{-1} is compact also.

1.2.3. The following properties of locally proper and proper actions are proved in Palais[**?**]. The proofs are easier for compact or discrete G. The use of nets can be avoided if one assumes that X is first countable. We will use these properties mostly for compact or discrete G. We suggest that the reader furnish his/her own proofs for discrete G.

For locally proper actions, we have:

1. *Each orbit is closed in X. Hence, $G\backslash X$ is T_1. In fact, $G\backslash X$ is locally completely regular. However, $G\backslash X$ may fail to be Hausdorff.*
2. *The map $gG_x \to gx$ is a homeomorphism of G/G_x onto Gx.*
3. *If K is the ineffective part of G, then G/K acts locally properly. (Similarly, for proper actions).*

(4) *If an x has a neighborhood U such that $\{g \in G : U \cap gU \neq \emptyset\}$ is finite, then Gx is discrete in X, and in fact, G itself is discrete.*
(5) *(G, X) is proper if and only if $G\backslash X$ is regular. In fact, $G\backslash X$ is completely regular when (G, X) is proper.*

Here are some additional facts for G-spaces.

(6) a) *If X and Y are G-spaces, and if X is a locally proper (resp. proper) G-space, then so is $X \times Y$.*
b) *Let Y be a locally proper (resp. proper) G-space and X a (resp. completely regular) space. If $f : X \longrightarrow G\backslash Y$ is a map, then the pullback $\widetilde{f} : \widetilde{X} \longrightarrow Y$ is a locally proper (resp. proper) G-space whose orbit space is X.*
c) *If X is a locally proper (resp. proper) G-space, H a closed subgroup of G, and Y an H-invariant subspace of X, then Y is locally proper (resp. proper) H-space. (Note, the Example 1.1.7 shows that H must be closed, for we may take $H = p(G) \subset T^2$).*
(7) *If X is a locally compact G-space, the following are equivalent:*
  (a) *(G, X) is locally proper and $G\backslash X$ is Hausdorff*
  (b) *(G, X) is proper*
  (c) *For each compact subset C of X, the closure of $\{g \in G : gC \cap C \neq \emptyset\}$ is compact.*
(8) *Let X be a proper G-space and H a closed normal subgroup of G. Then $H\backslash X$ is a proper G/H-space.*

Note: All citations are from Palais' paper[**?**]: [1] follows from (1.1.4) and Corollary 2 of (1.2.8); [2] is (1.1.5); [3] is (1.1.6); [5] (1.2.5) says that locally proper and $G\backslash X$ completely regular imply properness, and (1.2.8) states properness implies $G\backslash X$ regular; [6a] is (1.3.3), [6b] is (1.3.4); and [6c] is (1.3.1); and [7] is (1.2.9); The item [7] above is a common criterion for proper action. We state it as a corollary.

1.2.4 Corollary. *Let X be a completely regular, locally compact Hausdorff space. A G-action on $X$is proper if and only if, for each compact subset C of X, the closure of $\{g \in G : gC \cap C \neq \emptyset\}$ is compact.*

1.2.5 Remark. Even if (G, X, φ) is not necessarily a proper G-space, there is a natural homomorphism $\widetilde{\varphi} : G \to \mathrm{TOP}(X)$, where $\mathrm{TOP}(X)$ is the group of all self-homeomorphisms of X. We may topologize $\mathrm{TOP}(X)$ so that $\widetilde{\varphi}$ becomes continuous if we choose to do so. For example, $\widetilde{\varphi}$ will be continuous if we take the compact-open topology on $\mathrm{TOP}(X)$, see [**?**, §9.4, p.75] and, consequently, also continuous if we take the smaller point-open topology (i.e., the topology of point-wise convergence). We shall often denote $\widetilde{\varphi}$ by $\rho : G \to \mathrm{TOP}(X)$, thinking of ρ as a representation.

$\mathrm{TOP}(X)$ becomes a topological group under the point-open topology and then ρ becomes a homomorphism of topological groups. Under the compact-open topology, $\mathrm{TOP}(X)$ is almost a topological group but fails only in that inversion may not be continuous. However, if X is assumed to be locally compact Hausdorff and either connected or locally connected, then $\mathrm{TOP}(X)$ under the compact-open topology is a topological group. cite—-. If and when the topology on $\mathrm{TOP}(X)$ becomes an issue, we shall be explicit about it. [Arens]

H-kernel
H-slice
global H-slice
slice at x

For a *locally proper* G-action on a Hausdorff space with G locally compact, we have the following

1.2.6 PROPOSITION ([**?**, (1.1.7)]). *$\rho : G \to \mathrm{TOP}(X)$ is a continuous and relatively open map of G when $\mathrm{TOP}(X)$ is given the point-open topology or the compact-open topology. Furthermore, the image $\rho(G)$ is closed in both topologies. Thus, if ρ is injective (i.e., the G action is effective), ρ is an isomorphism onto a closed subgroup of $\mathrm{TOP}(X)$ under the point-open topology and, similarly under the compact-open topology if $\mathrm{TOP}(X)$ is a topological group with this topology.*

1.3. Fiber Bundles

1.4. Tubular neighborhoods and Slices

The most important tool for analyzing Lie group actions is the existence of a *slice*. Slices give us the complete equivariant structure of an invariant tubular neighborhood of each orbit.

1.4.1 DEFINITION. Let X be a G-space and H a closed subgroup of G with a local cross-section. A subset S of X is an *H-kernel* if there exists an equivariant map $f : GS \to G/H$ such that $f^{-1}(H) = S$. If, in addition, GS is open in X, we call S an *H-slice* in X. If $GS = X$, we call S a *global H-slice* for X. For $x \in X$, by a *slice at x*, we mean a G_x-slice in X which contains x.

1.4.2. The following is a prototypical example of a global H-slice in X. In fact, it follows from Proposition 1.4.3 that *any two spaces with the same global H-slices are G-isomorphic.*

Prototypical Example: Let H be a closed subgroup of G with a local cross-section, and suppose H acts on S. On $G \times S$, define an action of $G \times H$ by

$$(g,h)(\overline{g},x) = (g\overline{g}h^{-1}, hs).$$

Denote the quotient of the "diagonal" H-action by $G \times_H S$, and the image of (g,s) by $\langle g,s\rangle \in G \times_H S$. Since H commutes with the G-action on $G \times S$, the G-action descends to $G \times_H S$, and is given by $(g, \langle \overline{g}, s\rangle) \to \langle g\overline{g}, s\rangle$. We get a commutative diagram of projections and orbit mappings:

$$\begin{array}{ccccc}
(G \times H, G) & \xleftarrow{\pi_1} & (G \times H, G \times S) & \xrightarrow{G\backslash} & (H,S) \\
\Big\downarrow{\scriptstyle H\backslash} & & \Big\downarrow{\scriptstyle H\backslash} & & \Big\downarrow{\scriptstyle H\backslash} \\
(G, G/H) & \xleftarrow{\overline{\pi}_1} & (G, G \times_H S) & \xrightarrow{G\backslash} & H\backslash S = G\backslash(G \times_H S)
\end{array}$$

Clearly, $G \times_H S$ is a G-space whose orbit space is $H\backslash S$ and also a fiber bundle over G/H with fiber S and structure group H/K, where K is the ineffective part of the action of H on S. The associated principal bundle has total space G/K and structure group H/K. There is the obvious G-isomorphism between $(G, G \times_H S)$ and $(G, G/K \times_{H/K} S)$. *The map $\overline{\pi}_1$ induced from π_1, the projection onto the first* *factor, is G-equivariant such that* $\overline{\pi}_1^{-1}(H) = S = \langle e, S\rangle$. Thus, the G-space $G \times_H S$ has a global slice $\langle e, S\rangle$.

G-invariant tube about the orbit Gx

G-invariant tubular neighborhood of Gx

We have the following converse:

1.4.3 Proposition. *Suppose H is a closed subgroup of G with a local cross-section, and X is a G-space with a G-equivariant map $f : X \to G/H$. Let $S = f^{-1}(H)$. Then there exists a G-isomorphism* $\varphi : (G, G \times_H S) \longrightarrow (G, X)$.

Proof. We first show that there exists a G-map φ which is continuous, one-one and onto. The set S is H-invariant because $f(hs) = hf(s) \in H$ for all $h \in H$ and $s \in S$. Define an action of $G \times H$ on $G \times S$ as above and a G-map $\widetilde{\varphi} : G \times S \to X$ by $\widetilde{\varphi}(g, s) = gs$. The map is easily seen to be continuous. We show $\widetilde{\varphi}$ is also onto. For any $x \in X$, $f(x) = gH \in G/H$ for some $g \in G$. Then $f(g^{-1}x) = H$. Therefore, $s = g^{-1}x \in S$ for some $s \in S$, and $x = gs = \widetilde{\varphi}(g, s)$.

If $\widetilde{\varphi}(g, s) = \widetilde{\varphi}(g', s')$, then $gs = g's'$, hence $s' = {g'}^{-1}gs$. But as $f(s') = {g'}^{-1}gf(s) = H$, ${g'}^{-1}g \in H$. Thus, $g' = gh^{-1}$, $s' = hs$ for some $h = {g'}^{-1}g$. Consequently, $\widetilde{\varphi}(g', s') = \widetilde{\varphi}(gh^{-1}, hs)$ and $\widetilde{\varphi}$ factors through $\varphi : (G, G \times_H S) \longrightarrow (G, X)$.

In fact, we have actually shown that φ is continuous one-one and onto. Furthermore, it is G-equivariant. To show φ^{-1} is continuous when G is compact first, we observe that $G \times S \longrightarrow GS$ is a *closed* mapping since S is closed in X. We defer the continuity of φ^{-1} in the non-compact case until Corollary 1.4.11. □

The proposition suggests the following

1.4.4 Definition. If a G-space has a slice S_x at x, and G_x has a local cross-section in G, then GS_x, which is G-isomorphic to $G \times_{G_x} S_x$, by Proposition 1.4.3, is a fiber bundle over the orbit Gx with structure group $G_x / \bigcap_{s \in S} G_s$ (the ineffective part of the action of G_x on the slice S_x) and fiber S_x. The set GS_x is called a *G-invariant tube about the orbit Gx* or a *G-invariant tubular neighborhood of Gx*. The fiber over gx is gS_x.

1.4.5. Proposition 1.4.3 says that if a G-action has a slice at x, then there exists a G-invariant tubular neighborhood about G_x. Conversely, the G-invariant map $\overline{\pi}_1$ in 1.4.2 shows that if there exists a G-invariant neighborhood of $G(x)$ in X, of the type $(G, G \times_{G_x} S)$ with S containing x, then S is a slice at x. Therefore *the existence of a slice at x is equivalent to the existence of a G-invariant tubular neighborhood at x.*

1.4.6 Example. Let (G, X) be a group of regular covering transformations, $\nu : X \longrightarrow G\backslash X$, the covering projection. For $\nu(x) = x^* \in G\backslash X$, let U^* be a neighborhood of x^* which is evenly covered. That is, $\nu^{-1}(U^*)$ is the disjoint union of copies of open sets homeomorphic to U^*. If U denotes the copy containing x, then $\nu^{-1}(U^*) = GU$ which is isomorphic to $G \times U$, and forms a G-tubular neighborhood of $\nu^{-1}(x^*)$.

1.4.7 Example. Consider the affine transformations of $\mathbb{Z} \rtimes \mathbb{Z}_2 = G$ on the real line as given by

$$(n, \epsilon)x = \epsilon x + n, \qquad \epsilon = \pm 1.$$

Brieskorn variety

The stabilizer at x is trivial if x is not an integer or a half integer. If $x = \frac{m}{2}$ for some integer m, then $G_x = \{(0,1),(m,-1)\} \cong \mathbb{Z}_2$. For a slice at 0, one can choose the set $S = (-\frac{1}{2}, \frac{1}{2}) \subset \mathbb{R}$. One has that $(\mathbb{Z}\rtimes\mathbb{Z}_2) \times_{\mathbb{Z}_2} (-\frac{1}{2}, \frac{1}{2})$ is a tubular neighborhood V of the orbit. It consists of all of $\mathbb{R}$ except for the orbit through $\frac{1}{2}$. That is, $G \times_{\mathbb{Z}_2} S = V = GS = \mathbb{R} - (\frac{1}{2} + \mathbb{Z})$. We can define a $(\mathbb{Z}\rtimes\mathbb{Z}_2)$-equivariant map

$$f : G \times_{\mathbb{Z}_2} S = V = GS \longrightarrow G/G_x = \mathbb{Z}\rtimes\mathbb{Z}_2/\mathbb{Z}_2$$

by $v = (n,\epsilon)(s) = (n,-\epsilon)(-s) \xrightarrow{f} \{(n,1) \cup (n,-1)\}$, then $f^{-1}\{(0,1) \cup (0,-1)\} = f^{-1}(\mathbb{Z}_2) = f^{-1}(G_0) = S$. (Note that the equivariant mapping f can not be extended to all of $\mathbb{R}$).

1.4.8(**Brieskorn Varieties**). Consider an action $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ given by

$$z \times (z_1, \cdots, z_n) \to (z^{b_1} z_1, \cdots, z^{b_n} z_n)$$

where b_i are positive integers ≥ 1.

Notice the restriction to $S^1 \times S^{2n-1} \to S^{2n-1}$ where $z \in S^1 \subset \mathbb{C}^*$, and $(z_1, \cdots, z_n) \in S^{2n-1} \subset \mathbb{C}^n$ is well defined.

The b_i's that we shall use are arrived at as follows: Define a set

$$V(a_1, a_2, \cdots, a_n) = \{(z_1, \cdots, z_n) \mid z_1^{a_1} + \cdots + z_n^{a_n} = 0\}$$

where a_i are integers ≥ 2. Put $a = \operatorname{lcm}\{a_1, \ldots, a_n\}$ and define $b_i = a/a_i$.

Then V is invariant under the $\mathbb{C}^*$ action for

$$(z^{b_1} z_1)^{a_1} + \cdots + (z^{b_n} z_n)^{a_n} = z^a (z_1^{a_1} + \cdots + z_n^{a_n}) = 0$$

if $(z_1, \cdots z_n) \in V$. Note that the set

$$K(a_1, \ldots, a_n) = V(a_1, \ldots a_n) \cap S^{2n-1}$$

is also S^1 invariant. Then

$$K(a_1, \ldots, a_n) = \{(z_1, \ldots, z_n) \mid z_1\overline{z_1} + \cdots + z_n\overline{z_n} = 1 \quad \text{and} \quad z_1^{a_1} + \cdots z_n^{a_n} = 0\}.$$

Let

$$p(z_1, \cdots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n}.$$

The polynomial function $p : \mathbb{C}^n - \mathbf{0} \to \mathbb{C}$ has 0 as a regular value. Therefore $p^{-1}(0) = V(a_1, \cdots, a_n) - \mathbf{0}$ is a *complex manifold* of dimension $n-1$ and $K(a_1, \cdots, a_n)$ is a *real analytic manifold* of dimension $2n-3$. It is not difficult to see that $K(a_1, \cdots, a_n) \times \mathbb{R}^1 \approx V(a_1, \cdots, a_n) - \mathbf{0}$.

Define $\varphi : S^{2n-1} - K \to S^1$ by

$$\varphi(\vec{z}) = \frac{p(\vec{z})}{|p(\vec{z})|} \in S^1.$$

If we define an S^1-action on this image by

$$z \times \frac{p(\vec{z})}{|p(\vec{z})|} \mapsto z^a \frac{p(\vec{z})}{|p(\vec{z})|},$$

we see that this map φ is S^1-equivariant. Therefore there exists a global S^1-slice $Y = \varphi^{-1}(1)$, $1 \in S^1$ for the $(S^1, S^{2n-1} - K)$ action. That is,

$$(S^1, S^{2n-1} - K) \xrightarrow[\approx]{} (S^1, S^1 \times_{\mathbb{Z}_a} Y)$$

an S^1-equivariant homeomorphism. Thus, $S^{2n-1} - K$ fibers over $S^1/\mathbb{Z}_a$ equivariantly with fiber $Y = \varphi^{-1}(1)$ and with structure group $\mathbb{Z}_a$.

It can be shown citeMilnor that $Y \cup K$ is a compact manifold with boundary. Milnor p.76
Y is parallelizable of dimension $2(n-1)$ and has the homotopy type of K. In many cases, K is a smoothly embedded topological sphere in $\mathbb{C}^n$ not diffeomorphic to the standard sphere.

As a special case, take $\mathbb{C}^2$ and the action $z \times (z_1, z_2) \to (z^2 z_1, z^3 z_2)$. On S^3 this action results in a fixed point free action where the stabilizer on $(z_1, 0)$ is $\mathbb{Z}_2$ and, on $(0, z_2)$ it is $\mathbb{Z}_3$. The 1-manifold K is the trefoil knot. From above, $S^3 - K$ fibers over the circle with fiber a 2-manifold having the homotopy type of $S^1 \vee S^1$. Since its boundary is K, F is a punctured torus T', so $S^3 - K = S^1 \times_{\mathbb{Z}_6} T'$, which gives the smooth fibered structure of the complement of the $(2,3)$ torus knot K.

1.4.9 Lemma. [**?**, 2.1.2] *Let S be an H-kernel in the G-space X, and $\eta : U \to G$ be a local cross-section from G/H to G, $(\eta(H) = 1)$. Then if $g_0 \in G$, the map $F : (u, s) \mapsto g_0 \eta(g_0^{-1} u) s$ is a homeomorphism of $g_0 U \times S$ onto an open neighborhood of $g_0 S$ in GS. Moreover, $f(F(u, s)) = u$ when f is the equivariant map defining the H-kernel.*

Proof. $f(F(u, s)) = g_0(\eta(g_0^{-1} u) H) = g_0(g_0^{-1} u) = u$. Therefore, $F(g_0 U \times S) = f^{-1}(g_0 U)$, which is an open neighborhood of $g_0 S$ in GS. Note that F is one-one and continuous. We claim that F^{-1} is continuous by showing that if $F(u_\alpha, s_\alpha)$ converges to $F(u, s)$, then $u_\alpha \to u$ and $s_\alpha \to s$, where we use nets if there is no countable neighborhood base. Now because f is continuous, $u_\alpha = f(F(u_\alpha, s))$ converges to $u = f(F(u, s))$. Therefore, $\eta(g_0^{-1} u_\alpha)^{-1}$ converges to $\eta(g_0^{-1} u)^{-1}$ because η is continuous. Now, $\eta(g_0^{-1} u_\alpha) s_\alpha = g_0^{-1} F(u, s_\alpha)$ which converges to $g_0^{-1} F(u, s) = \eta(g_0^{-1} u) s$, which implies s_α converges to s.

Note, taking $g_0 = 1$, the argument shows that if W is open in S, then $G(W)$ is open in GS. □

1.4.10 Proposition. [**?**, 2.1.3] *Let S_1 and S_2 be H-kernels in G-spaces X_1 and X_2 respectively, and let f_0 be an H-equivariant map of S_1 into S_2. Assume H has a local cross-section from G/H to G. Then there exists a unique G-equivariant map f of GS_1 onto GS_2 such that $f|S_1 = f_0$; namely, $f(gs) = g f_0(s)$ for $g \in G$, $s \in S$. Moreover, if f_0 embeds S_1 into S_2, then f embeds GS_1 into GS_2.*

Proof. We are able to extend f_0 to f because f_0 being H-equivariant implies $H_s \subset H_{f_0(s)}$, $s \in S$, and so, $G_s \subset G_{f(s)}$. To check continuity of f we use the previous Lemma. Let $\eta : U \to G$ be a local cross section, U a neighborhood of H in G/H. Now $F_i : (u, s) \mapsto g_0 \eta(g_0^{-1} u) s$ is a homeomorphism of $g_0 U \times S_i$ onto a neighborhood of $g_0 S_i$ in GS_i, $i = 1, 2$, and $g_0 \in G$. Since $f(F_1(u, s)) = f(g_0(\eta(g_0^{-1}(u)) s)) = g_0 \eta(g_0^{-1} u) f_0(s) = F_2(u, f_0(s))$, the continuity of f follows. Also if f_0^{-1} exists and is continuous, then f^{-1} is continuous by symmetry. □

1.4.11(Proof of Proposition 1.4.3). Suppose (G, X) and $f : X \to G/H$ and $S = f^{-1}(H)$ are given as in the Proposition 1.4.3. Choose the H-equivariant identity map $\mathrm{id} : S \to S$ and extend it to $\varphi : G \times_H S \to (G, X)$. This extension is unique. Therefore, φ is a G-isomorphism.

1.4.12 COROLLARY. *If (G, X) is a completely regular proper G-space and G has a slice S at x, and G_x has a local cross-section on G, then S can be chosen so small such that each $y \notin G_x$, has a neighborhood V_y for which the closure of $\{g \in G : GS \cap V_y \neq \emptyset\}$ is compact.*

PROOF. By Lemma 1.4.9, there is a homeomorphism of $U \times S$ onto a neighborhood of S in GS where U is a neighborhood of G_x in G/G_x. Therefore, $\nu : S \to G_x \backslash S$ has open image on $G \backslash X$. Take a sufficiently small neighborhood W of $\nu(x)$ and put $\nu^{-1}(W) \cap S = S'$. Then $\eta(U)S'$ is a neighborhood of x. If U is also to be sufficiently small, then the definition of proper will imply that for each $y \in X$, there exists V_y so that the closure of $\{g \in G : GS' \cap V_y \neq \emptyset\}$ is compact. □

1.5. Existence of Slices

As one can see from Proposition 1.4.3, the existence of a slice at x means there exists a G-equivariant open tube about the orbit Gx which is G-isomorphic to $(G, G \times_{G_x} S_x)$, where S_x is the slice at x. Therefore, the global structure of the G-action can be achieved by successfully piecing together the tubes about the orbits. But of course, slices do not exist without restrictions on G as the example 1.1.7 shows. Showing that slices exist for various situations has had an illustrious history with various significant contributions having been made by Montgomery, Zippin, Yang, Kozul, Mostow, Palais[**?**] and others. Excellent expositions for the existence of slice for compact Lie group actions can be found in [**?**] or in [**?**]. We will sketch a proof of this in this section when the action is smooth.

For the general Lie groups, we have the

1.5.1 THEOREM ([**?**]). *If (G, X) is a locally proper Lie group action on a completely regular space X, then G has a slice at each $x \in X$.*

1.5.2 COROLLARY. *If G is a Lie group acting properly on a completely regular space X and, x and y lie on different orbits of G, then there exist slices S_x and S_y at x and y respectively such that the closures of GS_x and GS_y are disjoint.*

PROOF. Since G is a Lie group, G/H has a local cross-section to G for any closed subgroup H. By Theorem 1.5.1, G has a slice at each $x \in X$. By 1.2.3(5), $G \backslash X$ is completely regular. We can separate $\nu(x)$ and $\nu(y)$ by open sets whose closures are disjoint. Now apply the procedure used in the proof of Corollary 1.4.12 to obtain the desired conclusion. □

1.5.3 COROLLARY. *For a completely regular G-space where G is a Lie group, the following are equivalent:*

(i) *G acts locally properly,*
(ii) *G has a G_x-slice at each $x \in X$, and G_x is compact,*
(iii) *The closure $\{g \in G : gS_x \cap S_x \neq \emptyset\}$ is compact for each $x \in X$.*

For our purposes, we shall deal mostly with compact Lie group actions and two types of non-compact Lie group actions. If G is not compact and not discrete, then G will usually be acting freely as left translations on a principal G-bundle. In this case, the existence of a slice is built into the definition of a principal G-bundle. If

G is discrete, we shall give a direct elementary proof of the existence of a slice. Let us begin with this latter case.

Again, Example 1.1.7 shows if we take the usual copy of $\mathbb{Z}$ in $\mathbb{R}$, the free smooth action on the descent of the irrational line is *not* locally proper and no slices exist for this action.

1.5.4 PROPOSITION. *Let (G, X) be a locally proper G-space with G discrete and X Hausdorff. For each $x \in X$, there exists a G_x-invariant neighborhood S such that $\{g \in G : gS \cap S \neq \emptyset\} = G_x$ and $gS \cap \overline{g}S \neq \emptyset$ if and only if g and $\overline{g}$ belong to the same G_x-coset.*

PROOF. By the property 1.2.3, which is valid if X is Hausdorff without assuming completely regular, the orbits are closed subsets of X. Since the action is locally proper, G discrete, then all orbits are discrete and all their stability groups are finite. Let $x \in X$, then there exists U such that $F = \{g \in G : gU \cap U \neq \emptyset\}$ is a finite set. If $g \notin G_x$, and $g \in F$, then $gx \neq x$. There exist neighborhoods ${V_x}^g$ of x and V_{gx} of gx such that ${V_x}^g \cap V_{gx} = \emptyset$, ${V_x}^g \subset U$ and $g{V_x}^g \subset V_{gx}$. Put $V = \cap_{g \in F - G_x} {V_x}^g$. Then we have $\{g \in G - G_x : gV \cap V \neq \emptyset\} = \emptyset$. Take W a neighborhood of x, $W \subset V$ and such that $gW \subset V$ for all $g \in G_x$. Put $S = G_x(W)$, then $S \subset V$ and S is G_x-invariant, and $gS \cap S = \emptyset$ if and only if $g \in G - G_x$. Furthermore, observe that $gG_x \cap \overline{g}G_x \neq \emptyset$ if and only if g and $\overline{g}$ belong to the same G_x coset. Thus, $\{gG_xS\}$, as gG_x runs through the disjoint G_x-cosets, forms an invariant neighborhood of the orbit with each distinct gG_x being disjoint from all the others. □

1.5.5 THEOREM. *Let (G, X) be a locally proper G-space with G discrete and X Hausdorff. Then for each $x \in X$, there exists a slice S and (G, GS), a G-invariant neighborhood of Gx, is G-isomorphic to $(G, G \times_{G_x} S)$ and homeomorphic to $G/G_x \times S$.*

PROOF. For S, we choose the S from the Proposition 1.5.4. Define $f : GS \to G/G_x$ by

$$f(gs) = gG_x, \quad g \in G, \quad s \in S.$$

If $gs = g's'$, then ${g'}^{-1}g \in G_x$ by Proposition 1.5.4. Therefore, $f(gs) = f(g's')$ and so f is well defined. It is continuous since G/G_x is discrete, GS is a product space homeomorphic to $G/G_x \times S$ since GS is a disjoint collection of open sets each homeomorphic to P and indexed by G/G_x. The G-map

$$\varphi : (G, G \times_{G_x} S) \longrightarrow (G, GS)$$

defined by $(g, \langle \overline{g}, s\rangle) \xrightarrow{\varphi} g\overline{g}s$ is clearly one-one, continuous, G-invariant and onto. The inverse is also clearly continuous. □

1.5.6 REMARK. Note this form of the slice theorem does not require that X be completely regular. Even when X is a smooth manifold and the action is smooth, the orbit space can fail to be Hausdorff. However if the action is *proper*, $G\backslash X$ will be completely regular and, in particular, Hausdorff. Here is an

1.5.7 EXAMPLE. Let the free $\mathbb{R}$-action on the strip $Z = \{(x, y) : -1 \leq y \leq 1\}$ be that whose orbits are pictured below:

Figure 1: Flow

This action is locally proper for if we delete either $y = 1$ or $y = -1$, then the deleted strip is a (locally trivial) principal $\mathbb{R}$-bundle with a cross-section (i.e., a global slice). However, there is no global cross-section. The action is proper on $\{(x, y) : -1 < y < 1\}$ but properness on the strip is violated for if U is a neighborhood of $(x, 1)$ and V is a neighborhood of $(x', -1)$, then closure $\{g \in G : gU \cap V \neq \emptyset\}$ is not compact. The orbit space is a half open interval with a double point at the end point, corresponding to the two lines $y = \pm 1$. This is a non-Hausdorff 1-manifold with boundary. We shall reserve the term n-manifold to mean a Hausdorff space with a countable basis with each basis element homeomorphic to an open subset of $\mathbb{R}^n$. For a discrete example, just choose $\mathbb{Z} \subset \mathbb{R}$ in this example.

1.5.8 Theorem. *Let (G, M) be a smooth action of a compact Lie group on a smooth manifold. Then each point $x \in M$ has a smooth G_x-slice.*

Proof. [Sketch] (cf. [**?**, p.108]) Introduce a Riemannian metric ρ on M. Then average this metric over the compact Lie group G:

$$\rho'(v, v') = \int_G \rho(gv, gv')_{gx} d\mu(g)$$

to get a new G-invariant Riemannian metric ρ'. The G-action via the differential on the tangent bundle $T(M)$ now acts as isometries with respect to the new metric ρ'. The exponential map $\exp : T(M) \to M$ is G-equivariant. That is,

$$g \exp(v) = \exp(g_* v),$$

where g_* denotes the differential of the diffeomorphism $g \in G$. For a (compact) orbit $G(x)$, the tangent bundle along the orbit splits equivariantly into the tangent vectors along the orbit and those normal to the orbit. For $\epsilon > 0$, sufficiently small, exponential maps the vectors of length less than ϵ of the normal bundle of the compact orbit Gx diffeomorphically onto an open neighborhood consisting of those points of M whose distance from Gx is less than ϵ. The normal vectors at $x \in Gx$ are invariant under the action of G_x and so in any normal coordinate system about x, G_x acts orthogonally. For a slice S at x, take the exponential of the normal vectors of length less than ϵ to Gx at x. Then S is an open disk centered at x. Clearly $G_x S = S$ and GS is an open tubular neighborhood of Gx. We claim

$$gS \cap S \neq \emptyset \Longrightarrow g \in G_x.$$

Suppose $g(\exp v) = \exp w$, with $v, w \in \exp_x^{-1}(S)$. Because the restricted exponential is a diffeomorphism on GS, $\exp(g_* v) = g \exp(v) = \exp(w)$, and so $g_*(v) = w$. But $w \in \exp^{-1}(S)$, hence $g_*(v) \in \exp^{-1}(S)$ which implies that $g \in G_x$.

Next, we claim there is a G-isomorphism between $(G, G \times_{G_x} S)$ and (G, GS). Define a smooth G-equivariant map $(G, G \times S) \xrightarrow{f} (G, GS)$ by $(\overline{g}, (g, s)) = (\overline{g}g, s) \mapsto \overline{g}(gs) = \overline{g}g(s)$. This map is smooth, G-equivariant and onto. It is not one-one in general. In fact, f factors through $(G, G \times_{G_x} S)$: Suppose $(g, s) \mapsto gs$ and $(g', s') \mapsto g's' = gs$. Then ${g'}^{-1} g = h \in G_x$. Consequently, $(g', s') = (gh^{-1}, s') = (gh^{-1}, {g'}^{-1} gs) = (gh^{-1}, hs)$. Conversely, $(gh^{-1}, hs) \mapsto gs$ for all $h \in G_x$. Thus, the map f factors through $(G, G \times_{G_x} S)$. Note that G-action descends to $G \times_{G_x} S$ since the diagonal G_x action commutes with the G-action on $G \times S$. If we denote the image of (g, s) by $\langle g, s\rangle \in G \times_{G_x} S$, then $\overline{g}\langle g, s\rangle = \langle \overline{g}g, s\rangle$ and is well defined. It is clear that the induced $\overline{f} : G \times_{G_x} S \longrightarrow GS$ is now one-one. It suffices to show $\overline{f}$ is open to conclude that $\overline{f}$ is a diffeomorphism. Let $t : U \longrightarrow G$ be a local cross-section in G/G_x and define $K : (u, v) \mapsto K_*(u)(v)$ on $U \times \exp^{-1}(S)$ diffeomorphically onto an open set in the normal bundle to $G(x)$. Then $\widetilde{K} = \exp \circ K$ is a diffeomorphism onto an open set in X. Since $\exp(t_*(u)v) = t(u)\exp(v)$, then $(u, s) \mapsto t(u)s$ is a diffeomorphism of $U \times S$ onto an open set in GS. □

1.6. Actions of $G \cdot \Pi$ (G Lie group, Π discrete)

The following proposition will be extremely useful in our study of Seifert fiberings.

1.6.1 PROPOSITION. *Let G be a Lie group acting effectively and properly on a completely regular space X. Suppose there also exists a discrete group Π acting effectively on X. Assume*

(1) *Π normalizes G in* $\mathrm{TOP}(X)$, *and*
(2) *$\Gamma = G \cap \Pi$ is a closed discrete subgroup of G.*

Put $Q = \Pi/\Gamma$. Then there exists an induced action of Q on $G\backslash X = W$ which is effective. If Π acts properly and Γ is cocompact in G, then the Q action on W is proper. Conversely, if the Q action on W is proper, then the group, $G\Pi$, generated by G and Π acts properly on X, and consequently, Π acts properly on X.

PROOF. Let $\nu : X \to W$ denote the G-orbit mapping. Since Π normalizes G, Π acts effectively on W as a Q-action. Since G was proper on X, the orbit space W, by the property 1.2.3(5), is completely regular. The map ν is Π-equivariant.

(not assuming X is locally compact) Later!!

(assuming X is locally compact) We shall prove (Q, W) is proper. Let $K \subset W$ be compact. One can find a set $K' \subset X$ such that $K' = (\bigcup S_i) \cap \nu^{-1}(K)$ (where S_i is a slice) and $\nu(K') = K$. This is possible because the action (G, X) is proper. Let $e \in F \subset G$ be a compact set such that $\Gamma \cdot F = G$ (say, a fundamental domain). Then $F \cdot K'$ is compact. Suppose $\alpha \in Q$ satisfies $\alpha(K) \cap K \neq \emptyset$. Then, by adjusting by an element of Γ if necessary, we can find $\widetilde{\alpha} \in \Pi$ such that $\widetilde{\alpha}(F \cdot K') \cap (F \cdot K') \neq \emptyset$. Since the Π action is proper, there are only finitely many such $\widetilde{\alpha}$'s in Π, and hence there are only finitely many such α's in Q.

For the converse, let $C \subset X$ be a compact set. Then $\nu(C) \subset W$ is compact. Since (Q, W) is proper, there exist only finitely many elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in Q$ such that $\alpha_i(\nu(C)) \cap \nu(C) \neq \emptyset$. For each i, pick a preimage $\widetilde{\alpha}_i \in \Pi$ of α_i, and let

$$D_i = \widetilde{\alpha}_i(C) \cup C.$$

Then D_i is a compact subset of X.

Suppose $f \in G \cdot \Pi$ satisfies $f(C) \cap C \neq \emptyset$. Let $\overline{f} \in Q$ denote the image of f. Then

$$\overline{f}(\nu(C)) \cap \nu(C) = \nu(f(C)) \cap \nu(C) \supset \nu(f(C) \cap C) \neq \emptyset.$$

covering projection
evenly covers
covering space!equivalent
covering transformation

This implies that $\overline{f}$ is one of the α_i's; that is, $f = g\widetilde{\alpha}_i$ for some $g \in G$. Then $g\widetilde{\alpha}_i(C) \cap C \neq \emptyset$. This implies

$$g(D_i) \cap D_i = g(\widetilde{\alpha}_i(C) \cup C) \cap (\widetilde{\alpha}_i(C) \cup C) \supset g\widetilde{\alpha}_i(C) \cap C \neq \emptyset.$$

Consequently we have

$$\begin{aligned}\{f \in G \cdot \Pi : f(C) \cap C \neq \emptyset\} &\subset \bigcup_i \{g\widetilde{\alpha}_i \in G \cdot \Pi : g(D_i) \cap D_i \neq \emptyset\} \\ &= \bigcup_i \{g \in G : g(D_i) \cap D_i \neq \emptyset\}\widetilde{\alpha}_i.\end{aligned}$$

Since (G, X) is proper, the last term is compact, and the proof is complete. □

1.6.2 Example. Let $X = \mathbb{R}^2$, $G = \mathbb{R}$ and acting on $\mathbb{R} \times \mathbb{R}$ by translation on the first factor. Let $\Pi = \mathbb{Z}^2$ with generators $(1, 1)$ and $(1, \sqrt{2}) \in \mathbb{R} \times \mathbb{R}$ and act on $\mathbb{R}^2$ by translating as a subgroup of $\mathbb{R}^2$. The actions commute and $\Pi \cap G = 0 \in \mathbb{R}$. Both G and Π act properly on $\mathbb{R}^2$ with quotients $W = \mathbb{R}$ and $\Pi\backslash\mathbb{R}^2 = T^2$, respectively. Neither of the induced actions $(\frac{G\times\Pi}{G}, G\backslash X) = (\mathbb{Z}^2, W) = (\mathbb{Z}^2, \mathbb{R})$ or $(G, \Pi\backslash\mathbb{R}^2) = (\mathbb{R}, T^2)$ are locally proper.

1.7. Covering Spaces

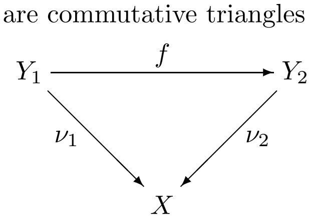
Covering spaces play a crucial role in this book. The subject is very familiar and to set notation, we adopt the terminology of covering projections as in [**?**, §2.5 and §2.6]. Unless specified otherwise, a *covering projection* will be a map

$$\nu : (Y, y) \longrightarrow (X, x),$$

where Y is path-connected, X is locally path-connected, and semi-1-connected (each x has a neighborhood U such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial) such that

(i) ν is onto,
(ii) each x has a neighborhood such that $\nu^{-1}(U)$ is a disjoint collection of open sets each of which maps, by ν, homeomorphically onto U (Y *evenly covers* X).

The category of covering spaces has objects which are covering projections $\nu : Y \to X$ and morphisms are commutative triangles



where ν_1 and ν_2 are covering projections. In this category, *every morphism is also a covering projection*. If f is a homeomorphism, the covering spaces Y_1 and Y_2 are called *equivalent*. If $Y_1 = Y_2$ and f is a homeomorphism, we call f a *covering transformation*. It is well known that f is determined by what it does to a single point.

1.7.1(Construction of covering spaces). Let K be a subgroup of $\pi_1(X,x)$. We define

$X_K =$ the covering space of X associated with the subgroup K

so that $\pi_1(X_K, \widehat{x}_0) = K$ as follows:

covering space!construction
universal covering space

Denote by $P(X,x)$ the set of all paths in X with initial point x. Define an equivalence relation on $P(X,x)$ by $p_1 \sim_K p_2$ if and only if $p_1(1) = p_2(1)$ and the closed loop

$$p_1 * \overline{p_2} = \begin{cases} p_1(2t), & 0 \le t \le 1/2, \\ p_2(2-2t), & 1/2 \le t \le 1 \end{cases}$$

represents an element of K. Finally let

$$X_K = P(X,x)/ \sim_K .$$

A basis for the topology of X_K consists of the collection $\{\langle p, U\rangle\}$, where U is open in X, p is a path such that $p(1) \in U$, $p(0) = x$, and $\langle p, U\rangle$ denotes all equivalence classes of paths having a representative of the form $p * p'$, where $p'(0) = p(1)$ and $p'(t) \in U$. (If X is Hausdorff, this topology is the same as that induced by the identification topology where $P(X,x)$ is given the compact-open topology).

The map $p \mapsto p(1)$ induces the projection map $\nu : X_K \to X$. If $\widehat{x}_0 \in X_K$ denotes the equivalence class of the constant path at x, then $\nu_* : \pi_1(X_K, \widehat{x}_0) \to \pi_1(X,x)$ has image K.

If K is a *normal* subgroup, then a free and locally proper action of the quotient group $\pi_1(X,x)/K$ on X_K can be defined to show that $\nu : X_K \to X$ is a regular covering: Recall that, the group operation of $\pi_1(X,x)$ is defined by juxtaposition. That is, given loops $\ell_1(t), \ell_2(t)$ at x, $[\ell_1\ell_2]$ is the homotopy class of

$$(\ell_1 * \ell_2)(t) = \begin{cases} \ell_1(2t), & 0 \le t \le 1/2, \\ \ell_2(2t-1), & 1/2 \le t \le 1. \end{cases}$$

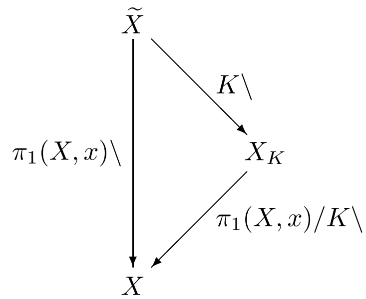
Let $\alpha \in \pi_1(X,x)/K$ and $\widehat{y} \in X_K$. Take a path p in $P(X,x)$ representing $\widehat{y}$, and a loop ℓ in $P(X,x)$ representing α. Define

$$\alpha \cdot \widehat{y} = \text{the “}\sim\text{” equivalence class represented by } \ell * p \in P(X,x).$$

Suppose $p' \in P(X,x)$ and $\alpha' \in P(X,x)$ are other elements representing $\widehat{y}$ and α respectively. Then $p * \overline{p'}$ and $\ell * \overline{\ell'}$ represent elements of K. Thus, $(\ell * p) * (\overline{\ell' * p'}) \simeq \ell * (p * \overline{p'}) * \overline{\ell'}$. Since $[p * \overline{p'}] \in K$ and K is normal in $\pi_1(X,x)$, $[\ell * (p * \overline{p'}) * \overline{\ell'}] = [\ell] * [p * \overline{p'}] * [\overline{\ell'}] = ([\ell] * [p * \overline{p'}] * [\ell]^{-1})([\ell * \ell']^{-1}) \in K$. Consequently, the two paths $\ell * p$ and $\ell' * p'$ represent the same point of X_K. Finally, K acts trivially on all of X_K so that $\pi_1(X,x)/K$ is the group of covering transformations.

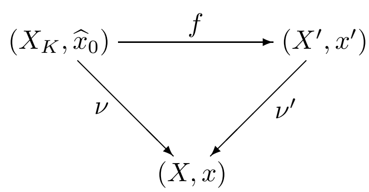
If, in particular, K is the trivial group, X_K is the *universal covering space* $\widetilde{X}$. In general, we have a commuting diagram of spaces with free, locally proper lifting-sequence

actions:



Let $\nu' : (X', x') \to (X, x)$ be a covering projection such that ${\nu'}_* : \pi_1(X', x') = K \subset \pi_1(X, x)$. Let X_K be the covering space of X associated with the based space (X, x). If $p \in P(X, x)$, then its "$\sim$" equivalence class represents a point $\widehat{y} \in X_K$. We may lift the path p to a path $\widetilde{p}$ in X' with initial point x' and terminal point $\widetilde{p}(1)$. The projections $\nu(\widehat{y}) = p(1)$ and $\nu'(\widetilde{p}(1))$ are equal.

1.7.2 Exercise. Show that the assignment $\widehat{y} \xrightarrow{f} \widetilde{p}(1)$ defines an equivalence

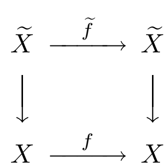


We may now transfer all of our constructions with X_K to X' by just lifting the paths used in the construction of X_K to X'.

1.8. Lifting Group Actions to Covering Spaces

In this section, we shall describe how we may lift actions on a space X to actions on covering spaces of X.

1.8.1(Lifting sequence). Let $\nu : \widetilde{X} \longrightarrow X$ be the universal covering projection, and $f : X \longrightarrow X$ a homeomorphism. Then f lifts to a homeomorphism $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{X}$, making the diagram



commutative. Since $\pi_1(X, x)$ acts on $\widetilde{X}$ effectively as the group of covering transformations, we view $\pi_1(X, x)$ as a subgroup of $\mathrm{TOP}(\widetilde{X})$, the group of all self-homeomorphisms of $\widetilde{X}$.

Since each element $\alpha \in \pi_1(X, x)$ induces the identity map on X, there are many such lifts $\widetilde{f}$; namely, $\widetilde{f} \circ \alpha$, for all $\alpha \in \pi_1(X, x)$, are lifts of f.

Let $\rho : G \to \mathrm{TOP}(X)$ be an action of G on X, and let $K \subset G$ be the kernel of ρ. That is, K is the ineffective part of the action. The group of all liftings of elements of G is denoted by G^*, and fits the following commuting diagram

action!lifting exact sequence
lifting exact sequence
extended-G-lifting

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X,x) & \longrightarrow & \pi_1(X,x)\times K & \longrightarrow & K & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle =} & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \pi_1(X,x) & \longrightarrow & G^* & \longrightarrow & G & \longrightarrow & 1 \\
 & & & & \big\downarrow{\scriptstyle \rho^*} & & \big\downarrow{\scriptstyle \rho} & & \\
 & & & & \mathrm{TOP}(X) & \xrightarrow{=} & \mathrm{TOP}(X) & &
\end{array}$$

In fact, the lifts of all of $\mathrm{TOP}(X)$ is $N(\pi_1(X,x)) = N_{\mathrm{TOP}(\widetilde{X})}(\pi_1(X,x))$, the normalizer of $\pi_1(X,x)$ in $\mathrm{TOP}(\widetilde{X})$. Note that $N(\pi_1(X,x)) \subset \mathrm{TOP}(\widetilde{X})$ is the group of all homeomorphism of $\widetilde{X}$ which induce homeomorphisms on X. The normalizer acts on $\pi_1(X,x)$ by conjugation yielding a homomorphism $\theta : N(\pi_1(X,x)) \longrightarrow \mathrm{Aut}(\pi_1(X,x))$. This induces a homomorphism into the outer automorphism group $\psi : \mathrm{TOP}(X) \longrightarrow \mathrm{Out}(\pi_1(X,x))$ yielding the commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X,x) & \longrightarrow & N(\pi_1(X,x)) & \longrightarrow & \mathrm{TOP}(X) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \psi} & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(X,x)) & \longrightarrow & \mathrm{Aut}(\pi_1(X,x)) & \longrightarrow & \mathrm{Out}(\pi_1(X,x)) & \longrightarrow & 1
\end{array}$$

Now with the action $\rho : G \to \mathrm{TOP}(X)$, we pull back the top exact sequence to get (see section **??**

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X,x) & \longrightarrow & G^* & \longrightarrow & G & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \rho^*} & & \big\downarrow{\scriptstyle \rho} & & \\
1 & \longrightarrow & \pi_1(X,x) & \longrightarrow & N(\pi_1(X,x)) & \longrightarrow & \mathrm{TOP}(X) & \longrightarrow & 1
\end{array}$$

The top exact sequence is called the *lifting exact sequence* of the group action (G, X); the action $(G^*, \widetilde{X})$ is called the *extended-G-lifting* of (G, X). When the lifting exact sequence splits (i.e., $G^* = \pi_1(X,x) \rtimes G$), the action $(G, \widetilde{X})$ is called a *G-lifting* of (G, X).

1.8.2 Exercise. Show that if $\mathbb{Z}_2$ acts on the circle group $G = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ by $z \mapsto \overline{z}$, then $G^* = \mathbb{Z} \rtimes \mathbb{Z}_2$ acts on $\mathbb{R}$ as in Example 1.4.7.

1.8.3 Proposition. *Suppose G is a discrete group, acting properly on a completely regular space X. Then the extended-G-lifting G^* on the universal covering space $\widetilde{X}$ is proper.*

Proof. The universal covering will be completely regular. Since $G\backslash X$ is regular by the property 1.2.3(5), and $G^*\backslash \widetilde{X} = G\backslash X$, we need only show that $(G^*, \widetilde{X})$ is locally proper. Let $\widetilde{x} \in \widetilde{X}$ and $x = \nu(\widetilde{x})$, where $\nu : \widetilde{X} \to X$ is the covering projection.

Take a G_x-slice U at x so small that $\nu^{-1}(U)$ is an even cover and also so that $G \times_{G_x} U$ forms a tube about the G-orbit. If $gx \neq x$, then gU is a slice about gx and

$\nu^{-1}(gU)$ is an even covering of gU. Thus we get a tube around $G^*(\widetilde{x})$ by taking $\bigcup_{g\in G} \nu^{-1}(gU)$. This is a disjoint union of homeomorphic copies of U, one for each point on the orbit $G^*\widetilde{x}$. Let W be the lift of U to $\widetilde{x}$. Let $1 \to \Pi \to G_x^* \to G_x \to 1$ be the lifting sequence of the stabilizer G_x. Note that G_x is finite because (G, X) is proper. For $h \in \Pi \subset G_x^*$, if $hW \cap W \neq \emptyset$, then $h = \mathrm{id}$. Therefore, the number of elements $h \in G_x^*$ such that $hW \cap W \neq \emptyset$ is exactly the same as the order of G_x, finite. □

1.8.4 Exercise. Suppose K is a normal subgroup of $\pi_1(X, x)$ and conjugation by G^* on $\pi_1(X, x)$ leaves K invariant. Then there is an induced action of G^*/K on $K\backslash\widetilde{X}$. This action is proper if G is discrete acting properly on a completely regular X. A proof follows from previous Exercise and 1.2.3 (8).

1.8.5 Remark. The lifting exact sequence can be made very explicit. Take x as a base point, and for each $g \in G$, take a path $p_g \in P(X, x)$ so that $p_g(1) = gx$. For a point in $b \in \widetilde{X}$, take a path $p_b \in P(X, x)$ representing b. A lift of g can be given as follows: Define a path

$$(p_g * (g \cdot p_b))(t) = \begin{cases} p_g(2t), & 0 \le t \le 1/2, \\ g(p_b(2t-1)), & 1/2 \le t \le 1 \end{cases}$$

so that $p_g * (g \cdot p_b) \in P(X, x)$. Then the map $p_b \mapsto p_g * (g \cdot p_b)$ defines a map $\widetilde{X} \to \widetilde{X}$, which is a lift of $g : X \to X$. A careful description and construction of an explicit lifting exact sequence as an extension of $\pi_1(X, x)/K$ and G can be found in [**?**, §2]. The explicit and rather technical construction enables one to give a detailed analysis of the group structure of G^* in terms of the choice of the paths p_g. The explicitness then leads to several interesting applications.

If we assume that G has a fixed point at x, then p_g can be chosen to be the trivial path and one constructs the extended-G-lifting G^* as $\pi_1(X, x) \rtimes_\varphi G$. The automorphism $\varphi : G \longrightarrow \mathrm{Aut}(\pi_1(X, x))$ is induced from sending a loop class $[\ell(t)] \in \pi_1(X, x)$ to the loop class $[g(\ell(t))] \in \pi_1(X, x)$. Thus, $\varphi(g)([\ell(t)]) = [g(\ell(t))]$. We examine this type of lifting in the next section.

1.9. Lifting an action of G when G has a fixed point

In this section, $(G, (X, x), \varphi)$ denotes a topological group G acting on X together with a *base point x which is fixed under the action of G*. Each element $g \in G$, considered as a continuous map $g : (X, x) \to (X, x)$, induces an automorphism $g_* : \pi_1(X, x) \longrightarrow \pi_1(X, x)$ by $g_*[\ell(t)] = [g \cdot \ell(t)]$. Clearly, $g \mapsto g_*$ defines a homomorphism

$$G \longrightarrow \mathrm{Aut}(\pi_1(X, x)).$$

Observe—Frank?? that this homomorphism factors through $G \longrightarrow G/G_0$, where G_0 is the path-component of G containing the identity element.

Furthermore, since $x \in X^G$, G operates naturally on the space of paths issuing out of x, $P(X, x)$. We wish to explicitly describe the liftings of this action to covering spaces. We shall always denote the base point of a covering space by the equivalence class of the trivial path and denote it by $\widehat{x}$. We will denote $\varphi(g)y$ by $g \cdot y$, for every $y \in X$.

1.9.1 THEOREM. [?, 3.1] *If $K \subset \pi_1(X,x)$ is invariant under the action of G on $\pi_1(X,x)$, then there is a covering action*

$$(G,(X_K,\widehat{x})) \xrightarrow{\ \nu\ } (G,(X,x))$$

for which ν is G-equivariant. If K is normal in $\pi_1(X,x)$, then

$$g\cdot(\alpha\widehat{y}) = g_*(\alpha)(g\cdot\widehat{y})$$

for all $g \in G$, $\widehat{y} \in X_K$ and $\alpha \in \pi_1(X,x)$. The extended-G-lifting on X_K is a semi-direct product $(\pi_1(X,x)/K)\rtimes G$ which operates as

$$(\alpha,g)(\widehat{y}) = \alpha(g\cdot\widehat{y}).$$

Thus,

$$1 \longrightarrow \pi_1(X,x)/K \longrightarrow (\pi_1(X,x)/K)\rtimes G \longrightarrow G \longrightarrow 1$$

is exact, where $\pi_1(X,x)/K$ is the deck transformation group of the covering space $X_K \longrightarrow X$.

PROOF. G acts on $P(X,x)$ leaving the trivial path fixed. The G-invariance of K allows us to introduce the action on $(X_K,\widehat{x})$: If $p(t)$ represents the equivalence class of a point $\widehat{y}$ in X_K (a path issuing from $x \in X$), then $g\cdot\widehat{y}$ is the equivalence class of $g\cdot p(t)$.

To show that this is well-defined, suppose $q(t)$ represents the same equivalence class of the point $\widehat{y}$ in X_K; that is, $[p(t)*\overline{q}(t)] \in K$. Then,

$$[g\cdot p(t)*\overline{g\cdot q(t)}] = [g\cdot(p(t)*\overline{q}(t))] \in g_*[(p(t)*\overline{q}(t))] \in g_*(K) = K,$$

which shows $g\cdot p(t)$ and $g\cdot q(t)$ represent the same point in X_K.

Now let $\ell(t)$ be a loop at x (representing α), then

$$\begin{aligned} g\cdot(\ell*p)(t) &= \begin{cases} g\cdot\ell(2t), & 0\le t\le 1/2,\\ g\cdot p(2t-1), & 1/2\le t\le 1.\end{cases}\\ &= (g\cdot\ell*g\cdot p)(t)\end{aligned}$$

which yields the formulas.

Let us now define an action of $(\pi_1(X,x)/K)\rtimes G$ on X_K by

$$(\alpha,g)\widehat{y} = \alpha(g\cdot\widehat{y}).$$

This defines an action because

$$\begin{aligned}(\beta,h)(\alpha,g)(\widehat{y}) &= (\beta,h)(\alpha(g\cdot\widehat{y}))\\ &= \beta(h\cdot(\alpha(g\cdot\widehat{y})))\\ &= \beta(h_*(\alpha)(h\cdot(g\cdot\widehat{y})))\\ &= (\beta h_*(\alpha))((hg)\cdot\widehat{y})\\ &= (\beta h_*(\alpha),hg)(\widehat{y}).\end{aligned}$$

Also

$$\nu((\alpha,g)\widehat{y}) = \nu(\alpha(g\cdot\widehat{y})) = \nu(g\cdot\widehat{y}) = g\cdot\nu(\widehat{y}).$$

The map ν is equivariant under the homomorphism which projects $(\pi_1(X,x)/K)\rtimes G$ onto its second coordinate. The lifting of G described above is just the restriction to $1\rtimes G \subset (\pi_1(X,x)/K)\rtimes G$. This is clearly the full lifting sequence of all lifts of G-actions covering the given G-action. □

1.9.2 PROPOSITION. *Suppose G acts on X, leaving the normal subgroup K of $\pi_1(X,x)$ invariant. Put $F = X^G$, and $E = (X_K)^G$. Let $\alpha \in \pi_1(X,x)/K$, $\alpha \neq 1$. Then $g_*(\alpha) = \alpha$ for all $g \in G$ if and only if $\alpha E \cap E \neq \emptyset$. In particular, $\alpha E = E$.*

PROOF. Suppose $g_*(\alpha) = \alpha$. Let $\widehat{y} \in E$. Then $g \cdot (\alpha\widehat{y}) = g_*(\alpha)(g \cdot \widehat{y}) = g_*(\alpha)(\widehat{y}) = \alpha\widehat{y}$. This implies $\alpha E \subset E$. For $\widehat{y} \in E$, consider $\alpha^{-1}\widehat{y}$. Then $g \cdot (\alpha^{-1}\widehat{y}) = g_*(\alpha^{-1})(g \cdot \widehat{y}) = \alpha^{-1}\widehat{y}$ so that $\alpha^{-1}E = E$. Consequently, $\alpha E = E$.

Conversely, suppose there exists $\widehat{y} = \alpha\widehat{z}$, with $\widehat{y} \in E$ and $\widehat{z} \in E$. Then $\alpha(\widehat{z}) = \widehat{y} = g \cdot (\widehat{y}) = g \cdot (\alpha\widehat{z}) = g_*(\alpha)(g \cdot \widehat{z}) = g_*(\alpha)(\widehat{z})$. Since α, $g_*(\alpha)$ are elements of deck transformation, $g_*(\alpha) = \alpha$. □

1.9.3 COROLLARY. *Suppose G acts on X, leaving the normal subgroup K of $\pi_1(X,x)$ invariant. Put $F = X^G$, and $E = (X_K)^G$. If $\Gamma = \{\alpha \in \pi_1(X,x)/K : g_*(\alpha) = \alpha,$ for all $g \in G\}$, then $\Gamma\backslash E = \nu(E) \subset X^G$.*

1.9.4 LEMMA. *Suppose G acts on X, leaving the normal subgroup K of $\pi_1(X,x)$ invariant. Put $F = X^G$, and $E = (X_K)^G$. Let $E_{\widehat{x}}$ denote the path component of E that contains $\widehat{x}$ and F_x the path component of $F = X^G$ that contains x. Then $\nu(E_{\widehat{x}}) = F_x$.*

PROOF. Clearly $\nu(E_{\widehat{x}}) \subset F_x$. Suppose $y \in F_x$ and p is a path in F_x from x to y. Then the lift of this path is a path starting at $\widehat{x}$ and ending at $\widehat{y}$, where $\nu\widehat{y} = y$. Since $g \circ p(t) = p(t)$ for each t, this lift is in $E_{\widehat{x}}$. So ν maps $E_{\widehat{x}}$ onto F_x. □

1.9.5 COROLLARY. *Suppose G acts on X, leaving the normal subgroup K of $\pi_1(X,x)$ invariant. Put $F = X^G$, and $E = (X_K)^G$. If E is path connected, then the image of $\pi_1(F,x) \longrightarrow \pi_1(X,x)/K$ is the subgroup*

$$\Gamma = \{\alpha \in \pi_1(X,x)/K : \ g_*(\alpha) = \alpha \text{ for all } g \in G\} = (\pi_1(X,x)/K)^G.$$

If, in addition, $\Gamma = \pi_1(X,x)/K$, then $\nu^{-1}(F_x) = E$.

PROOF. We have $\Gamma\backslash E = F_x$, the path component of F containing x. For $\alpha \in \Gamma$, if $\alpha\widehat{x} = \widehat{y}$, then $\widehat{y} \in E$ and there is a path $\alpha(t)$ in E starting at $\widehat{x}$ and ending at $\widehat{y}$ and representing α. Then $\nu(\alpha(t))$ is a loop in F_x based at x. This represents a non-trivial element of $\pi_1(X,x)/K$ if and only if the image of this class does not lie in K, or equivalently, $\widehat{x} \neq \widehat{y}$. Conversely, a loop ℓ in F_x based at x, lifts to a path in E starting at $\widehat{x}$ and ending at $\alpha(\widehat{x})$, where $\alpha \in \text{Image}(\pi_1(F_x,x))$ in $\pi_1(X,x)/K$ is represented by ℓ. If $\Gamma = \pi_1(X,x)/K$, then $\nu^{-1}(F_x) = E$ for $\alpha(\widehat{x})$ is in E, for each $\alpha \in \Gamma$. □

1.9.6 PROPOSITION. *Suppose G is compact and acts on the locally compact Hausdorff X with fixed point, leaving the normal subgroup K of $\pi_1(X,x)$ invariant. Then $(\pi_1(X,x)/K) \rtimes G = G^*$ acts properly on X_K.*

PROOF. Let C be a compact subset of X_K. Then we want to show that closure $\{(\alpha,g) \in G^* : (\alpha,g)C \cap C \neq \emptyset\}$ is compact. We have $\{\alpha \in \pi_1(X,x)/K : \alpha(G \cdot C) \cap (G \cdot C) \neq \emptyset\}$ is finite since $\pi_1(X,x)/K$ acts properly on X_K and $G \cdot C$ is compact. (It acts locally properly and X is Hausdorff). Every point in $(\alpha,g)C$ for evaluation map

any $(\alpha, g) \in G^*$ is of the form $\alpha G \cdot C$. If $(\alpha, g)C \cap C \neq \emptyset$, then $\alpha(G \cdot C) \cap (G \cdot C) \neq \emptyset$. Hence we are essentially counting a finite number of copies of G. □

1.9.7 Example. [**?**, 3.7] Let X be a 2-sphere S^2 and consider the rotation of 180^o around the north-south axis. This action is equivariant with respect to the antipodal map and so induces an action of $\mathbb{Z}_2$ on the projective plane P_2 with fixed point equal to an isolated point (the image of the poles) and a circle which generates the fundamental group (the image of the equator). If we now lift this induced action on P_2 to its universal covering space, using the isolated fixed point as a base point, we obtain an action equivalent to the original rotation about the polar axis. Notice that $E \neq \pi^{-1}(F)$ and E consists of only two points; $\pi_1(\text{pt}) \to \pi_1(P_2)$ is surely not onto. On the other hand, if we were to lift the action but based at one of the points on the circle we would induce an action on S^2 equivalent to reflecting across the equator. In this case since E projects onto the component containing the base point, and $\pi_1(F_x) \to \pi_1(P_2)$ is onto.

The reader may wish to further explore liftings of groups with fixed points to covering spaces in [**?**] and in [**?**, Appendix]. For example, it is shown that the number of connected components of the fixed set for a p-group action (G, M) on spherical space-forms or aspherical manifolds is in $1-1$ correspondence with $H^1(G; \pi_1(X))$, the set of equivalence classes (cohomology classes) of crossed homomorphisms of G into the fundamental group $\pi_1(X)$.

1.10. Evaluation Homomorphism

We are also interested in getting a more precise description of lifting group actions when G is path-connected.

Let G be a path-connected group acting on a path-connected Hausdorff space X. Fix $x \in X$. The evaluation map $\text{ev}^x : (G, e) \to (X, x)$ is defined by

$$\text{ev}^x(g) = g \cdot x\,.$$

This induces the evaluation homomorphism

$$\text{ev}^x_{\#} : \pi_1(G, e) \to \pi_1(X, x)\,.$$

The main reference for the following is [**?**, §4]. We will see that in general the image of $\pi_1(G, e)$ in $\pi_1(X, x)$ is a central subgroup of $\pi_1(X, x)$ and independent of the base point.

1.10.1 Lemma. *Let $g \in P(G, e)$ and $p \in P(X, x)$ (paths emanating from e and x, respectively). Then the three paths*

$$(A) \quad \begin{cases} g(2t) \cdot x, & 0 \leq t \leq 1/2, \\ g(1) \cdot p(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

$$(B) \quad g(t) \cdot p(t), \quad 0 \leq t \leq 1,$$

$$(C) \quad \begin{cases} p(2t), & 0 \leq t \leq 1/2, \\ g(2t-1) \cdot p(1), & 1/2 \leq t \leq 1 \end{cases}$$

are homotopic by fixed end-point homotopies.

PROOF. Schematically, we will have a diagram

Figure 2: middle:$g(t) \cdot p(t)$

Introduce a path in $P(G, e)$ by

$$(g * c_{g(1)})(t) = \begin{cases} g(2t), & 0 \le t \le 1/2, \\ g(1), & 1/2 \le t \le 1 \end{cases}$$

and a path in $P(X, x)$ by

$$(c_x * p)(t) = \begin{cases} x, & 0 \le t \le 1/2, \\ p(2t-1), & 1/2 \le t \le 1, \end{cases}$$

(c denotes the constant path). Now $g * c_{g(1)} \simeq g$ implies $(g * c_{g(1)}) \cdot p \simeq g \cdot p$ relative ∂I. But $(g * c_{g(1)}) \cdot p = (g \cdot p) * g(1) \cdot p$ is the path labeled (A), while $g \cdot p$ is the path labeled (B). Thus the path(A) is homotopic to (B) via a fixed end-point homotopy. A similar argument relates (C) to (B). □

1.10.2 THEOREM. *The image of* $\mathrm{ev}^x_\# : \pi_1(G, e) \to \pi_1(X, x)$ *is a central subgroup of* $\pi_1(X, x)$ *which is independent of choices of* x.

PROOF. Suppose $g \in P(G, e)$ and $p \in P(X, x)$ are closed loops. Then $\mathrm{ev}^x_\#$ is induced by

$$g(t) \mapsto g(t) \cdot x.$$

Noting that $g(1) = e$ and $p(1) = x$, we have

$$g(t) \cdot x * p(t) = \begin{cases} g(2t) \cdot x, & 0 \le t \le 1/2, \\ g(1) \cdot p(2t-1), & 1/2 \le t \le 1 \end{cases}$$

while

$$p(t) * (g(t) \cdot x) = \begin{cases} p(2t), & 0 \le t \le 1/2, \\ g(2t-1) \cdot p(1), & 1/2 \le t \le 1. \end{cases}$$

In view of Lemma 1.10.1, these two loops represent the same element of $\pi_1(X, x)$,

$$[g] \cdot [p] = [p] \cdot \mathrm{ev}^x_\#([g]),$$

thus the image of $\mathrm{ev}^x_\#$ lies in the center of $\pi_1(X, x)$.

Let y be another point in X and C a path from x to y. Then C induces an isomorphism $C_\# : \pi_1(X, y) \longrightarrow \pi_1(X, x)$ by sending a loop ℓ based at y to a loop $C * \ell * \overline{C}$ based at x. If $\ell' = g(t) \cdot x$, with $\ell = g(t) \cdot y$, then $C * \ell * \overline{C} \simeq \ell'$ (for the homotopy just moves along C). Choosing a different path C' will send ℓ to a loop homotopic to ℓ'. Therefore, $\mathrm{ev}^x_*(\pi_1(G, e))$ is independent of choice. □

1.11. Lifting Connected Group Actions

1.11.1 Theorem ([**?**, §4], [**?**, Chapter I, §9]). *let G be a path-connected topological group acting on a space X which admits covering space theory. Let K be a normal subgroup of $\pi_1(X,x)$ containing the image of $\mathrm{ev}^x_\# : \pi_1(G,e) \to \pi_1(X,x)$ and put $Q = \pi_1(X,x)/K$. Then*

(1) *The G-action on X lifts to a G-action on X_K which commutes with the covering Q-action.*
(2) *If in addition, X is completely regular, G is a connected Lie group acting properly on X, then the G-action on X_K is proper and the induced action of Q on $W = G\backslash X_K$ is also proper.*♣

Consequence of previous theorems??

Proof. (1) The G-action on X_K is described as follows. Given $u \in G$ and $\widehat{x} \in X_K$, select a path $g \in P(G,e)$ with $g(1) = u$, and choose a path p which represents $\widehat{x}$. We define $u \cdot \widehat{x}$ to be the common equivalence class of the three paths listed in Lemma 1.10.1. In particular, $u\cdot\widehat{x}$ is represented by a path $(g\cdot x)*(u\cdot p)$ connecting x and $u \cdot y$, where $y = p(1)$. Suppose $g' \in P(G,e)$ also has $g'(1) = u$, and p' also represents $\widehat{x}$. Then

$$\begin{aligned}((g\cdot x)*(u\cdot p))*\overline{(g'\cdot x)*(u\cdot p')} &\simeq (g\cdot x)*((\overline{g'}\cdot x)*(g'\cdot x))*(u\cdot p)*(u\cdot\overline{p'})*(\overline{g'}\cdot x)\\ &\simeq ((g*\overline{g'})\cdot x)*((g'\cdot x)*u\cdot(p*\overline{p'})*(\overline{g'}\cdot x)).\end{aligned}$$

Since K contains $\mathrm{image}\{\mathrm{ev}^x_\# : \pi_1(G,e) \to \pi_1(X,x)\}$, $(g*\overline{g'})\cdot x$ represents an element of K.

Now consider the path $(g'\cdot x)*u\cdot(p*\overline{p'})*(\overline{g'}\cdot x) = (g'\cdot x)*g'(1)\cdot(p*\overline{p'})*(\overline{g'}\cdot x)$. The map

$$F(t,s) = (g'(st)\cdot x)*g'(s)\cdot(p(t)*\overline{p'}(t))*(\overline{g'}(st)\cdot x)$$

gives a homotopy from $F(t,0) = p(t)*\overline{p'}(t)$ to $F(t,1) = (g'\cdot x)*g'(1)\cdot(p*\overline{p'})*(\overline{g'}\cdot x)$. Also, since $p*\overline{p'}$ represents an element of K, the definition of $u\cdot\widehat{x}$ does not depend on the choices of g and p.

Recall that the covering action of Q on X_K is induced from the action of $\pi_1(X,x)$ on $\widetilde{X}$ by juxtaposition. Suppose $\alpha \in \pi_1(X,x)$ is represented by a closed loop ℓ at x. By Lemma 1.10.1, the part $0 \le t \le 3/4$ of the two paths

$$\begin{cases} g(2t)\cdot x, & 0 \le t \le 1/2,\\ u\cdot\ell(4t-2), & 1/2 \le t \le 3/4,\\ u\cdot p(4t-3), & 3/4 \le t \le 1;\end{cases}$$

and

$$\begin{cases} \ell(2t), & 0 \le t \le 1/2,\\ g(4t-2)\cdot x, & 1/2 \le t \le 3/4,\\ u\cdot p(4t-3), & 3/4 \le t \le 1.\end{cases}$$

are equivalent so that $u\cdot(\alpha\cdot\widehat{x}) = \alpha\cdot(u\cdot\widehat{x})$ in X_K. This shows the G action and Q action on X_K commute with each other. We have not used complete regularity or that G is a Lie group so far, but these assumptions are needed for (2).

(2) Let the equivalence class of $\beta : (I,0,1) \to (X,x,y)$ represent a point $\widehat{y}$ in X_K. We will show that the G-action on X_K is locally proper at $\widehat{y}$. Since the G-action on X is proper, we can choose a slice S_y at y which we can assume to be path-connected and small enough so that when we lift a neighborhood of y to X_K at $\widehat{y}$, the slice lifts homeomorphically to $\widehat{S}$. The stabilizer of $\widehat{y}$ is $G_{\widehat{y}} \subset G_y$.

By continuity and the G-equivariance of ν, $\widehat{S}$ is $G_{\widehat{y}}$-invariant. In fact, the map $G\widehat{S} \xrightarrow{\overline{\nu}} GS_y \xrightarrow{f} G/G_y$ is G-invariant and $(f \circ \overline{\nu})^{-1}(G_y)$ gives a G_y-kernel. (Here f is the G-invariant map defining the G_y-slice at y and $\overline{\nu}$ denotes the restriction of ν to $G\widehat{S}$). $(f \circ \overline{\nu})^{-1}(G_y)$ will be disjoint copies of $\widehat{S}$, one for each element of $G_y/G_{\widehat{y}}$ which is a finite group since G_y is compact, $G_{\widehat{y}}$ is a closed subgroup and $G_y/G_{\widehat{y}}$ is isomorphic to a subgroup of the discrete group Q. In other words, we get a G_y-slice (an H-slice, $H = G_y$), $\bigcup_{g \in G_y/G_{\widehat{y}}} g\widehat{S})$, in X_K. This slice $G_y/G_{\widehat{y}}$ disjoint covering of the slice S_y at $y \in X$.

Now since X is completely regular, we need only show that (G, X_K) is locally proper at the arbitrary point $\widehat{y}$. But this is clear as $G \times_{G_{\widehat{y}}} \widehat{S}$ is an open path-connected G-invariant tube about the orbit $G\widehat{y}$ which covers the tube $G \times_{G_y} S_y$. Therefore we have $\widehat{S}$ is a $G_{\widehat{y}}$-slice at $\widehat{y}$.

The action of G^*, the group of homeomorphisms generated by the commuting groups G and the covering transformations Q is also locally proper because $(f \circ \nu)^{-1}(G \times_{G_y} S_y)$ is a G^*-invariant disjoint union of components each homeomorphic to the component $GS_{\widehat{y}} = G\widehat{S}$. Consequently, the action of Q on $W = G\backslash X_K$ is also proper by 1.2.3 (8). □

1.11.2. If $\nu : (X', x') \to (X, x)$ is a covering projection of the G-space X with $\nu_*(\pi_1(X', x')) = K \supset \mathrm{ev}_*^x(\pi_1(G, e))$, then we can construct the lift of the G-action on X' by lifting the paths used in the construction of (G, X_K) to X'. We may then easily see what happens if we choose a different base point to lift paths.

Choose base points $y' \in X'$ such that $\nu(y') = y \in X$. Take a path $\beta : (I, 0, 1) \longrightarrow (X', y', x')$. If b is a point in X', then let $\gamma : (I, 0, 1) \longrightarrow (X', x', b)$. Then gb is given by the lift of the path $\nu(\gamma(t)) * g(t) \cdot \nu(\gamma(1))$ with initial point x'. The point b can also be represented by the end point of $\beta(t) * \gamma(t)$. Then $g(b)$ using y' and y as base points is represented by the lift of the paths $\gamma(\beta(t) * \gamma(t)) * g(t)\nu(\gamma(1))$ with initial point y'. Thus we see that we have the same image gb. *So our lifting construction can be described at* any base point *and the lifted G-action is independent of the choice of the base point.*

1.11.3 Remark. Suppose G is path-connected and acts on X with fixed point x, and on $\pi_1(X, x)$ leaving the normal subgroup K invariant. Then the lifted action of G described in section 1.10 and section 1.11 are identical. In particular, G^* is just $G \times Q$, since it is a commuting semi-direct product.

1.11.4(Examples of lifting actions). (cf. Example 1.9.7) (1) Consider the standard action of S^1 on the 2-sphere S^2 by rotation around the axis joining the north and south poles. The action of $\mathbb{Z}_2$ by antipodal map commutes with the S^1 action. Thus, the S^1 action induces an action on $\mathbb{R}P_2 = S^2/\mathbb{Z}_2$.

For the action $(S^1, \mathbb{R}P_2)$, the evaluation homomorphism

$$\mathrm{ev}_{\#}^{x_*} : \pi_1(S^1) = \mathbb{Z} \to \mathbb{Z}_2 = \pi_1(\mathbb{R}P_2)$$

is trivial. [So it lifts to the 2-sphere S^2, and it lifts to the standard rotations about the polar axis. The N and S poles project to the fixed point set.] The projective line (corresponding to the "boundary" of identification of the disk lifts to a semi-circular arc. On $\mathbb{R}P_2$ this orbit has $\mathbb{Z}/2$ stabilizer and so on $\mathbb{R}P_2$ when $e^{2\pi it}(a)$ has gone from a to a as $0 \leq t \leq \frac{1}{2}$ the lifted action has gone $\frac{1}{2}$ around the equator. On $\mathbb{R}P_2$ we have a section (an arc going from 0 to a) and on S^2, there is a section from N to S on which the covering transformation acts and "covers" the section on $\mathbb{R}P_2$. Check explicitly that the lifts are independent of the base points in $\mathbb{R}P_2$.

(2) $SO(3)$ action on $\mathbb{R}P_3$. Since $\mathbb{R}P_3 \approx \mathrm{SO}(3)$, SO(3) acts on $\mathbb{R}P_3 \cong \mathrm{SO}(3)$ as just left translation. Moreover $ev^x_{\#} : \mathbb{Z}_2 \to \mathbb{Z}_2$ is an isomorphism. So this action *cannot* be lifted to an action on S^3.

However, $S^3 = \mathrm{Spin}(3)$ doubly covers as a group SO(3), and Spin(3) is simply connected. So the non-effective action of S^3 on $\mathbb{R}P_3$ via $S^3 \to \mathrm{SO}(3)$ can be lifted to S^3. On S^3 it is a free transitive action.

(3) Let us try the S^1 action on the Klein bottle K which is just the induced S^1 action as we put together equivariantly two S^1 actions on two copies of the Möbius band.

By Van Kampen's theorem, $\{a, b, t \mid t = a^2 = b^{-2}\} = \{a, b \mid a^2b^2 = 1\}$. Now the center is generated by t which is also a generator of $\mathrm{ev}^x_{\#}(\pi_1(S^1))$. We can lift now all the way up to $K_{\mathbb{Z}}$ with $\pi_1(K_{\mathbb{Z}}) \cong \mathbb{Z}$, but no further. This is an annulus $K_{\mathbb{Z}} = S^1 \times \mathbb{R}$ and the action is free and just translates along the first factor.

The orbit space of this S^1 action is $\mathbb{R}$ and there is an action of $\pi_1(\mathrm{K})/\mathrm{im}(\mathrm{ev}^x_*) \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ on $\mathbb{R}$. This induced action on $\mathbb{R}$ is just the action of the infinite dihedral group: $(n, \epsilon)(x) = \epsilon x + n, \epsilon = \pm 1$. The orbit space is an interval which is identical with the orbit space of the S^1-action on the Klein bottle. The covering action on $S^1 \times \mathbb{R}^1$ is given by the isotropy groups on the Klein bottle $(n, \epsilon)(z, x) = (\epsilon z, \epsilon x + n)$.

1.11.5 Example. We will sketch what happens with S^1 actions on the connected sums of $S^2 \times S^1$. There is an action of S^1 on $S^2 \times S^1$ obtained by rotating S^2 about its north and south pole and acting trivially on the second factor. Clearly the orbit space is an $\{\mathrm{arc}\} \times S^1$, an annulus with the boundary corresponding to the 2 circles of fixed points. Conversely, if S^1 acts on a space such that an annulus is its orbit space and all orbits are free in its interior and fixed on the boundary, the action must be equivalent to that just described. For, over the interior, the action is free and so the space is a bundle over $S^1 \times (0, 1)$. All such principal S^1 bundles are trivial since they are classified by the elements of $H^2(S^1 \times \mathbb{R}; \mathbb{Z}) = 0$. Thus there is a section to this product space $S^2 \times \mathbb{R}^1$. With care, this section can be modified so that it extends to all of $S^1 \times I$. Basically we are then dealing with a free action on $S^1 \times (S^1 \times I)$ with the action just translation on the first factor. But we collapse each orbit over the points on $S^1 \times 0$ and $S^1 \times 1$ to a point. This gives us the required action on $S^2 \times S^1$. There is a 3-cell, invariant tubular neighborhood about a point on the fixed set. This can be represented as the inverse image of the orbit mapping of the shaded region in the orbit space as shown.

To "Appendix" to the end

Let us take 2 copies of $S^2 \times S^1$, remove the interiors of an invariant 3-ball and match equivariantly their boundaries by an orientation reversing homeomorphism as shown. This gives us an S^1-equivariant connected sum of $S^2 \times S^1$ with $S^2 \times S^1$.

Figure 3: E1–1

Figure 4: E1-2

We can do this any number of times, say $n > 1$, getting $M_n = S^2 \times S^1 \# \cdots \# S^2 \times S^1$ whose orbit space is a disk with n-holes and $n+1$ boundary components, corresponding to the $n+1$ circles of fixed points.

We may describe *all* the S^1 actions on M_n, $n \geq 0$, as follows. Take an oriented compact surface A of genus g with $k(>0)$ boundary components such that $2g + k - 1 = n$. Form $S^1 \times A$ and let S^1 act as standard rotations on the first factor and trivially on the second factor. The boundary consists of $S^1 \times \partial A$. Now if we take each orbit $S^1 \cdot a$, where $a \in \partial A$ and collapse it to a distinct point, we obtain on the identification space an S^1-action with exactly k connected components of fixed points homeomorphic to the boundary curves on A. The action is free elsewhere. To see that this space is homeomorphic to M_n, let us take a punctured torus for A written as:

A': picture 1

Make the construction above to create M. Above the dotted line we have an S^1-invariant 2-sphere. Cut open the 3-manifold along this 2-sphere and add two S^1-invariant 3-balls. The action on each of the 3-balls being just the cone of the S^1 action on the boundary 2-sphere. The S^1 action now extends to this new 3-manifold. It has now two boundary components of fixed points and the orbit space looks like

$ab, a'b'$: picture 2

which is an annulus and from the argument as before is diffeomorphic to $S^2 \times S^1$. Now we can reconstruct our original M by just removing the interiors of the S^1-invariant 3-balls and identifying equivariantly the boundaries. This operation is connected sum with $S^2 \times S^1$. So our $M' = M_2 = S^2 \times S^1 \# S^2 \times S^1$. This action on M_2 is different from the one obtained using A, a twice punctured disk, since the orbit spaces are different.

To identify M with M_n, we just observe that any A can be obtained from doing boundary connected sums with an annulus or a punctured torus with one boundary component. The corresponding M_n is obtained by doing the corresponding S^1-equivariant connected sums.

Choosing a fixed point as base point, the S^1-action can be lifted to an S^1-action on the universal covering $\widetilde{M}_n$. If $n = 1$, this is clearly $S^2 \times \mathbb{R}^1$ with fixed set $\{N\} \times \mathbb{R} \cup \{S\} \times \mathbb{R}$ with $\{N\}$ and $\{S\}$ being north and south poles. Each of these copies of $\mathbb{R}^1$ covers a component of the fixed set $\{N\} \times S^1 \cup \{S\} \times S^1$ of $S^2 \times S^1$. For $n > 1$, we get $\pi_1(M_n) = \mathbb{Z} * \cdots * \mathbb{Z}$, free product of n copies of $\mathbb{Z}$. $\widetilde{M}_n$ is a simply connected 3-manifold with a semi-free S^1-action.

We use 3 facts from transformation groups:

(1) If G is connected, $\nu_* : \pi_1(X, x) \to \pi_1(G \backslash X, x)$ is always onto.
(2) If G has a fixed point, then ν_* is onto.

(3) If I is an arc in $G\backslash X$, X completely regular, and G acts properly, then there exists an arc I^* in X, $\nu : I^* \to I$ a homeomorphism and $\nu^{-1}(I) = G \times I^*/\sim$ where the identification is made as follows. If $y^* \in I^*$, then the points on $G(y^*)$ are identified to G/G_{y^*}. (An arc I is supposed to be homeomorphic to unit interval $\{t : 0 \leq t \leq 1\}$.)

The orbit space $S^1\backslash\widetilde{M} = W$ is simply connected by (1). By the nature of the slice theorem, $S^1\backslash\widetilde{M}$ is a (simply connected) 2-manifold with boundary components comprised of the fixed orbits on M. Thus $S^1\backslash\widetilde{M} - \{\text{boundary}\}$ is an open 2-cell and the S^1-action is free over this open cell. Hence as it is a principal S^1-bundle, it must be a product S^1-action. Thus $\widetilde{M} - \{\text{fixed set}\}$ is S^1-equivalent to $S^1 \times \mathbb{R}^2$ where action is just translation along the first factor. There is a section, over the open 2-cell, to this principal S^1-action. With care this section extends over the entire boundary of W. What this says is that we can reconstruct $(S^1, \widetilde{M})$ by taking $S^1 \times W$ and collapsing each $S^1 \times b$ to a point, where b is a boundary point of W. This is equivalent to the standard circle rotation about the z-axis except that we must delete a closed subset C from the z-axis so that (z-axis $- C$) is homeomorphic to $\widetilde{M}^{s^1}$.

On W there is induced a proper action of the free product of n copies of $\mathbb{Z}$. All such actions are "geometric" in the sense that they are equivalent to an action of isometries on the hyperbolic plane extended to the boundary of components of W. The quotient is the surface A of genus g with $(k+1)$ boundary components and $n = 2g + k - 1$. Schematically we have:

$$\begin{array}{ccc}
(S^1, \mathbb{R}^3 - C) & \xrightarrow{S^1\backslash} & (\overset{n}{*}\mathbb{Z}, W) \\
\nu \big\downarrow \overset{n}{*}\mathbb{Z}\backslash & & \big\downarrow \overset{n}{*}\mathbb{Z}\backslash \\
(S^1, M) & \xrightarrow{S^1\backslash} & S^1\backslash M
\end{array}$$

as a commutative diagram of orbit mappings. The so-called "geometry" on $M - \{\text{fixed set}\}$ is an $\mathbb{R} \times \mathbb{H}$-geometry if $n > 1$, and on M, we will show later that it has a flat conformal structure.♣ Later...

1.11.6 Example. Let S^3 be the unit sphere in $\mathbb{C}^2$. Let us parametrize it by taking

$$S^3 = \{(\rho z_1, (1-\rho^2)^{1/2} z_2) : \ z_1, z_2 \in S^1, 0 \leq \rho \leq 1\}.$$

Note $\rho^2 z_1 \overline{z}_1 + (1-\rho^2) z_2 \overline{z}_2 = 1$, $0 \leq \rho \leq 1$. For a fixed $\rho \neq 0$ or 1, $(\rho z_1, (1-\rho^2)^{1/2} z_2) = T_\rho$ is a torus. If $\rho = 0$ or 1, $T_0 = \{(0, z_2)\}$ and $T_1 = \{(z_1, 0)\}$ are circles. Define an effective action of $S^1 \times \mathbb{Z}_p$ on S^3 by

$$(z, \lambda^s)(\rho z_1, (1-\rho^2)^{1/2} z_2) = (\rho z_1 \lambda^s, (1-\rho^2)^{1/2} z \lambda^{sq} z_2)$$

where $\lambda = e^{\frac{2\pi i}{p}}$, $z \in S^1$, $0 < q < p$, $(p, q) = 1$. The action of $\mathbb{Z}_p$ is free and $\mathbb{Z}_p\backslash S^3$, the quotient 3-manifold is called the (p, q)-lens space and is denoted by $L(p, q)$. The action of S^1 on S^3 is semi-free. That is, it is free away from the circle fixed set $\{(z_1, 0)\}$. The orbit space of this S^1-action is a 2 disk with the fixed set projecting onto the boundary. Note each fixed ρ-level T_ρ is $S^1 \times \mathbb{Z}_p$-invariant and if $\rho \neq 0$ or 1, the action is free. The quotient by $\mathbb{Z}_p$ is again a torus and the quotient by S^1 is a circle. The part $\underset{0 \leq \rho \leq \frac{1}{2}}{\cup} T_\rho$ is a solid torus with $\{(0, z_2)\}$ the core circle and

$\underset{\frac{1}{2}\leq\rho\leq 1}{\cup} T_\rho$ is a complementary solid torus. Thus $L(p,q)$ is again the union of 2 solid tori $V_0 \cup V_1$, where

$$V_0 = \mathbb{Z}_p\backslash\left(\underset{0\leq\rho\leq\frac{1}{2}}{\cup} T_\rho\right), \quad V_1 = \mathbb{Z}_p\backslash\left(\underset{\frac{1}{2}\leq\rho\leq 1}{\cup} T_\rho\right).$$

On $L(p,q)$ there is induced an action of S^1, which is free whenever $\rho \neq 0$ or 1. On $\mathbb{Z}_p\backslash T_0$, this is a single S^1 orbit with stability group isomorphic $\mathbb{Z}_p$. On $\mathbb{Z}_p\backslash T_1$, the circle is fixed by the S^1-action. On V_0, the action is equivalent to $(S^1, S^1 \times_{S^1{}_{\langle 0,1\rangle}} D^2)$ where D^2 is the 2-disk and $S^1_{\langle 0,1\rangle}$ is the stability group of the image of $S^1_{(0,1)}$ in $L(p,q)$. This is isomorphic to $\mathbb{Z}_p \subset S^1$ and it operates diagonally as $\lambda\cdot(z,u) \mapsto (z\lambda^{-1}, \lambda^s u)$, where $u \in \mathbb{C}$, $\|u| \leq 1$. The quotient space is $S^1\backslash(\mathbb{Z}_p\backslash S^3) = \mathbb{Z}_p\backslash(S^1\backslash S^3)$. So it is a disk with the boundary corresponding to the fixed set and an interior point which is the image of the core circle of V_0. In terms of the orbit mapping, we have

$$\begin{array}{ccc} (S^1\times\mathbb{Z}_p, S^3) & \xrightarrow{\mathbb{Z}_p\backslash} & L(p,q) \\ {\scriptstyle S^1\backslash}\big\downarrow & & \big\downarrow{\scriptstyle S^1\backslash} \\ D^2 & \xrightarrow{\mathbb{Z}_p\backslash} & \mathbb{Z}_p\backslash D^2 = S^1\backslash L(p,q) = (\mathbb{Z}_p\times S^1)\backslash S^3 \end{array}$$

Thus the lift of the $(S^1, L(p,q))$ to S^3 is just rotation about the z-axis in $\mathbb{R}^3 \cup \infty = S^3$. Let us denote the orbit space $\mathbb{Z}_p\backslash L(p,q)$ by $\widehat{L}(p,q)$. Suppose we have $(S^1, L(p,q))$ and $(S^1, L(p',q'))$, two actions with orbit spaces $\widehat{L}(p,q)$ and $\widehat{L}(p',q')$. If we take the inverse images of the shaded parts in the orbit spaces we get an invariant closed 3-cell in each lens space. We can then delete the interior of these cells and match S^1-equivariantly the boundaries of these cells by an orientation reversing homeomorphism. We get an equivariant connected sum. Now on $L(p,q)\#L(p',q')$, there is constructed an S^1-action with a circle of fixed points and all the other orbits are free except for 2 circles which were the cores of the respective V_i's. The orbit space is again a disk.

Figure 5: E2-4

The fundamental group of $L(p,q)\#L(p',q')$ is $\mathbb{Z}_p * \mathbb{Z}_{p'}$ by the Van-Kampen theorem. Obviously we may continue to take equivariant connected sums constructing S^1-actions on $L(p_1,q_1)\#\cdots\#L(p_n,q_n) = M$ with orbit space a disk and n singular orbits and one circle (boundary) of fixed points.

Now if we wish to lift this S^1-action to its universal covering $\widetilde{M}$, we must have a non-compact universal covering if $n \neq 1$. The fixed set in $\widetilde{M}$ projects onto the fixed set in M.

The stability groups of the S^1-action at $\widehat{y}$ must be a subgroup of the stability groups at y. Therefore if $\widehat{y}$ has finite stability, it must be cyclic and order divides p_j for some j.

We shall sketch the lifted action on the universal covering space. Details can be found in [**?**]. If we take an arc $p(t)$ as shown in $S^1\backslash M$,

Figure 6: E2-5

we can lift this path to an arc $r(t)$ such that $S^1(r(t))$ is the complete inverse image under the orbit mapping of $p(t)$. It is a 2-disk with boundary identified by a period p_j homeomorphism. If $\nu(\widehat{y}) = y \in M$, where $S^1_y = \mathbb{Z}_{p_j}$, then $S^1_{\widehat{y}}$ is cyclic of order dividing p_j. Lifting an arc from M to $\widetilde{M}$ as we did from $S^1\backslash M$ to M to obtain an arc whose S^1-image is a disk with boundary identified by a homeomorphism of period equal to the order of $S^1_{\widehat{y}}$. If this order is not 1, then this disk with identifications C carries non-trivial torsion in its second cohomology which will be impossible as a simply connected 3-dimensional manifold [$0 \to H^2(C) \to H^3_c(\widetilde{M - C}) \to \mathbb{Z} \to 0$ is exact and $H^3_c(M - C)$ is torsion free.] Therefore the action of S^1 on $\widetilde{M}$ is semi-free (all orbits are free or fixed). This enables one to prove a cross-sectioning theorem. The orbit space $S^1\backslash\widetilde{M}$ is a simply connected 2-manifold with boundary the fixed points. It therefore is a disk with a closed set removed from the boundary. The cross sectioning theorem tells us that $(S^1, \widetilde{M})$ is smoothly equivalent to $\mathbb{R}^3$ with a closed set removed from the z-axis and S^1 rotates the yz-plane, $y > 0$, about this deleted z-axis.

The fundamental group $\pi_1(M) \cong \mathbb{Z}_{p_1} * \cdots \mathbb{Z}_{p_n}$ commutes with the S^1-action on $\widetilde{M}$ and so acts effectively on $S^1\backslash\widetilde{M}$, a 2-disk with part of the boundary removed. This action is smoothly equivalent to a proper discrete action on the unit disk with parts of the boundary removed. On the interior, it actually is smoothly equivalent to hyperbolic isometries or equivalently to a holomorphic action. The quotient space being the disk with n exceptional points in its interior ($= S^1\backslash M$).

We may reconstruct the action on $L(2,1)\#L(3,2)$ from an S^1 action on S^3 as follows. Take $S^1 \times S^3 \to S^3$ as defined by

$$z \times (\rho z_1, (1-\rho^2)^{1/2} z_2) \mapsto (z^2 \rho z_1, z^3 (1-\rho^2)^{1/2} z_2).$$

This gives a "linear" circle action on S^3 which is free away from $(z_1, 0)$ and $(0, z_2)$. On these 2 circles it has stabilizer $\mathbb{Z}_3$ and $\mathbb{Z}_2$ respectively. If we take a free orbit and remove an S^1 invariant cube about it and sew in equivariantly $S^1 \times D^2$ with S^1-action just rotating on the second factor, we get an action of S^1 on M^3 with a circle of fixed points, 2 orbits with stabilizer $\mathbb{Z}_3$ and $\mathbb{Z}_2$ and the rest free. It is shown in [**?**] that $M^3 = L(3,2)\#L(2,1)$, and removing the set of fixed points is just $S^3 - \{\text{a single free orbit}\}$. Since this orbit lies in a torus, it follows that $M^3 - \{\text{fixed points}\}$ is just the complement of the $(2,3)$ torus knot in S^3.

We have described a circle action on S^3 with fixed points. The other possible circle actions (up to equivalence) are obtained as follows:

$$z \times (\rho z_1, (1-\rho^2)^{1/2} z_2) \mapsto (z^p \rho z_1, z^q (1-\rho^2)^{1/2} z_2), \quad (p,q) = 1.$$

Note that each of free cyclic actions yielding lens spaces are inside some S^1-action. This action has stabilizer $\mathbb{Z}_p$ if $\rho = 1$ and $\mathbb{Z}_q$ of $\rho = 0$. Away from the 2 core circles of the union of the 2 solid tori forming the 3-sphere, each orbit is free and lies on one of the S^1 invariant tori T_ρ. The orbit is a (p,q) torus knot. In fact this action is nothing but the action defined in subsection 1.11.4(5) (on Brieskorn varieties) for S^3 except that here we use a join representation of S^3.

1.11.7 Remark. It is clear, that in the presence of fixed points, we can do a boundary connected sum of the orbit spaces A in Example 1.11.5 with those of figure 6. In this way, we get an S^1 action on a connected sum of $S^2 \times S^1$'s and lens spaces. In [?] it is shown that this is the only possible way that S^1 can operate with fixed points on a closed orientable 3-manifold. Further, of these 3-manifolds, only S^3 and $S^2 \times S^1$ admit effective S^1 actions without fixed points. Thus, if S^1 acts effectively on a closed oriented 3-manifold M with fixed points, then M is homeomorphic to a connected sum of lens spaces and $S^2 \times S^1$'s.♣

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1.12. Slice Representations

In section 1.9 we discussed lifting actions when the action had fixed points and in section 1.11 when the group was path-connected. If G is path-connected and, a closed subgroup $H \subset G$ fixes $x \in X$, then H may be lifted by the method of section 1.9 and also by the method of section 1.11. Exactly how these liftings may be compared is the subject of this section.

1.12.1 Lemma ([?, Lemma 4.5]). *Let X be a path-connected space; G a path-connected topological group acting on X. Let H be a closed subgroup of G fixing $x \in X$. Then there is a natural homomorphism $\psi : H \longrightarrow \pi_1(X,x)/\mathrm{Im}(\mathrm{ev}_*^x)$ such that*

$$h_*(\alpha) = \psi(h)\ \alpha\ \psi(h)^{-1} \text{ for all } h \in H, \text{ and } \alpha \in \pi_1(X,x),$$

where $h_* \in \mathrm{Aut}(\pi_1(X,x)/\mathrm{Im}(\mathrm{ev}_*^x))$ is induced by $h_* \in \mathrm{Aut}(\pi_1(X,x))$ given by $h_*([\ell(t)] = [h \cdot \ell(t)]$.

Proof. Let $h(t) \in P(G,e)$ such that $h(1) = h \in H \subset G$. Then by examining the homotopy diagram as in subsection 1.10.1

we have $\ell(t) * h(t) \cdot x \simeq h(t) \cdot x * h(\ell(t))$. Consequently, $\overline{h(t) \cdot x} * \ell(t) * h(t) \cdot x \simeq h(\ell(t))$ represents $h_*(\alpha)$. If $h(t) \cdot x$ represents $\beta_h \in \pi_1(X,x)$, then $h_*(\alpha) = \beta_h^{-1}\alpha\beta_h$. To see that this is well defined, let $h'(t) \in P(G,e)$ with $h'(1) = h$. Then $h'(t) \cdot x$ represents $\beta'_h \in \pi_1(X,x)$ and $\beta_h {\beta'}_h^{-1} \in \mathrm{Im}(\mathrm{ev}_*^x) \subset \mathcal{Z}(\pi_1(X,x))$. Therefore ${\beta'}_h^{-1}\alpha{\beta'}_h = \beta_h^{-1}\alpha\beta_h$ and so $h_*(\alpha) = \beta_h^{-1}\alpha\beta_h$ is a well defined formula.

The function $h \mapsto \beta_h$ is not well defined as an element of $\pi_1(X,x)$ but it is well defined by taking the $\pi_1(X,x)/\mathrm{Im}(\mathrm{ev}_*^x)$-path class of $h(t) \cdot x$. Therefore, we define $\psi : H \to \pi_1(X,x)/\mathrm{Im}(\mathrm{ev}_*^x)$ by

$$\psi(h) = [\beta_h^{-1}].$$

Examine the homotopy diagram

Figure 7: middle: $h_1(t) \cdot h_2(t) \cdot x$

which is obtained from section 1.10.1 by $g(t) = h_1(x)$ and $p(t) = h_2(x) \cdot x$. Note that $h_1(t) \cdot h_2(1) \cdot x = h_1(t) \cdot x$. This shows that ψ is a homomorphism. The formula $h_*(\alpha) = \psi(h)\ \alpha\ \psi(h)^{-1}$ is clear from the construction. □

1.12.2 COROLLARY. *Let (G, X) be an action of a path-connected topological group on a path-connected space with H, a closed subgroup of G, fixing $x \in X$. Let K be a normal subgroup of $\pi_1(X, x)$ containing the image of $\mathrm{ev}^x_\# : \pi_1(G, e) \to \pi_1(X, x)$. Let $Q = \pi_1(X, x)/K$. Then there is a natural homomorphism $\psi : H \longrightarrow Q$ such that*

$$h_*(\alpha) = \psi(h)\ \alpha\ \psi(h)^{-1} \textit{ for all } h \in H, \textit{ and } \alpha \in \pi_1(X, x),$$

where $h_* \in \mathrm{Aut}(Q)$ is induced by $h_* \in \mathrm{Aut}(\pi_1(X, x))$ given by $h_*([\ell(t)] = [h \cdot \ell(t)]$.

1.12.3. Since H leaves K invariant and $K \supset \mathrm{Im}(\mathrm{ev}^x_*)$, we may lift the H-action on X_K by the method of section 1.9. Denote this lifted H-action by

$$h \times \widehat{y} \mapsto h \diamond \widehat{y}.$$

H also lifts by the method of section 1.11, to the restriction of the lifted G-action to H. This will be denoted, as usual, by

$$h \times \widehat{y} \mapsto h \bullet \widehat{y}.$$

Let $p(t) \in P(X, x)$ represent $\widehat{y}$; $h(t) \in P(G, e)$ is a path in G from e to h. Then

$$\begin{array}{lll} h \diamond \widehat{y} & \text{is represented by} & h \cdot p(t), \\ h \bullet \widehat{y} & \text{is represented by} & h(t) \cdot p(t). \end{array}$$

Considering the homotopy diagram

Diagram

we get the

1.12.4 LEMMA. $h \diamond \widehat{y} = h \bullet (\psi(h) \cdot \widehat{y})$.

Consider the homotopy diagram

This shows

$$(h(t) \cdot x) * (h \cdot p(t)) \simeq h(t) \cdot p(t),$$

which implies $\psi(h)^{-1} \cdot (h \diamond \widehat{y}) = h \bullet \widehat{y}$. Thus, $h \diamond \widehat{y} = \psi(h) \cdot (h \bullet \widehat{y}) = h \bullet (\psi(h) \cdot \widehat{y})$.

1.12.5. Let $\rho : (G, X_K) \xrightarrow{G\backslash} (G\backslash X_K) = W$ denote the orbit mapping. If $\widehat{y} \in X_K$, put $\rho(\widehat{y}) = w$. Since the covering transformations $Q = \pi_1(X, x)/K$ commutes with the lifted G-action, there is induced a Q-action on W, denoted by $\alpha \times w \mapsto \alpha \cdot w$, and ρ is Q-equivariant as well as G-equivariant (using trivial G-action on W). H also acts on W via $\psi : H \to \psi(H) \subset Q$.

1.12.6 PROPOSITION. $\rho : (H, X_K, \diamond) \longrightarrow (H, W, \psi)$ *is an equivariant map.*

For, $\rho(h \diamond \widehat{y}) = \rho(h \bullet (\psi(h) \cdot \widehat{y})) = \rho(\psi(h) \cdot (h \bullet \widehat{y})) = h \cdot w = h \cdot \rho(\langle \widehat{y} \rangle)$. □

locally injective

1.12.7 Corollary. *Let (G, X) be as above with G a connected Lie group acting properly on a completely regular X which admits covering space theory. Let $x \in X$, and $K \supset \mathrm{Im}(\mathrm{ev}^x_*)$. Assume $G_{\widehat{x}} = 1$, $\widehat{x} \in X_K$, $\nu(\widehat{x}) = x$ and $w = \langle \widehat{x} \rangle \in W = G \backslash X_K$. Then $G_x \xrightarrow{\psi} Q_w$ is an isomorphism and there exists slices S_x at x and Σ_w at w such that the slice actions are G_x-equivalent.*

Proof. Let S_x be a slice at x and assume it is path-connected and sufficiently small as to be included in an evenly covered neighborhood. Then S_x lifts homeomorphically to a slice $S_{\widehat{x}}$ at $\widehat{x}$. Put $G_x = H$, then H action on S_x lifts exactly to the $(H, \diamond)$ action on $S_{\widehat{x}}$, by: $y \mapsto hy$ lifts to $\widehat{y} \mapsto h \diamond \widehat{y}$. Put $\rho(S_{\widehat{x}}) = \Sigma_w$. Σ_w is a slice at w and $\rho(h \diamond \widehat{y}) = \psi(h) \cdot \rho(\widehat{y}) = h \cdot \widehat{y}$. Note as $G_{\widehat{y}} \subset G_{\widehat{x}}$ for $\widehat{y} \in S_{\widehat{x}}$, $\rho : S_{\widehat{x}} \longrightarrow \Sigma_w$ is an H-equivariant homeomorphism. □

1.12.8 Remark. Note also if the (G, X) action is smooth, analytic or holomorphic, then

$$S_x \longrightarrow S_{\widehat{x}} \longrightarrow \Sigma_w$$

are also smoothly, analytically or holomorphically equivalent.

In the smooth case, the slice representations (S_x is a cell, normal to the orbit at x) are smoothly equivalent to linear representations, and it can easily be shown that as linear representations, they are linearly equivalent.

If $G_x \neq 1$, then $Q_w = \psi(G_x) \cong G_x/G_{\widehat{x}}$ and $(G_x, \widehat{S}_{\widehat{x}})$ factors as

$$\begin{array}{ccc}
(H, S_{\widehat{x}}) = (G_x, S_{\widehat{x}}) & \xrightarrow{G_{\widehat{x}}\backslash} & (G_x/G_{\widehat{x}}, G_{\widehat{x}}\backslash S_{\widehat{x}}) = (\psi(G_x), \Sigma_w) \\
\nu \big\downarrow & & \big\downarrow \psi(G_x)\backslash \\
(G_x, S_x) & \xrightarrow{G_x} & G_x \backslash S_x
\end{array}$$

It is significant to note that when (G, X_K) is a principal action, then the stabilizers of the Q-action on W encodes all the slice information of (G, X). We shall address the reverse of this procedure in a later chapter. Namely, if we are given (Q, W) acting properly with Q discrete, and $Y \to W$ a fixed principal G-bundle, then we can ask how may we construct a covering Q-action on Y which commutes with the principal G-action. This reverse process may or may not be possible, and if possible, it may not be unique.

1.12.9 Exercise. The two H-actions on X_K in subsection 1.12.3 are subgroups of the group, $(\pi_1(X, x)/K) \rtimes H$, generated by all lifts of the H-action on X. Determine explicitly these two subgroups in $(\pi_1(X, x)/K) \rtimes H$. [cf. Exercise **??**.]

1.12.10 Definition. Suppose a path-connected G acts on X admitting covering space theory. If, at each $x \in X$, $\psi : G_x \to \pi_1(X, x)/\mathrm{Im}(\mathrm{ev}^x_*)$ is a monomorphism, then (G, X) is called a *locally injective* G-action.

1.12.11 Proposition. *Let (G, X) be a locally injective action of a connected Lie group on a space X. Suppose that*

$$Q = \pi_1(X, x)/\mathrm{Imev}^x_*(\pi_1(G, e))$$

is torsion free. Then the G action must be free.

classifying space

Proof. Let $y \in X$, then $Q' = \pi_1(X,y)/\mathrm{Imev}^y_*(\pi_1(G,e))$ is isomorphic to Q and so Q' is torsion free. Since (G,X) is locally injective, $\eta_y : G_y \to Q'$ must be injective. Since G_y is finite, $G_y = 1_G$. □

1.12.12 Proposition. *If (G,X) is a locally injective G-action, then the lifted G-action $(G, X_{\mathrm{Im}(\mathrm{ev}^x_*)})$ is a free action and conversely. Moreover, if G is a Lie group acting properly on X, then $(G, X_{\mathrm{Im}(\mathrm{ev}^x_*)})$ is proper so the action is a principal G-action on $X_{\mathrm{Im}(\mathrm{ev}^x_*)}$.*

Proof. Pick a base point $x \in X$ and let $H = \mathrm{Im}(\mathrm{ev}^x_*)$. Then (G,X) lifts to (G,X_H) and put $\widehat{x}$ to be the path class of the constant loop at x. We have seen $\psi(G_x) = Q_{\rho(\widehat{x})} \subset \pi_1(X,x)/H$. The kernel is easily seen to be $G_{\widehat{x}}$ but this by hypothesis is trivial. For each $y \in X$, $G_y \to \pi_1(X,y)/\mathrm{Im}(\mathrm{ev}^y_*)$ is also monic. We get around using the x and $\widehat{x}$ for base point of X_H by appealing to subsection 1.7.1 and 1.11.2. And so we have $\psi : G_y \to Q_{\rho(\widehat{y})}$ is an isomorphism implying $G_{\widehat{y}}$ is also trivial. In order that (G,X_H) be a principal G-action, we just need to observe, by Theorem in 1.11.1, that the action is proper if (G,X) is proper.

The converse is also true. If Π centralizes $\ell(G)$ and Π acts as covering transformation, then the G-action on X must be locally injective. Note here G is not necessarily T^k or $SU(n)$. It can even be non-compact (e.g., $\mathbb{C}^*$). □

We remark that free actions are always locally injective. Also, injective toral actions (see section 1.14 for a definition) are locally injective, but locally injective toral actions are not necessarily injective toral actions. Any T^k action with a global H-slice where H is a finite subgroup of T^k is an injective T^k action. For example, the $\mathbb{C}^*$ and S^1 actions $\mathbb{C}^* \times \mathbb{C}^n - V \longrightarrow \mathbb{C}^n - V$, $S^1 \times S^{2n-1} - K \longrightarrow S^{2n-1} - K$, from Example 1.4.8, have global $\mathbb{Z}_a$ slices. However, the extended action (S^1, S^{2n-1}) has finite isotropy but is not locally injective.

1.12.13 Proposition. *Suppose P is a principal G-bundle where G is a connected Lie group, and $\Pi \subset \mathrm{TOP}_G(P)$ is a group of covering transformations of P acting properly, that centralizes $\ell(G)$ and $\ell(G) \cap \Pi = 1$. Then the induced G-action on $\Pi \backslash P = X$ is locally injective.*

Proof. Since Π commutes with $\ell(G)$, there is induced a G-action on $\Pi\backslash P = X$, which is covered by $\ell(G)$ on P. Because G is connected, this lift is the unique lift to P covering the induced G-action on X. Therefore, $pi_1(G,e) \to \mathrm{ev}^x_*(\pi_1(G,e)) \subset \pi_1(P,\widehat{x})$. We have seen that $G_x \to \pi_1(P,\widehat{x})/\mathrm{Im}(\mathrm{ev}^x_*)$ is a monomorphism in section 1.12.7. Since the choice of x is arbitrary, the G-action must be locally injective. □

1.13. Classifying Spaces

1.13.1. Let us recall some facts about classifying spaces for principal G-bundles. Good references are Steenrod's book [**?**] and Dold's paper [**?**].

For a topological group G, we have a *principal G-bundle* $E_G \longrightarrow G\backslash B_G = B_G$, called the universal G-bundle. All the principal G-bundles over a paracompact X (or numerable principal G-bundles over arbitrary W) are obtained by pulling back the universal G-bundle $E_G \to B_G$ by a continuous map from W into B_G. Any two

Borel space

pullbacks $f^*(E_G)$ and $g^*(E_G)$ are equivalent principal G-bundles if and only if, f is homotopic to g. Thus the set of equivalence classes of principal G-bundles over W is in one-one correspondence with the elements $[W, B_G]$, the homotopy classes of maps from W into B_G.

Let (G, X) be a G-space and E_G a contractible space on which G acts freely and properly. A technique due to A. Borel for studying G-actions is the so-called *Borel space* $E_G \times_G X$ associated to the G-space X. On $E_G \times X$, there is the diagonal G-action given by

$$g(e, x) = (ge, gx).$$

We define the Borel space to be the quotient space $G\backslash(E_G \times X)$. This is usually written as either $E_G \times_G X$ or X_G. This leads to the commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\overline{\pi}_2} & E_G \times X & \xrightarrow{\overline{\pi}_1} & E_G \\ \downarrow & & \downarrow{\scriptstyle G\backslash} & & \downarrow{\scriptstyle G\backslash} \\ G\backslash X & \xleftarrow{\pi_2} & E_G \times_G X = X_G & \xrightarrow{\pi_1} & B_G \end{array}$$

where π_1 is a fiber bundle mapping with fiber X and structure group G/K where K is the ineffective part of the G action on X, π_2 is a mapping such that $\pi_2^{-1}(x^*) = B_{G_x}$, where $x^* \in G\backslash X$, and $x \in x^*$. Therefore, if F is the set of fixed points in $G\backslash X$, then we claim $\pi_2^{-1}(F) = F \times B_G \subset X_G$. For, over each point $b \in B_G$, we have the fiber X and if $x \in X^G \neq \emptyset$, then there is a unique point x_b in the fiber over b corresponding to x. Since G acts on the fiber and fixes x_b, then $b \mapsto x_b$ defines a cross-section of $X_G \to B_G$. Then $\pi_1^* : H^*(B_G; L) \to H^*(X_G; L)$ is a *direct summand for any* PID L. If we view x as in $G\backslash X$, then $\pi_2^{-1}(x) = B_G$ and so $\pi_2^{-1}(F) = F \times B_G \subset X_G$.

1.13.2. To construct E_G, when G is a Lie group, one can construct the n-fold join of G, $G^{(n)}$, with G acting diagonally. This is an approximation of E_G in the sense that $G^{(n)}$ is an $(n-1)$ connected simplicial complex if G is connected. E_G is just the infinite join of G with itself and under a suitable topology, it is contractible. It is often a technical convenience to replace E_G by $G^{(n)} \subset E_G$, for n very large. This will be satisfactory for classifying purposes of bundles over X, if the dimension of X is finite.

The convenience arises in that we can use Čech cohomology with various supports without worrying if things such as the universal coefficient formulas remain valid in its most abstract setting. We will not mention this minor technicality any further and we refer to the reader A. Borel et al "Seminar on Transformation Groups" [**?**] for the various ways one gets around these technicalities. Since our interest is mainly in finite dimensional geometric situations, any problem can always be avoided by dealing with $G^{(n)}$, n large, instead of E_G.

1.13.3. If $G = T^k$, then $B_G = \mathbb{C}P_\infty \times \cdots \mathbb{C}P_\infty$, k copies. This is a $K(\mathbb{Z}^k, 2)$. When $k = 1$, $E_G = S^\infty$, the infinite join of circles and $(T^1)^{(n)} = S^{2n-1}$. Explain more

$$[X, B_{T^k}] = [X, K(\mathbb{Z}^k, 2)] \cong H^2(X, \mathbb{Z}^k).$$

If $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where each $\mathbb{Z}_{n_i}$ is a finite cyclic group of order n_i, then $BG = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \backslash S_1^\infty \times \cdots \times S_k^\infty$, where each $\mathbb{Z}_{n_i}$ acts freely as covering transformations on S_i^∞ and as a subgroup of T^1. Then, $[X, B_G] = [X, K(G,1)] \cong H^1(X;G)$. Note, there is a principal T^k fibering over B_{T^k} with total space B_G. $(B_G)^{(n)}$ is a product of $(2n-1)$-dimensional lens spaces with the ith factor being $\mathbb{Z}_{n_i}\backslash S^{2n-1}$. action!injective torus

So all the principal T^k (respectively, $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$)-bundles over X, up to bundle equivalence, are in one-one correspondence with the elements of $H^2(X;\mathbb{Z}^k)$ (repectively, $H^1(X;\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$).

1.14. Injective Torus Actions

1.14.1 Definition. A torus action (T^k, X) is said to be *injective* if $\mathrm{ev}^x_\# : \pi_1(T^k, e) \to \pi_1(X, x)$ is injective.

Suppose T^k acts injectively on a path connected, locally path connected, semi 1-connected paracompact Hausdorff space X. Then, by Theorem 1.11.1, the T^k action lifts to an action on the covering space $X_{\mathbb{Z}^k}$, where $\mathbb{Z}^k$ is the image of $\pi_1(T^k) \to \pi_1(M)$. We show that the lifted action $(T^k, X_{\mathbb{Z}^k})$ is free and splits.

1.14.2 Theorem ([**?**, §3.1]). *If T^k acts injectively on X, then $X_{\mathbb{Z}^k}$ (the covering space of X with $\pi_1(X_{\mathbb{Z}^k}) = \mathbb{Z}^k$) splits into $T^k \times W$ so that $(T^k, X_{\mathbb{Z}^k}) = (T^k, T^k \times W)$, where the T^k action on $T^k \times W$ is via translation on the first factor and trivial on the simply connected W factor.*

Proof. Let $G = T^k$ and S^1 be a circle subgroup of G. (Note that S^1 is a direct factor). Lift the S^1 action to $X' = X_{\mathbb{Z}^k}$. Let $y' \in X'$ and suppose $S^1_{y'} \neq 1$ is the stabilizer of the lifted S^1 action at y'. Choose paths γ in X' from the base point x' over x to y' and $\alpha : (I, 0, 1) \to (S^1, 1, z)$ where z is the "first" element $\neq 1$ of S^1 for which $z \cdot y' = y'$. Then $\alpha(t)y'$ defines a loop at y'; and $\gamma * \alpha * \overline{\gamma}$ is the associated loop based at x'. Now $(\gamma * \alpha * \overline{\gamma})^n \simeq \gamma * \alpha^n * \overline{\gamma}$ represents a generator of $\mathbb{Z}^k = H$. Hence n is the order of z in S^1. This implies that $\gamma * \alpha * \overline{\gamma}$ represents an nth root of a generator of $\mathbb{Z}^k$ which is impossible unless $n = 1$. Thus $S^1_{y'} = \{1\}$, for all $y' \in X'$ and all circle subgroups S^1 of T^k. Hence the lifted toral action (T^k, X') must be a free action. Since T^k is compact, the action (T^k, X') is proper and therefore $X \to W = T^k\backslash X'$ is a principal T^k bundle.

We shall now show that this bundle is trivial. Since $\mathrm{ev}^{x'}_* : \pi_1(T^k, 1) \to \pi_1(X', x')$ is an isomorphism, $\pi_1(W) = 1$. Let $\mathbb{R}^k$ act on X' through the covering projection $\mathbb{R}^k \to T^k$. Then the action of $\mathbb{R}^k$ lifts to the universal covering $\widetilde{X}$ of X'. In fact, as X' is a principal T^k-bundle, the lift $(\mathbb{R}^k, \widetilde{X})$ is a principal $\mathbb{R}^k$-bundle over W. This is seen by observing that $\pi_1(T^k, x') \to \pi_1(X', x')$ is an isomorphism (and is independent of x'). Thus each T^k orbit in X' lifts to an $\mathbb{R}^k$ orbit in $\widetilde{X}$. Since $\mathbb{R}^k$ is contractible, the $\mathbb{R}^k$ action on X' splits into a product $\mathbb{R}^k \times W$ with the $\mathbb{R}^k$ action just translation on the first factor. The bundle $X' \to W$ is recaptured by dividing out by the covering transformations $\mathbb{Z}^k \subset \mathbb{R}^k$ on each $\mathbb{R}^k$ orbit. Therefore, $(T^k, X') = (T^k, T^k \times W)$, where T^k just acts as left translations on the first factor. □

We remark that the group $T^k \times Q$ is acting properly on $T^k \times W$, and W is simply connected. The T^k action does not lift to the universal covering $\widetilde{X}$ of X

but the induced ineffective $\mathbb{R}^k$ action on $T^k \times W$ lifts to an effective $\mathbb{R}^k$ action on $\mathbb{R}^k \times W$ and commutes with the group $\pi_1(X, x)$ of covering transformations on $\mathbb{R}^k \times W$.

For a different and more general proof of the above theorem, see Chapter **??**.

1.15. Compact Lie group Actions

In this section we will investigate actions of compact connected Lie groups G on closed aspherical manifolds M. It will follow that if G acts effectively, then G must be a torus and the torus must act injectively. Therefore, the isometry group of such a Riemannian manifold M will have its connected component of the identity equal to the k-torus where $k \leq$ rank of center of $\pi_1(M)$.

There is also lots of control for G/G_0, G_0 being the connected component of the identity of a compact Lie group G. For example if $\mathcal{Z}(\pi_1(M)) = 1$, then $G_0 = 1$ and G/G_0 injects into $\mathrm{Out}(\pi_1(M))$. If M is closed Riemannian and $\kappa < 0$ ($\kappa =$ sectional curvature), then M is aspherical and center of $\pi_1(M)$ is trivial. Hence, if a finite group H acts effectively, then H injects into $\mathrm{Out}(\pi_1(M))$. Furthermore, if κ is *constant* then $\mathrm{Out}(\pi_1(M))$ is finite and every subgroup $H \subset \mathrm{Out}(\pi_1(M))$ can be realized as a group of isometries of M.

Consequently, if we can show that $\mathcal{Z}(\pi_1(M)) = 1$, and $\mathrm{Out}(\pi_1(M))$ has no torsion, then the closed aspherical manifold admits no action of (not necessarily connected) compact Lie groups whatsoever (totally rigid). Such manifolds exist even in dimension 3, and there they even fiber over the circle.♣ In fact, in every cobordism class, there exists a closed manifold without any finite group actions if dimension is ≥ 3.

See CRW, Tollefson, Schultz, Puppe ??

1.15.1 Lemma. *Let $\gamma : G \times X \to X$ be an action of a path connected group G on a path connected space X. Suppose the G action lifts to a covering space $p : \widehat{X} \to X$. Suppose $F = X^G$, the set of points of X fixed by G, is non-empty and F' is a path component of F. Similarly, let $E = \widehat{X}^G$. Then $p^{-1}(F) = E$. The path components of E which project into F' (and hence onto) are exactly the path components of $p^{-1}(F')$.*

Proof. Let $x \in F$ be a base point, and choose any $\widehat{x} \in \widehat{X}$ such that $p(\widehat{x}) = x$ as the base point for $\widehat{X}$. We claim that $g(\widehat{x}) = \widehat{x}$ for all $g \in G$. First $p(g\widehat{x}) = g(p(\widehat{x})) = x$. But the orbit of $\widehat{x}$ is connected and $g\widehat{x} \in p^{-1}(x)$, a discrete set. So $g\widehat{x} = \widehat{x}$. Since the choice of base point does not alter the lifted G-action (see section 1.11.2), we see that $p(E) \supset F$. Since $p(E) \subset F$, we have that $p(E) = F$ and $p^{-1}(F) = E$.

Let F' be a path component of F, and for $x \in F'$, choose $\widehat{x}$ such that $p(\widehat{x}) = x$ as base points. We observe that E' must project onto F'. Let $\widehat{x}$ be fixed and let E' be the path component of E containing $\widehat{x}$. Then $p(E') \subset F'$. Moreover, if we lift a path in F' starting at x and lift to a path in $\widetilde{X}$ starting at $\widehat{x}$, then it lifts to a path in E'. Hence $p(E') \supset F'$ and $p^{-1}(F') = E'$. □

1.15.2 Example (cf. 1.11.5). Let S^1 act on $S^2 \times S^1$ with a 2 fixed circles. Then the lifted S^1 action to $S^2 \times \mathbb{R}^1$ has 2 components of fixed points each homeomorphic to $\mathbb{R}^1$ and which project to the 2 circles. However if we take an S^1-action on $S^2 \times S^1$ #

$S^2 \times S^1$ with 3 components of fixed points (3-circles), but on the universal covering there is an infinite number of fixed components (each is line) projecting onto each circle.

1.15.3 LEMMA. (1) *If S^1 acts on a $\mathbb{Q}$-acyclic paracompact finite dimensional space X, then $F = X^{S^1} \neq \emptyset$ and is also $\mathbb{Q}$-acyclic.*

(2) *If a finite p-group (p prime) G acts on a $\mathbb{Z}_p$-acyclic paracompact finite dimensional space X, then $F = X^G \neq \emptyset$ and is also $\mathbb{Z}_p$=acyclic.*

PROOF. Form the Borel space and the diagram from section 1.13:

$$\begin{array}{ccccc} X & \xleftarrow{\pi_2} & S^\infty \times X & \xrightarrow{\pi_1} & S^\infty = E_{S^1} \\ \big\downarrow S^1\backslash & & \big\downarrow S^1\backslash & & \big\downarrow {}' S^1\backslash \\ S^1\backslash X & \xleftarrow{\pi_2} & X_{S^1} = S^\infty \times_{S^1} X & \xrightarrow{\pi_1} & \mathbb{C}P_\infty = B_{S^1} \end{array}$$

π_1 is a fibering with X as fiber and structure group S^1. Since X is $\mathbb{Q}$-acyclic, π_1^* is an isomorphism on cohomology. Suppose $F = \emptyset$, then $\pi_2^{-1}(x^*) = S^1_x\backslash S^\infty = B_{S^1_x}$, where $p(x) = x^*$, and S^1_x is trivial or finite cyclic. But then $B_{S^1_x}$ is also acyclic and so π_2^* is an isomorphism in rational cohomology by the Vietoris mapping theorem. See Remark 1.15.6.

Since $S^1\backslash X$ is finite dimensional, its $\mathbb{Q}$-cohomology vanishes above the dimension of $S^1\backslash X$. This contradicts that π_2^* is an isomorphism, for $H^*(X;\mathbb{Q}) \cong H^*(\mathbb{C}P_\infty;\mathbb{Q})$. Therefore $F \neq \emptyset$.

For $x^* \in S^1\backslash X - F$, $\pi_2^{-1}(x^*)$ is still $\mathbb{Q}$-acyclic. This again implies that $\pi_2^* : H^q(S^1\backslash X, F;\mathbb{Q}) \to H^*(X_{S^1}, F_{S^1};\mathbb{Q})$ is an isomorphism in Čech rational cohomology. Recall $F_{S^1} \cong F \times \mathbb{C}P_\infty$. So we examine

$$\xrightarrow{\delta} H^q(X_{S^1}, F_{S^1};\mathbb{Q}) \xrightarrow{j^*} H^q(X_{S^1};\mathbb{Q}) \xrightarrow{i^*} H^q(F \times \mathbb{C}P_\infty;\mathbb{Q}) \xrightarrow{\delta}$$

Since $H^q(S_1\backslash X, F;\mathbb{Q})$ vanishes for $q > \dim(S^1\backslash X)$, $H^q(X_{S^1}, F_{S^1};\mathbb{Q}) = 0$ and i^* is an isomorphism. We have seen $\pi_1^* : H^q(X_{S^1};\mathbb{Q}) \to H^q(\mathbb{C}P_\infty;\mathbb{Q})$ is an isomorphism and so, by the Künneth rule, we have

$$H^q(F_{S^1};\mathbb{Q}) = H^q(F \times \mathbb{C}P_\infty;\mathbb{Q}) = \Sigma_{i+j=q} H^i(F;\mathbb{Q}) \otimes H^j(\mathbb{C}P_\infty;\mathbb{Q}).$$

This implies that F must be $\mathbb{Q}$-acyclic.

Essentially the same argument used for S^1 now holds for $\mathbb{Z}_p$ if S^1, $\mathbb{Q}$ and B_{S^1} are replaced by $\mathbb{Z}_p$, $\mathbb{Z}_p$ and $B_{\mathbb{Z}_p}$. Note $(\mathbb{Z}_p)_x = 1$ or $\mathbb{Z}_p$. For the general case of a finite p-group, use the fact that G is solvable and contains a non-trivial normal subgroup $H \subset G$. Consequently, $X^G = (X^H)^{G/H}$. Now, reduce this case to $G = \mathbb{Z}_p$ by induction. □

1.15.4 COROLLARY. *If X is a finite dimensional aspherical space, then $\pi_1(X, x)$ is torsion free.*

PROOF. The universal covering $\widetilde{X}$ of X is contractible and finite dimensional. If the covering transformations of $\widetilde{X}$ contained a cyclic group of prime order, then it would contradict the Lemma. □

1.15.5 COROLLARY ([**?**, Theorem 5.6], see also Theorem **??**). *If $G \neq e$ is a compact connected Lie group acting effectively on a closed aspherical manifold M, then G is a toral group and acts injectively on M.*

PROOF. Let T^k be a maximal torus in G. If the restriction of the evaluation homomorphism $\mathrm{ev}^x_* : \pi_1(G,e) \to \pi_1(X,x)$ to the torus T^k is not an injection, then there exists a circle subgroup C in T^k, for which the restriction of the evaluation homomorphism is not injective. This contradicts the Lemma. Consequently, the evaluation homomorphism $\mathrm{ev}^x_* : \pi_1(T^k,e) \to \pi_1(M,x)$, which factors through $\pi_1(G,e)$, is injective. But, $\mathrm{ev}^x_* : \pi_1(T^k,e) \to \pi_1(G,e)$ (under the homomorphism induced by inclusion) is injective if and only if G itself is T^k. $\square$

1.15.6 REMARK. There is a subtlety in the argument above, namely, the validity of the Künneth rule for Čech cohomology (with closed support). This is no problem if F has the homotopy type of a CW-complex or if F is compact. Even if F is neither, the rule is still valid because $\mathbb{C}P_\infty$ (resp. $B_{\mathbb{Z}_p}$) is a nice space, see [**?**, Chapter XVI, §5].

The Vietoris mapping theorem states that under a closed mapping for which inverse images of points are acyclic in Čech cohomology, then the mapping induces an isomorphism in cohomology. Here we replace $B_{S^1_x} = S^1_x\backslash\mathbb{C}P_\infty$ by $S^1_x\backslash\mathbb{C}P_n$, where n is very large. Then $S^1_x\backslash\mathbb{C}P_n$ is compact and $\mathbb{Q}$-acyclic up to dimension $n-1$. These technical concerns all vanish if we assume that (S^1, X) is a smooth action. In fact, then F is a smooth $\mathbb{Q}$-acyclic submanifold. A different proof of the lemma, using Smith theory, can be found in [**?**, III,§10]. The lemma is also valid with cohomology with integral coefficients and with the provision that (S^1, X) has only a finite number of non-isomorphic stability groups.

1.15.7 LEMMA. *If (S^1, M) is a non-trivial action where M is a closed connected aspherical manifold, then the action must be injective.*

PROOF. Assume that the kernel of $\mathrm{ev}^x_\#$ is not trivial. Then there exists a finite covering group $'S^1 \to S^1$ which acts non-trivially on M, and for which image of $\mathrm{ev}^x_\#$ is trivial. Then this action of $'S^1$ lifts to the universal cover $\widetilde{M}$ of M. Since $\widetilde{M}$ is contractible, it is $\mathbb{Q}$-acyclic. The group of covering transformations acts on $\widetilde{M}$.

By Lemma 1.15.3 , $E = \widetilde{M}^{'S^1}$ must be $\mathbb{Q}$-acyclic and by Lemma 1.15.1, $\nu^{-1}(F) = E$, and $\pi_1(M)\backslash E = F = X^{'S^1}$. Since E is $\mathbb{Q}$-acyclic, then $H^*(\pi_1(M)\backslash E; \mathbb{Q}) = H^*(F;\mathbb{Q})$ is the same as the group cohomology $H^*(\pi_1(M);\mathbb{Q})$, which is the same as $H^*(F;\mathbb{Q})$ because M is aspherical.

If M is orientable, we have $H^n(M;\mathbb{Q}) = \mathbb{Q}$. But $F \neq M$, since the action (S^1, M) was assumed non-trivial, and $H^n(F;\mathbb{Q}) = 0$ because F is a proper closed subset of M. This is a contradiction so the S^1-action is injective.

If M is not orientable, we can lift the S^1-action to the orientable double cover $\widehat{M}$, because the elements of $\pi_1(M)$ which preserve the orientation of $\widehat{M}$ are left invariant by S^1. Thus the S^1-action lifts to orientable double cover $\widehat{M}$ of M. Then the S^1-action on $\widehat{M}$ is injective, and consequently it is injective on M. $\square$

1.15.8 PROPOSITION ([**?**, Corollary 6.2], see also Corollary **??**). *Let G be a finite group acting effectively on a closed connected aspherical n-manifold M with fixed point at $x \in M$. Then the representation $\theta : G \to \mathrm{Aut}(\pi_1(M,x))$ induced by the action of G on $\pi_1(M,x)$ is injective.*

PROOF. Suppose K is the kernel of θ. If $K \neq 1$, let $\mathbb{Z}_p$ be a cyclic subgroup of K of prime order p. Then the action of $\mathbb{Z}_p$ lifts to the universal covering $\widetilde{M}$ of M. Then $\widetilde{M}^{\mathbb{Z}_p} \neq \emptyset$ and is $\mathbb{Z}_p$-acyclic by Lemma 1.15.3. Furthermore, $\mathbb{Z}_p$ commutes with the covering transformations $\pi_1(M,x)$ on $\widetilde{M}$, by Theorem 1.9.1. By Corollary 1.9.5???, $\pi_1(M)$ acts freely as covering transformations on $\widetilde{M}^{\mathbb{Z}_p}$ and covers C, the path component of $M^{\mathbb{Z}_p}$ containing x. Therefore, $H^*(\pi_1(M,x);\mathbb{Z}_p) \cong H^*(M;\mathbb{Z}_p) \cong H^*(C;\mathbb{Z}_p)$. If M is orientable, this means that $H^n(M;\mathbb{Z}_p) \cong H^n(C;\mathbb{Z}_p) \cong \mathbb{Z}_p$. But C is a closed subset of M and this can only happen if $C = M$, which contradicts the effectiveness of G. Therefore K must be trivial and θ is faithful.

If M is not orientable, the action of $\mathbb{Z}_p$ lifts with fixed point to the orientable double covering M' of M. The action of $\mathbb{Z}_p$ on $\pi_1(M',x')$ is trivial and implies that the action of $\mathbb{Z}_p$ on M' is trivial by the argument for the orientable case. Thus, the action of $\mathbb{Z}_p$ on M would have to be trivial which again contradicts the effectiveness of the G-action. □

1.15.9 LEMMA ([**?**]). *If (T^k,M) is an effective action of a torus on a closed aspherical manifold and if $H \subset \pi_1(M,x)$ is a central subgroup which contains $\mathrm{Im}(\mathrm{ev}_*^x)$, then $H/\mathrm{Im}(\mathrm{ev}_*^x)$ contains no elements of finite order.*

PROOF. Let M_H be the covering space associated to the subgroup H (i.e., with $\pi_1(M_H) = H$). By Theorem 1.11.1, the T^k action lifts to M_H. We will show first that (T^k, M_H) is free. Let $b \in M_H$ and suppose T_b be the stabilizer of the action (T^k, M_H). Let $p : (M_H, b) \to (M, p(b))$ be the covering projection. If $T_b \neq 1$, then there exists a finite cyclic subgroup $F \subset T_b \subset T_{p(b)}$. Let

$$g : (I,0,1) \longrightarrow (T^k, e, f),$$

where f is a generator of F. The path $p(g(t)\cdot b)$ is the projection of a loop based at $b \in M_H$ to a loop based at $p(b)$ in M. The homotopy class of $p(g(t)\cdot b)$ is an element of H which is in the center of $\pi_1(M)$, because H is central and $p_* : H \to \pi_1(M,x)$ is injective to center. Let

$$\alpha : (I,0,1) \longrightarrow (M, p(b), p(b))$$

be a loop in M based at $p(b)$. By Lemma 1.12.1,

$$[f\cdot\alpha(s)] = [p(g(t)\cdot b) * \alpha(s) * \overline{p(g(t)\cdot b)}].$$

But as $p(g(t)\cdot b)$ is in the center of $\pi_1(M,p(b))$, we have $f_*(\alpha) = \alpha$. In other words, $F \longrightarrow \mathrm{Aut}(\pi_1(M,p(b)))$ is trivial. This contradicts that $F \longrightarrow \mathrm{Aut}(\pi_1(M,p(b)))$ must be injective since F fixes $p(b)$, see Corollary **??** (ii). So $T_b = 1$ for each $b \in M_H$.

Put $N = \pi_1(M_H)/\mathrm{Im}(\mathrm{ev}_*^b) = H/\mathrm{Im}(\mathrm{ev}_*^{p(b)})$. We can lift the free torus action (T^k, M_H) to the splitting $(T^k, T^k \times W)$-action; see Theorem 1.14.2. Here W is maximal torus action

contractible because M is aspherical. (Actually, W is a contractible manifold or a contractible cohomology manifold). Now $T_b/T_y \cong N_w$, by Corollary 1.12.7, where $y \mapsto b$ and $y \mapsto w$ under the projections $T^k \times W \to M_H$ and $T^k \times W \to W$. Since $T_b = 1$ for each $b \in M_H$, $N_w = 1$ for each $w \in W$. But, W is finite dimensional and contractible, so each prime order cyclic subgroup of N must fix some non-empty subset of W, by Lemma 1.15.3. Therefore, N is torsion free. Since $\mathrm{Im}(\mathrm{ev}_*^{p(b)}) \subset H \subset \pi_1(M,x)$, with H central and $\mathrm{Im}(\mathrm{ev}_*^b) = \mathrm{Im}(\mathrm{ev}_*^{p(b)})$, $H/\mathrm{Im}(\mathrm{ev}_*^b)$ contains no element of finite order. □

1.15.10 Corollary ([**?**, Lemma 2]). *If (T^k, M) is an effective action of a torus on a closed aspherical manifold and if $H \subset \pi_1(M,x)$ is a finitely generated central subgroup for which $\mathrm{Im}(\mathrm{ev}_*^x) \subset H$, then $\mathrm{Im}(\mathrm{ev}_*^x)$ is a direct summand of H.*

1.15.11 Corollary. *Let (T^k, M) be a free action on a closed aspherical manifold. Then $\pi_1(M,x)/\mathrm{ev}_*^x(\pi_1(T^k,1))$ is torsion free.*

Proof. Let $H = \mathrm{Imev}_*^x(\pi_1(T^k,1))$, $Q = \pi_1(M,x)/\mathrm{ev}_*^x(\pi_1(T^k,1))$. Then the (T^k, M) action lifts to $(T^k, M_H) = (T^k, T^k \times W)$ by Theorem 1.14.2, and commutes with the covering Q action. This induces a proper and effective action of Q on W. Since W is contractible, any p-subgroup of Q must fix a non-trivial subset of W (Lemma 1.15.3). Then by Corollary 1.12.7, T^k could not be free. □

1.15.12 Definition. Any compact, connected Lie group which acts effectively on a closed aspherical manifold is a torus T^k with $k \leq$ rank of $\mathcal{Z}(\pi_1(M))$, the center of $\pi_1(M)$. When $k =$ rank $\mathcal{Z}(\pi_1(M))$, the torus action is called a *maximal torus action*.

1.15.13 Corollary. *Let M be a closed aspherical maniflod for which the center of its fundamental group is finitely generated. If (T^k, M) is a maximal torus action, then $\mathrm{Im}(\mathrm{ev}_*^x) = \mathrm{Center}\ \pi_1(M,x)$. Conversely, if $\mathrm{Im}(\mathrm{ev}_*^x) = \mathrm{Center}\ \pi_1(M,x)$, then (T^k, M) is a maximal torus action on M.*

appl-chapter

In Chapter **??**,♣ we shall examine in detail maximal torus actions on many types of aspherical manifolds.

1.15.14 Remark. There are two unsolved problems here.

(1) Let M be a closed aspherical manifold and $\mathcal{Z} = \mathrm{Center}\ \pi_1(M)$. Is $\mathcal{Z}$ finitely generated? No examples of closed aspherical manifold with non-finitely generated $\mathcal{Z}$ are known ot us.
(2) Let M be a closed aspherical manifold. Suppose $\mathcal{Z} = \mathrm{Center}\ \pi_1(M) \neq 1$. Does M admit a non-trivial action of S^1? Does M admit a maximal torus action? Again no examples of closed aspherical manifolds without a maximal torus action are known to us.

1.16. Cohomology manifolds and the Smith theorems

cohomology m-manifold
cohomologically locally connected
local cohomology group
orientable
non-orientable

Lemma 1.15.3 us an example of one of the important Smith theorems. The technique of proof employed there can be expanded to prove much stronger results of Smith. These Smith theorems play a very important role in transformation groups. We shall now record some of them. For a full account, one can consult the original papers of P. A. Smith or more recent proof and/or expositions found in [**?**] or [**?**].

1.16.1. Let (G, M) be a non-trivial action on a connected m-manifold M, where $G = T^k$ or a finite p-group, p a prime. Then

(1) $F = M^G$ is a cohomology manifold over $\mathbb{Z}$ for $G = T^k$, (respectively, over $\mathbb{Z}_p$, for G a p-group) of dimension $\leq m-2$ (respectively, $\leq m-1$). F, itself, could be empty, in which case, we say $\dim F = -1$.
(2) $\dim F \equiv m (\text{mod } 2)$ if $G = T^k$, or p-group, $p \neq 2$.
(3) If the action is effective, then $\dim F \leq m - 2k$ for $G = T^k$ or $G = \mathbb{Z}_{p^k}$ with $k \neq 1$, if $p = 2$. If $p = 2$, $\dim(F) \leq n - 1$.
(4) $\chi(F) = \chi(M)$ if $G = T^k$ (respectively, $\chi(F) \equiv \chi(M) \bmod p$ if $G = \mathbb{Z}_p$) when $\chi(M)$ is defined.
(5) If M is oriented and G, a 2-group, preserves orientation, then $\dim F \equiv m (\text{mod } 2)$.
(6) If the action is smooth, then F is a smooth submanifold of M.
(7) If M is $\mathbb{Z}$-acyclic (respectively, $\mathbb{Z}_p$-acyclic), then M^{T^k} (respectively, M^G, G a p-group) is $\mathbb{Z}$-acyclic (respectively, $\mathbb{Z}_p$-acyclic).
(8) If M has the $\mathbb{Z}$-homology (respectively, the $\mathbb{Z}_p$-homology) of the m-sphere, then M^{T^k} (respectively, M^G, G a p-group) has the $\mathbb{Z}$-homology (respectively, the $\mathbb{Z}_p$-homology) of an $(m-r)$-sphere, $r \geq 0$.

1.16.2. A definition of a *cohomology m-manifold* X over a PID L can be given as:

(1) X is a locally compact space with a countable neighborhood basis.
(2) $H^{m+1}(X, A; L) = 0$ for all closed subsets A of X, (i.e, $\dim_L(X) \leq m$).
(3) If U is a neighborhood of x, then there exists a neighborhood V of x, contained in U and $H^*(U; L) \to H^*(V; L)$ is trivial, (i.e., *clc* = *cohomologically locally connected*).
(4) $H^p(X, X - x; L) = L$ if $p = m$, and 0 otherwise, and for all $x \in X$. [$H^p(X, X - x; L)$ is called the *local cohomology group* in dimension p at x].

Each U, open, connected, and with compact closure satisfies $H^m(X, X - U; L) = L$ or $L/2L$. If it is always L, then X is called *orientable* over L. If not, then X is *non-orientable*. In particular, if X is compact and connected, then $H^m(X; L) \cong L$ if orientable and $H^m(X; L) \cong L/2L$ if not. Furthermore, if $V \subset U$, then the homomorphism induced by the inclusion $H^m(X, X - V) \to H^m(X, X - U)$ is an isomorphism, when U and V are as above. If A is closed in X and $A \neq X$, X connected, then $H^m(A; L) = 0$.

Cohomology manifolds cannot be avoided if one wishes to study non-smooth actions. For, the fixed point set of a p-group or toral group acting non-smoothly

homologically locally connected
locally smooth

may fail to be locally Euclidean. However, it will be a cohomology manifold over the appropriate L. These cohomology manifolds behave homologically just as manifolds. They satisfy Poincaré duality, both globally and locally (i.e., relative duality). They have a fundamental class when orientable and a twisted fundamental class when not orientable. While these cohomology manifolds will appear in creating topological actions, all that we will actually use are the facts of subsection **??**. More details about cohomology manifolds can be found in [**?**] and [**?**].

Cohomology manifolds are not necessarily ANR's and so the convenient cohomology theory to use is the Čech or equivalently the Alexander-Spanier theory. This agrees with the singular theory if X is *hlc* (*homologically locally connected*) over L; that is, if every neighborhood U of $x \in X$ has a neighborhood V, $x \in\subset U$, so that $i_* : H_*(V; L) \to H_*(U; L)$ is trivial with respect to singular homology (for example, if X is locally contractible). The appropriate homology to use is the Borel-Moore homology (or, equivalently, the Čech homology whenever L is a field). Again we can substitute the singular homology if the *hlc* over L condition holds.

For a finite dimensional simplicial complex, this reduces to the condition

$$H^p(\mathrm{St}(v), \partial\mathrm{St}(v); L) \cong H^p(D^m, S^{m-1}; L)\ (\cong H^p(X; X - v; L))$$

for each vertex v, and $\mathrm{St}(v)$ is the star of v.

1.16.3. Examples of simplicial cohomology manifolds that arise as fixed points in non-smooth actions on smooth manifolds are: (a) Suspensions of smooth manifolds having the integral homology type of S^{n-1} but are not S^{n-1}; (b) Suspension of $\mathbb{R}P_n$ when $L = \mathbb{Z}_p$ or $\mathbb{Q}$, $p \neq 2$ and n odd.

For non-simplicial examples, one can take a badly (i.e., wild) embedded arc A in a smooth manifold M and collapse A to a point. Then M/A is a cohomology manifold of the same homotopy type as M, but is not locally Euclidean at the point $\{A/A\}$.

1.16.4. Another way to avoid the use of cohomology manifolds is to use the concept of "locally smooth actions" which was introduced by Bredon in [**?**]. Let (G, M) be a G-space. The action is called *locally smooth* if for each $x \in M$, there is a G_x-slice S at x such that the G_x-space S is equivalent to an orthogonal G_x-space. Consequently, the G-invariant tube $G \times_{G_x} S$ is a "linear" tube. Note S is homeomorphic to $\mathbb{R}^n$ and the action of G_x on S is topologically equivalent to an orthogonal action of G_x on $\mathbb{R}^n$. Thus, M is a topological manifold. It is not necessarily a smooth manifold, but each G_x-invariant tube is a smooth G-space. Moreover, if $H \subset G$ is a closed subgroup, then M is locally smooth as an H-space. A consequence of this is M^H is a (topological) submanifold of M, for each closed $H \subset G$. Of course, smooth actions are locally smooth. For further information, the reader should consult [**?**].

1.17. Manifolds on which only tori can act

Should this be in the first chapter?? — To a later Chapter??

The techniques of the preceding sections were developed in [**?**] and were then used to show that the only connected compact Lie groups G that could act effectively on a closed aspherical manifold were the tori, and they had to act injectively. In this section, we extend the class of manifolds on which only tori can act (see Theorem **??**). The material is taken from [**?**].

Aspherical
aspherical
Hyper-aspherical
K-manifold
admissible space

1.17.1. A connected, closed, oriented m-manifold M with $\Pi = \pi_1(M)$ is called:

(1) *Aspherical* if $\pi_i(M) = 0$, for all $i > 1$; M is therefore a $K(\Pi, 1)$.
(2) *Hyper-aspherical* [**?**] if there exists a closed aspherical m-manifold N and a map $f : M \to N$ of degree 1. This is equivalent to saying $f^* : H^m(N;\mathbb{Z}) \to H^m(M;\mathbb{Z})$ is onto.
(3) *K-manifold* [**?**] if there exists a *torsion-free* group Γ and a map $f : M \to K(\Gamma, 1)$ so that $f^* : H^m(K(\Gamma, 1), \mathbb{Z}) \to H^m(M, \mathbb{Z})$ is onto.
(4) *Admissible* [**?**] if the only periodic self-homeomorphisms of $\widetilde{M}$ commuting with the deck transformation group $\pi_1(M)$ are elements of the center of $\pi_1(M)$. Is this true in the non-orientable case?

1.17.2 LEMMA. [**?**, Lemma 2.5] *Let $G = \mathbb{Z}_q$ (q a prime) act non-trivially on a closed connected oriented n-manifold M. Let $p : M \to \mathbb{Z}_q\backslash M$ be the natural projection. Then the map $p^* : H^n(\mathbb{Z}_q\backslash M;\mathbb{Z}) \to H^n(M;\mathbb{Z})$ is not surjective.*

PROOF. Let F be the fixed point set of G. Then F is nowhere dense and the induced orbit map $p_1 : M - F \longrightarrow G\backslash M - F$ is a finite covering of topological manifolds. In particular, both (M, F) and $(G\backslash M, F)$ are relative topological n-manifolds; i.e., their differences are connected n-manifolds. Smith theory used here. If G acts orientation-preservingly, then $\dim F \leq n - 2$, as noted in [**?**, §1] and hence the horizontal arrows in diagram below

$$\begin{array}{ccc} H^n(G\backslash M, F) & \xrightarrow{j_0^*} & H^n(G\backslash M) \\ {\scriptstyle p_0^*}\downarrow & & {\scriptstyle p^*}\downarrow \\ H^n(M, F) & \xrightarrow{j^*} & H^n(M) \end{array}$$

are isomorphisms. Therefore it suffices to show that p_0^* is not surjective. But it is clear that $M - F$ and $G\backslash M - F$ are connected topological manifolds and p_1 has degree q. A diagram chase then shows that $H^n(G\backslash M, F) \cong \mathbb{Z} \cong H^n(M, F)$ and p_0^* corresponds to multiplication by q. Hence p^* is not surjective.

If G does not preserve orientation, then $G = \mathbb{Z}/2$. In this case the invariant cohomology group $H^n(M;\mathbb{Q})^{\mathbb{Z}/2} = 0$ since $H^n(M;\mathbb{Q}) = H^n(M;\mathbb{Z}) \otimes \mathbb{Q}$. Consider the following diagram.

$$\begin{array}{ccccc} H^n(M;\mathbb{Z}) & \xrightarrow{\alpha} & H^n(M;\mathbb{Z}) \otimes \mathbb{Q} & \xrightarrow{=} & H^n(M;\mathbb{Q}) \\ {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*\otimes 1}\uparrow & & \uparrow \\ H^n(G\backslash M;\mathbb{Z}) & \xrightarrow{\beta} & H^n(G\backslash M;\mathbb{Z}) \otimes \mathbb{Q} & \xrightarrow{=} & H^n(G\backslash M;\mathbb{Q}) \end{array}$$

where the last map $H^n(G\backslash M;\mathbb{Q}) \to H^n(M;\mathbb{Q})$ factors through

$$H^n(G\backslash M;\mathbb{Q}) \xrightarrow{p_1^*} H^n(M;\mathbb{Q})^{\mathbb{Z}/2} \xrightarrow{\text{incl}} H^n(M;\mathbb{Q})$$

The map α is injective, and β is injective on the free part of $H^n(G\backslash M;\mathbb{Z})$. Furthermore p_1^* is an isomorphism by a standard transfer argument. (Compare [4 or 5].) Therefore $H^n(G\backslash M;\mathbb{Q}) = 0$, so that $H^n(G\backslash M,\mathbb{Z})$ is a torsion group. In fact, similar considerations also show that the latter is at most a 2-torsion group. In any case, we have shown enough to guarantee that p^* is zero if G does not preserve orientation. $\square$

1.17.3 THEOREM. *Aspherical* $\Longrightarrow$ *Hyper-aspherical* $\Longrightarrow$ *K-manifold* $\Longrightarrow$ *Admissible.*

PROOF. We only need to prove K-manifold $\Longrightarrow$ Admissible. Let $\Pi = \pi_1(X)$. Suppose M is a K-manifold which is not admissible. Then there exists a homeomorphism h of $\widetilde{M}$ so that

(i) h commutes with Π,
(ii) $h^k = \mathrm{id}$, for some $k > 1$,
(iii) $h \notin \mathcal{Z}(\Pi)$, the center of Π.

Let p be the smallest integer so that $h^p \in \mathcal{Z}(\Pi)$, $1 < p \leq k$. Let $k = d \cdot p$. We may assume p is a prime by choosing a power of h if necessary. Then

$$\mathbb{Z}_k = \{h, h^2, \ldots, h^k\} \subset C_{\mathrm{TOP}(\widetilde{M})}(\Pi),$$

the centralizer of Π in $\mathrm{TOP}(\widetilde{M})$,

$$\mathbb{Z}_d = \{h^p, h^{2p}, \ldots, h^{dp}\} = \mathbb{Z}_k \cap \Pi = \mathbb{Z}_k \cap \mathcal{Z}(\Pi)\,.$$

Such an h defines an action of $\mathbb{Z}_p \cong \mathbb{Z}_k/\mathbb{Z}_d$ on M. The lifting sequence (see section 1.8.1) of $(\mathbb{Z}_p, M)$ is $1 \to \Pi \xrightarrow{i} E \to \mathbb{Z}_p \to 1$ and $\Pi \supset \mathcal{Z}(\Pi) \supset \mathbb{Z}_d$, $E \supset C_E(\Pi) \supset \mathbb{Z}_k$ so that $1 \to \mathbb{Z}_d \to \mathbb{Z}_k \to \mathbb{Z}_p \to 1$ is exact.

Assume $\mathrm{Fix}(\mathbb{Z}_p, M) = \emptyset$. Then $\Pi_1(M/\mathbb{Z}_p) = E$. The set of torsion elements of $C_E(\Pi)$ forms a fully invariant subgroup of $C_E(\Pi)$ coinciding with $tC_E(\Pi)$, the smallest normal subgroup containing all torsion elements, and

$$1 \to t(\mathcal{Z}(\Pi)) \to t(C_E(\Pi)) \to \mathbb{Z}_p \to 1$$

is exact [**?**, (1.2)]. Then $\Pi/t(\mathcal{Z}(\Pi)) \cong E/t(C_E(\Pi))$. The kernel of the homomorphism $f_* : \Pi \longrightarrow \Gamma$, induced from $f : M \to K(\Gamma, 1)$ contains the smallest normal subgroup containing all the torsion of Π. Therefore $\Pi \to \Gamma$ factors through $\Pi/t(\mathcal{Z}(\Pi))$. Consequently, we may extend the homomorphism $\Pi \to \Gamma$ to $E \to \Gamma$ via

$$\begin{array}{ccccc} \Pi & \longrightarrow & \Pi/t(\mathcal{Z}(\Pi)) = E/t(C_E(\Pi)) & \longrightarrow & \Gamma \\ & & \uparrow & & \\ & & E \cong \pi_1(M/\mathbb{Z}_p) & & \end{array}$$

If $\mathrm{Fix}(\mathbb{Z}_p, M) \neq \emptyset$, then $E \cong \Pi \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the stabilizer of E at a preimage of a fixed point. Now $\pi_1(M/\mathbb{Z}_p) \cong E/N$, where N is the smallest normal subgroup containing all the stabilizers [**?**]. Since $\mathbb{Z}_p \subset N$ already, $\pi_1(M/\mathbb{Z}_p)$ is a quotient of Π by a normal subgroup of Π generated by torsion elements. Thus the homomorphism $\Pi \to \Gamma$ again factors through $\pi_1(M/\mathbb{Z}_p) \to \Gamma$.

In either case, we have

$$\begin{array}{ccc} \pi_1(M) & \twoheadrightarrow & \pi_1(M/\mathbb{Z}_p) \\ \downarrow & \swarrow & \\ \Gamma & & \end{array}$$

This induces a homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{q} & M/\mathbb{Z}_p \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ K(\Gamma,1) & & \end{array}$$

where q is the orbit mapping. The map g can be constructed [**?**, p.428], because $M/\mathbb{Z}_p$ has the homotopy type of a CW-complex since Floyd has shown that $M/\mathbb{Z}_p$ is an ANR [**?**]. The induced diagram on cohomology $f^* = q^* \circ g^*$ in dimension m leads to a contradiction, for it was assumed that f^* was onto, but $g^* : H^m(M/\mathbb{Z}_p;\mathbb{Z}) \longrightarrow H^m(M;\mathbb{Z})$ is never onto, by Lemma **??**. □

1.17.4 COROLLARY. *If N is a closed aspherical manifold, then no non-trivial finite subgroup of homeomorphisms of the universal covering $\widetilde{N}$ of N can commute with the covering transformations of N.*

PROOF. If N is orientable, then N is admissible. The center of $\pi_1(N)$ is torsion-free, by Corollary 1.15.4, so the conclusion follows. If N is not orientable, then its orientable double cover $\widehat{N}$ is admissible. Any finite subgroup acting on $\widetilde{N}$ and commuting with $\pi_1(N)$ will commute with the subgroup $\pi_1(\widehat{N})$, which means the group is trivial. □

All statement involving "admissible" should extend to "non-orientable manifolds whose orientable double covering are admissible.

1.17.5 THEOREM. *Any effective torus action on an admissible closed connected manifold is injective:*

PROOF. If S^1 is a circle subgroup of T^k, then it is a direct summand of T^k. Therefore it is easy to see that an action of T^k is injective if and only if every circle subgroup of T^k acts injectively. Assume that there is a subgroup S^1 which fails to be injective. Then $\mathrm{ev}^x_*(\pi_1(S^1,1)) = C \subset \pi_1(X,x)$ is a finite cyclic subgroup. Lift the S^1-action to M_C, the covering space of M with $\pi_1(M_C) = C$. This action is still effective, for if $H \subset S^1$ fixes all of M_C, then it also must fix all of M. If $C = 1$, then M_C is the universal covering $\widetilde{M}$. Otherwise, there is a $|C|$-fold cover $'S^1$ of S^1, where $|C|$ is the order of C. There is an effective action of $'S^1$ on $\widetilde{M}$, covering the action of $'S^1$ on M. The action of $'S^1$ commutes with $\pi_1(M)$ on M and $C \subset {'S^1}$ with $'S^1/C \cong S^1$. The group $C \subset \mathit{\Pi}$, so we take a prime p so that action!almost effective

$\mathbb{Z}_p \subset {}'S^1$ satisfies $C \cap \mathbb{Z}_p = 1$. Therefore, $\mathbb{Z}_p \times \Pi$ acts effectively on $\widetilde{M}$ and $\mathbb{Z}_p$ is not in the center of Π. This contradicts admissibility. □

1.17.6 Definition. An action (G, X) is *almost effective* if the kernel of the action of G on X is a finite subgroup of G.

1.17.7 Exercise. Theorem **??** still holds if the T^k-action is assumed to be almost effective.

1.17.8. Rewrite? More generally, the theorem holds for all manifolds M^n with $\xi^* : H^n(K(\pi_1(M), 1); \mathbb{Q}) \to H^n(M; \mathbb{Q})$ surjective, where ξ^* is induced by the classifying map $\xi : M \to K(\pi_1(M), 1)$ is injective. See [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] and [**?**] for such generalizations. Also, the theorem holds for homologically Kähler manifold all of whose isotropy subgroups are finite (e.g., holomorphic actions), [**?**, p.170], [**?**, p.186].

1.17.9 Theorem. *Let G be a compact connected Lie group acting almost effectively on an admissible manifold M. Then*

(1) *G is a torus acting injectively with* $\dim(G)$ *at most the rank of the center of $\pi_1(M)$.*
(2) *all isotropy groups are finite.*
(3) $\chi(M) = 0$.

Proof. From the structure theory of compact connected Lie groups, there is a covering group G' which maps homomorphically onto G with kernel a finite subgroup of the center of G. Furthermore, G' splits as a product into $T^k \times G_1 \times \cdots \times G_n$, where each of the G_i are simple and simply connected. Thus G', via this homomorphism, acts almost effectively on M. Clearly the torus factor T^k must act injectively by Exercise **??**. Let G_i be a simple factor. Then G_i contains a non-trivial maximal torus T_i of dimension > 0. Thus $\mathrm{ev}_{\#}^* : \pi_1(T_i) \to \pi_1(G_i) \to \pi_1(M)$ is the trivial homomorphism contradicting admissibility. Therefore, G_i is trivial. Then (1) and (2) follow immediately from properties of injective actions. If $\chi(M) \neq 0$, then $\chi(M^{T^k}) = \chi(M)$ (subsection **??**; proved by [**?**] or [**?**], see also subsection **??** (4)) which would mean that $M^{T^k} \neq \emptyset$, and so the T^k-action could not be injective. [If we wish to avoid the reference for the equality $\chi(M^{T^k}) = \chi(M)$, we can take a $g \in S^1 \subset T^k$. Then, by the Lefschetz fixed point formula, there is an x such that $g(x) = x$ and if the powers of g is dense in S^1, then $x \in M^{S^1}$.] □

1.17.10 Exercise. Let $c : (M, x) \to (N, y)$ be a finite regular covering of N by an admissible manifold M. Let H be the image $c_*(\pi_1(M, x)) \subset \pi_1(N, y)$, and suppose there exists an action of a compact connected Lie group G on N whose image $\mathrm{ev}_{\#}^y(\pi_1(G, e)) \subset H$. Then the conclusions of Theorem **??** still hold. Show also, if N is non-orientable and M is the orientable double covering, then $\mathrm{ev}_*^y(\pi_1(G, e)) \subset H$. Hint: The lifted action to the universal covering (and that is by the group $G_{\ker(\mathrm{ev}_*)}$) preserves orientation, and translates into $\mathrm{ev}_*^y(\pi_1(G, e)) \subset H$.

1.17.11 EXERCISE. [**?**, Corollary 5.7] Suppose M is an aspherical m-manifold perhaps with boundary and *not* closed. Let k be the smallest integer such that $H^{m-k}(M;\mathbb{Z}) \neq 0$. If G is a compact connected Lie group acting effectively on M, and with fixed point, then rank of the maximal torus T^s of G is $\leq k/2$. (Hint: $\dim(M^{T^s}) \leq m - 2s$, where $s > 0$.)

1.17.12 COROLLARY. *If M is admissible and G is a compact Lie group which acts effectively, then*

(i) $\mathcal{Z}(\pi_1(M)) = 1$ *implies G is finite and $\Psi : G \to \mathrm{Out}(\pi_1(M))$ is injective.*
(ii) $\mathrm{Fix}(G, M) \neq \emptyset$ *implies G is finite and $\theta : G \to \mathrm{Aut}(\pi_1(M))$ is injective.*

Furthermore, if M is a non-orientable manifold whose orientable double cover M' is admissible, then statements (i) *and* (ii) *also holds for M.*

Here Ψ is the abstract kernel induced from the lifting sequence $1 \to \pi_1(M) \to E \to G \to 1$ and θ is the representation into $\mathrm{Aut}(\pi_1(M))$ when a base point is chosen to be fixed by G. The corollary extends well-known results of [**?, ?, ?**], [**?**] and a unpublished result of A. Borel.

PROOF. We first assume that M is admissible. (i) If G is compact, let G_0 denote the connected component of the identity. The maximal torus of G_0 must act injectively by theorems **??**, **??** and the hypothesis. This maximal torus action must be trivial since $\mathcal{Z}(\pi_1(M))$ is trivial. Hence $G_0 = 1$, so G is finite. Let $K = \ker(\Psi)$. Then if $K \neq 1$, there is a prime p such that $\mathbb{Z}_p \subset K$. Form the lifting sequence $1 \to \pi_1(M) \to E \to \mathbb{Z}_p \to 1$. With the center $\pi_1(M)$ trivial, and $\mathbb{Z}_p \subset K$, the group extension E must be a product (since $\pi_1(M) \times \mathbb{Z}_p$ is an extension when $\Psi(\mathbb{Z}_p) = 1$ and collection of all such extensions are given by $H^2(\mathbb{Z}_p; \mathcal{Z}(\pi_1(M))) = 0$). So there is just one such extension. See Chapter 1.11.5 for such details. But M is admissible and $\mathit{\Pi} \times \mathbb{Z}_p$ acts effectively. Therefore, we have a contradiction and so $K = 1$.

(ii) As above, the maximal torus of G_0 must act injectively. But as $\mathrm{Fix}(G, M) \neq \emptyset$, $G_0 = 1$, and G is finite. The lifting sequence is given by $\pi_1(M, x) \rtimes_\theta G$, where $x \in M^G$, and $\theta : G \to \mathrm{Aut}(\pi_1(M, x))$ is the induced homomorphism. If $g \in \ker(\theta)$, we may take some power m so that $g' = g^m$ has order p, a prime. Then we have $\pi_1(M, x) \times \mathbb{Z}_p$ acting effectively on $\widetilde{M}$ contradicting admissibility of M. Therefore θ is injective.

Orientability of M is built into the definition of admissible. So now assume M is non-orientable with M' its orientable double covering admissible. Let G_0 be the connected component of the identity of G. For case (i), the image of the evaluation homomorphism is trivial. Therefore, G_0 lifts to M' and must be a torus acting injectively on M'. The action of G_0 can be lifted to the universal covering $\widetilde{M}$ of M, but it cannot be lifted to the universal covering of M', unless G_0 is trivial. Since the universal coverings of M and M' are identical, G_0 must be trivial. Now as before, let $K = \ker(\Psi)$. Then, as above, $\mathit{\Pi} \times \mathbb{Z}_p$, $\mathbb{Z}_p \subset K$, acts effectively on $\widetilde{M}$, and $\mathbb{Z}_p$ commutes with $\pi_1(M', x) \subset \pi_1(M, x)$. Therefore, $\mathbb{Z}_p$ must be in the center of $\pi_1(M', x')$ since M' is admissible. Consequently, $\mathbb{Z}_p$ cannot act effectively on M and we have a contradiction, implying that K is trivial.

For part (ii), once again G_0 is trivial, since G_0 lifts to M', with fixed points, and M' is admissible. As before, $\pi_1(M, x) \times \mathbb{Z}_p$ acts effectively on $\widetilde{M}$. $\mathbb{Z}_p$ commutes with the action of $\pi_1(M', x')$. Since M' is admissible, $\mathbb{Z}_p$ will have to be in the center of $\pi_1(M', x')$. This will contradict that $\mathbb{Z}_p$ acts effectively on M. □

Examples of non-orientable closed manifolds whose orientable double covers are admissible, other than the obvious ones, are all the closed non-orientable aspherical manifolds.

[question] If M is admissible and $M \to M^*$ is a (finite) covering, is M^* admissible? If M is admissible and $M' \to M$ is a finite (regular) covering, is M' admissible?

[Second question: problematic] *If $M \to M^*$ is a finite regular covering and M^* is admissible, is M admissible?* Let us not assume M^* is orientable, just assume that every cyclic group action on the universal covering $\widetilde{M}^*$ and commutes with $\pi_1(M^*)$ must be in the center of the covering transformations. Then if C is some cyclic group actiong on $\widetilde{M}$, the universal covering of M, and commuting with the covering transformations of $\pi_1(M)$ on $\widetilde{M}$, then we must show it belongs to the center of $\pi_1(M)$ (as covering transformations). So our problem is to somehow construct or extend the induced action or part of it on M to one on M^* at least. But it is not likely that we are to find that $\imath_1(M)$ in $\pi_1(M^*)$ is invariant in general. So I think this direction is diffeicult (or tricky at least). It is an interesting question by it would seem to take some thought to settle it one way or another.

[newly added exer]

1.17.13 Exercise. Let $M \to M^*$ be a finite regular covering with M admissible. Then each effective action of a cyclic group on the universal covering of M^*, $\widetilde{M}^*$, that commutes with the covering transformations of $\widetilde{M}^*$ is a covering action. (That is, M^* is also admissible if M^* is orientable, or satisfies the crucial condition of admissibility if M^* is not orientable. Is the orientable assumption of M really necessary to obtain the crucial condition?)

Proof. [leave out?] Let C be a cyclic group acting effectively on $\widetilde{M}^*$. If C commutes with the covering transformations of $\widetilde{M}^*$, $\pi_1(M^*)$m then C commutes with the subgroup $\pi_1(M)$ of covering transformations. Since M is admissible, C belongs to the covering transformations of $\widetilde{M}$. Thus it also belongs tot he covering transformations of $\widetilde{M}^*$.

1.17.14 Exercise. Let X be a closed aspherical manifold and $\theta : N(\pi_1(X, x)) \to \mathrm{Aut}(\pi_1(X, x))$, where $N(\pi_1(X, x))$ is the normalizer of $\pi_1(X, x)$ in $\mathrm{TOP}(\widetilde{X})$, be as in section 1.8.1. Then θ is injective on any torsion subgroup. (The proof uses the fact that $\pi_1(X, x)$ is torsion free).

1.17.15 Example. Let G be a compact Lie group acting effectively on the m-torus so that $\Phi : G \to \mathrm{Out}(\pi_1(T^m)) = \mathrm{GL}(m, \mathbb{Z})$ is trivial. Then $G = T^s \times F$, and it acts freely on T^m, where F is finite abelian and isomorphic to a subgroup of T^{m-s}, for some $0 \le s \le m$. Furthermore, F acts as covering transformations and $F\backslash T^m$ is again a topological torus while the T^s action splits into a product action.

Proof. The connected component of the identity must be an s-torus for some $0 \le s \le m$, and acts injectively. Suppose there is a $y \in T^m$ so that T^s_y contains an element h of order p. Let $g(t)$ be a path in $T^s = G_0$ from 1 to h. Then

$h(t)y$ is a loop at y. For any loop $\ell(t)$ based at y, the loop $\ell(t)$ is homotopic to $g(t)(y) * h(\ell(t)) * \overline{g(t)(y)}$. But as $\pi_1(T^m)$ is abelian, $\ell(t) \simeq h(\ell(t))$. That is, the automorphism $h_* \in \mathrm{GL}(m, \mathbb{Z})$ is trivial. Hence in the lifted sequence, the semi-direct product $\mathbb{Z}^m \rtimes \mathbb{Z}_p$ is a direct product. But, this contradicts admissibility of T^m. So $T^s_y = 1$ for each $y \in T^m$. That is, the action of T^s is free on T^m.

Moreover, we claim it is a product action. Now put $H = \mathrm{Im}(\mathrm{ev}^x_*(\pi_1(T^s)))$ and $Q = \pi_1(T^m)/H \cong \mathbb{Z}^m/\mathbb{Z}^s$. Q acts properly on $T^s\backslash T^m_H = W$, a contractible (cohomology) manifold. Then, Q, which is abelian, must act freely since $Q_w = 1$ for every $w \in W$ by $Q_w \cong T^s_y/T^s_{\widehat{y}} = 1$ from Theorem 1.11.1. But as W is contractible, this means Q is torsion free, by Corollary 1.15.4 . So $\mathrm{Im}(\mathrm{ev}^x_*(T^s)) \cong \mathbb{Z}^s$ is a direct summand of $\mathbb{Z}^m$. This will force the action to split as we have seen earlier in Theorem 1.14.2, so $(T^s, T^m) = (T^s, T^s \times Y)$ where Y is a (cohomology) manifold of the homotopy type of T^{m-s}.

Now F acts on Y, and acts trivially on $\pi_1(Y)$, a summand of $\mathbb{Z}^m$. In the lifting sequence $1 \to \pi_1(Y) \to E \to F \to 1$, the group E acts trivially on $\pi_1(Y)$ because G acts trivially on $\pi_1(T^m)$. Thus, E is a central extension of $\pi_1(Y) \cong \mathbb{Z}^{m-s}$ and since it maps trivially into $\mathrm{GL}(m-s, \mathbb{Z})$, it must be torsion free. Otherwise, E would contain an effective action of $\pi_1(Y) \times \mathbb{Z}_p$, for some prime p, on $\widetilde{Y}$, a contractible (cohomology) manifold with compact quotient which is impossible. The Theorem **??** was proven for topological manifolds, but it is also true by the same argument for ANR cohomology manifolds. Then Y is an ANR aspherical cohomology manifold homotopy equivalent to $(m-s)$-torus.

[If the action is smooth (respectively, locally smooth), then Y is a smooth manifold (respectively, topological manifold), with the homotopy type of T^{m-s}. In particular, using standard surgery results, Y will be homeomorphic to T^{m-s} provided $m - s \neq 3$. For some pathological topological actions, Y could fail to be locally Euclidean.]

Therefore, E as we shall see later in ()♣ is isomorphic to $\mathbb{Z}^{m-s}$, and F is abelian, ???
Moreover, as it must act properly and freely, E acts as covering transformations on $\widetilde{Y}$. Actually we can now see the structure of G as $T^s \times F$ where F is an abelian group isomorphic to a subgroup of T^{m-s}. As G splits as a product, we let F act on T^m. Then it acts as covering transformations since it does so on the factor Y. Of course, $F\backslash T^m$ is again a topological torus. The T^s action can be treated as acting on T^m or $F\backslash T^m$. □

This result for the torus has a generalization to infra-nilmanifolds and will be treated in Chapter 10. Is this right??

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1.17.16 Example. There are no shortage of closed aspherical manifolds. For example, if Γ is discrete and acts properly and freely on $\mathbb{R}^n$, then $\Gamma\backslash\mathbb{R}^n$ is an aspherical manifold (a $K(\Gamma, 1)$-manifold). If Γ acts so that the quotient is compact, then $\Gamma\backslash\mathbb{R}^n$ is a closed aspherical manifold. Typical examples arise by taking Γ a torsion free cocompact discrete subgroup of a Lie group G with finitely many components and K a maximal compact subgroup of G. Since K is a maximal compact subgroup, G/K is diffeomorphic to $\mathbb{R}^n$ for some n, and Γ being torsion free implies $\Gamma \cap K = 1$ so that $\Gamma\backslash\mathbb{R}^n = \Gamma\backslash(G/K) = (\Gamma\backslash G)/K$ is a closed aspherical manifold.

For example, each closed 2-manifold whose Euler characteristic is negative is of the form $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})\rtimes\mathbb{Z}_2/\mathrm{O}(2)$ where Γ is the fundamental group of the surface and is isomorphic to a torsion free cocompact subgroup of the full group of isometries of the hyperbolic plane.

Hyper-aspherical manifolds (see section **??**) which are not aspherical are most easily obtained by taking any closed oriented aspherical n-manifolds N and forming the orientable connected sum with any other closed oriented m-manifold P. Then $M = P\#N$ maps onto N with a map of degree 1 by collapsing $P - (\mathrm{Ball})^0 \subset M$ to a point.

Each of the implications of Theorem **??** cannot be reversed. See [**?**] for a complete discussion. For example, the connected sum of two non-homeomorphic 3-dimensional spherical space forms which are also not lens spaces is admissible but is not a K-manifold.

1.17.17. There are many interesting examples of closed aspherical manifolds without any compact Lie group action. The first examples were given by Conner–Raymond–Weinberger, and E. Bloomberg in his thesis. The first 3-dimensional examples were given by Raymond–Tollefson.

Put in Refs

In every bordism class (dim ≥ 3), there exist such manifolds. This was proved by R. Schultz. It is possible to construct 7-dimensional solv-manifolds for which $\mathrm{Out}(\pi_1(M)) = 1$.[**?**]. In general the examples are obtained by showing that center of $\pi_1(M)$ is trivial and $\mathrm{Out}(\pi_1(M))$ is torsion free (in the aspherical case).

Schultz
[CR]

1.17.18 Exercise. Suppose that M is a hyper-aspherical m-manifold and $\pi_1(M)$ has trivial center, and $\mathrm{Out}(\pi_1(M))$ is torsion free. Let N be a simply connected closed m-manifold and form $N\#M$. Show that $N\#M$ admits no effective finite group action.

Puppe♣ in []♣ has shown that there exists a 6-dimensional *simply* connected closed manifold without any effective finite group action. ♣ This manifold would therefore be admissible.

First name
???
Check if he did only for $\mathbb{Z}_2$

1.17.19 Example. [**?**, p.43, §7.2] CRW??? The usual technique for constructing aspherical manifolds with few or no finite group actions is to construct a group which is the fundamental group of a closed aspherical manifold and for which one can prove that center is trivial and the torsion in the outer automorphism group is trivial or at least finite and small. Here is a rather elementary example of a non-orientable 3-manifold that admits no action of any finite group except for $\mathbb{Z}_2$. Furthermore, smoothly there is only one such action. Take the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. This linear transformation on $\mathbb{R}^2$ preserves the integral lattice (that is, the matrix normalizes the $\mathbb{Z}\times\mathbb{Z}$ group of standard translations

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+m \\ y+n \end{bmatrix}, \quad (m,n)\in\mathbb{Z}\times\mathbb{Z}),$$

and so induces a homomorphism h on the 2-torus $T = \mathbb{Z}^2\backslash\mathbb{R}^2$. Form $M = T^2 \times_{\mathbb{Z}} \mathbb{R}$, with $\mathbb{Z}$ acting diagonally. The generator of $\mathbb{Z}$ acts on T^2 as h and on $\mathbb{R}$ by sending

$r \mapsto r-1$. M is then an affinely flat 3-manifold which fibers over S^1 with fiber T^2 and structure group $\mathbb{Z}$. It is not too difficult to show that $\mathcal{Z}(\pi_1(M)) = 1$ and $\mathrm{Out}(\pi_1(M)) = \mathbb{Z}_2$, h and all its powers fix just one point on T^2. There is a $\mathbb{Z}_2$ action on M which leaves each fiber T^2 invariant (because $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ centralizes h). This action on M has exactly 2 circles of fixed points.

CHAPTER 2

Applications

rewrite this The Seifert Construction, which is a special embedding, $\theta : \Pi \to \mathrm{TOP}_G(G \times W)$, of the group Π into $\mathrm{TOP}_G(G \times W)$ such that Π acts properly on $G \times W$, preserves some of the properties of both G and W on $\theta(\Pi)\backslash(G \times W)$. Furthermore, the action of Π on $G \times W$ "twists" the topology and geometry of G and W to create the orbit space $\theta(\Pi)\backslash(G \times W)$ in the same way that the group structures of Γ and Q "twist" to create the group Π. In other words, this algebraic twisting of Π makes the geometric twisting of the "bundle with singularities"

$$\Gamma\backslash G \to \theta(\Pi)\backslash(G \times W) \to Q\backslash W,$$

where the homogeneous space $\Gamma\backslash G$ is a principal fiber. In the several applications, we have selected to include here this features seems especially prominent.

One of the important geometric problems that has motivated the development of Seifert fiberings is the construction of closed aspherical manifolds realizing Poincaré duality groups Π of the form $1 \to \Gamma \to \Pi \to Q \to 1$. The Seifert Construction enables one to find explicit aspherical manifolds $M(\Pi)$ when Q acts on a contractible manifold W and Γ is a torsion free lattice in a Lie group.

For a topological manifold, the homotopy classes of self-homotopy equivalences can be regarded as algebraic data. We shall show how the Seifert Construction can be used to lift these finite subgroups of homotopy classes to an action on the manifold.

2.1. Existence of Closed Smooth $K(\Pi, 1)$-manifolds

There are two difficult problems related to the title. They are:

- *Which groups can be the fundamental group of a closed aspherical manifold?* and
- *If Π is the fundamental group of an aspherical manifold, can we give an actual explicit construction of an aspherical manifold for the group Π?*

There are some general criteria for the first problem such as Π must be finitely presented, have finite cohomological dimension and satisfy Poincaré duality in that dimension. The Seifert Construction gives answers to both questions for large class of groups Π. The idea is that if $1 \to \Gamma \to \Pi \to Q \to 1$ is a torsion free extension where Γ is the fundamental group of a closed aspherical manifold and Q is a proper action on a contractible manifold W with compact quotient, then Π should be the fundamental group of a closed aspherical manifold. We have the following

2.1.1 Theorem. *Let Γ be a cocompact special lattice in G (see subsection **??**) and $\rho : Q \to \mathrm{TOP}(W)$ be a proper action of a discrete group on a contractible manifold W with compact quotient. If $1 \to \Gamma \to \Pi \to Q \to 1$ is a torsion free extension of Γ by Q, then for any Seifert Construction $\theta : \Pi \to \mathrm{TOP}_G(G \times W)$,*

(1) *$M(\theta(\Pi)) = \theta(\Pi)\backslash(G \times W)$ is a closed aspherical manifold if Γ is of type* (S3)*,*
(2) *$M(\theta(\Pi)) = \theta(\Pi)\backslash((G/K) \times W)$, where K is a maximal compact subgroup of G, is a closed aspherical manifold if Γ is of type* (S4)*.*

PROOF. We know from Theorem **??** that for each extension $1 \to \Gamma \to \Pi \to Q \to 1$, there exists a homomorphism θ of Π into $\mathrm{TOP}_G(G \times W)$ (resp., $\mathrm{TOP}_{(G,K)}(G/K \times W)$ in the case of type (S4)), see **??** or **??**. which one? We need only to check that θ is injective. Suppose Q_0 is the kernel of $\phi \times \rho : Q \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$. Then Q_0 is finite since the Q action on W is proper. Let $1 \to \Gamma \to \Pi_{Q_0} \to Q_0 \to 1$ be the pullback via $Q_0 \subset Q$. By Corollary **??**, θ is injective if and only if Π_{Q_0} is torsion-free. But the group Π_{Q_0} is torsion free since Π is assumed to be torsion free.

(S4)?

The restriction $\theta|_E$ defines an action of E on $G \times W$ (resp., $G/K \times W$). Since Γ is of finite index in E, no non-trivial element of E can fix $G \times W$ (resp., $G/K \times W$). For if it did, then some power would be a non-trivial element of Γ which does not fix G (resp., G/K). Therefore, θ is injective; $\theta(\Pi)$ acts properly and freely since it is torsion free. □

2.1.2 REMARK. (1) The proof shows that the theorem still holds under the weaker assumptions that $\phi \times \rho : Q \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$ has finite kernel ($\rho : Q \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$ may have infinite kernel) and that the image $\rho(Q) \subset \mathrm{TOP}(W)$ acts properly on W with compact quotient.
(2) If W is a smooth contractible manifold and $\rho : Q \to \mathrm{Diff}(W)$, then the construction can be done smoothly and $M(\theta(\Pi))$ is smooth.
(3) If ρ_1 and ρ_2 are rigidly related (i.e., there exists $h \in \mathrm{TOP}(W)$ for which $\rho_2 = \mu(h) \circ \rho_1$) and Γ is characteristic in Π, then $M(\theta_1(\Pi))$ and $M(\theta_2(\Pi))$ are homeomorphic via a Seifert automorphism. Moreover, if we fix ℓ and ρ, then the constructed $M(\theta(\Pi))$ are all strictly equivalent.
(4) When $W = \{p\}$ is a point (a 0-dimensional contractible manifold), then Q must be finite for Q to act properly, and every $\rho : Q \to \mathrm{TOP}(\{p\})$ is rigidly related. The closed aspherical manifolds constructed are infra-G-manifolds. cf. Example **??**.
(5) One important application of these constructions is that they provide model aspherical manifolds with often strong geometric properties. If one wants to study the famous conjecture that two closed aspherical manifolds with isomorphic fundamental groups are homeomorphic via the methods of controlled surgery, then the constructed aspherical Seifert manifolds are excellent model manifolds.

The above procedure can be extended for even more general extensions. As an example,

2.1.3 THEOREM. *Let Π be a torsion-free extension of a virtually poly-$\mathbb{Z}$ group Γ by Q, where Q acts on a contractible manifold W properly with compact quotient. Then there exists a closed $K(\Pi, 1)$-manifold.*

PROOF. A torsion-free virtually poly-$\mathbb{Z}$ group Γ has a unique maximal normal nilpotent subgroup Δ, which is called the *discrete nilradical* of Γ. See [**?**]. Then the quotient Γ/Δ is virtually free abelian of finite rank. Furthermore, since Δ is a characteristic subgroup of Γ, it is normal in $\mathit{\Pi}$. Consider the commuting diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Delta & \xrightarrow{=} & \Delta & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \mathit{\Pi} & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \Gamma/\Delta & \longrightarrow & \mathit{\Pi}/\Delta & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Since Γ/Δ is virtually free abelian of finite rank (say of s), it contains a characteristic subgroup $\mathbb{Z}^s$. Let $Q' = (\mathit{\Pi}/\Delta)/\mathbb{Z}^s$. Then the natural projection $Q' \longrightarrow Q$ has a finite kernel. Therefore, if we let Q' act on W via Q, the action will still be proper.

One can do a Seifert fiber space construction with the exact sequence

$$1 \longrightarrow \mathbb{Z}^s \longrightarrow \mathit{\Pi}/\Delta \longrightarrow Q' \longrightarrow 1$$

which yields a proper action of $\mathit{\Pi}/\Delta$ on $\mathbb{R}^s \times W$ with compact quotient. Using this action of $\mathit{\Pi}/\Delta$ on $\mathbb{R}^s \times W$, one does a Seifert fiber space construction with the exact sequence

$$1 \longrightarrow \Delta \longrightarrow \mathit{\Pi} \longrightarrow \mathit{\Pi}/\Delta \longrightarrow 1.$$

This gives rise to a proper action of $\mathit{\Pi}$ on $N \times (\mathbb{R}^s \times W)$, where N is the unique simply connected nilpotent Lie group containing Δ as a lattice, with compact quotient.

If the space W is smooth, and the action of Q on W is smooth, both constructions can be done smoothly so that the proper action of $\mathit{\Pi}$ on $N \times (\mathbb{R}^s \times W)$ is smooth.

In any case, since the group $\mathit{\Pi}$ is torsion free, the resulting action of $\mathit{\Pi}$ on $N \times (\mathbb{R}^s \times W)$ is free. Consequently, we get a closed $K(\mathit{\Pi},1)$-manifold

$$M = \mathit{\Pi}\backslash(N \times \mathbb{R}^s \times W).$$

It has a Seifert fiber structure

$$F \longrightarrow M \longrightarrow Q\backslash W$$

where the principal fiber F itself has a Seifert fiber structure

$$\Delta\backslash N \longrightarrow F \longrightarrow T^s = \mathbb{Z}^s\backslash\mathbb{R}^s.$$

In fact, since the action of the characteristic subgroup $\mathbb{Z}^s$ on $\mathbb{R}^s$ is free, F is a genuine fiber bundle, with fiber a nilmanifold $\Delta\backslash N$ over the base torus T^s. □

The space W does not have to be aspherical. As far as the action of discrete Q is proper, the construction works. The resulting action of $\mathit{\Pi}$ is free if and only if the pre-image of Q_w (the stabilizer of the Q action at $w \in W$) in $\mathit{\Pi}$ is torsion

abstract kernel
lifting
geometrically realizable
Nielsen's realization problem
admissible

free. In this case, the space $\Pi\backslash(G \times W)$ will not be aspherical. See **??**. Cf. also Theorem **??** and **??**.

2.2. Lifting Problem for Homotopy Classes of Self-Homotopy Equivalences

Let M be a closed aspherical space and $\mathcal{E}(M)$ be the H-space of homotopy equivalences of M into itself. Any $f \in \mathcal{E}(M)$ induces an isomorphism $f_* : \pi_1(M, x) \to \pi_1(M, f(x))$. By choosing a path ω from x to $f(x)$, we have an automorphism f_*^ω of $\pi_1(M, x)$, defined by $f_*^\omega([\tau]) = [\omega^{-1} \cdot (f \circ \tau) \cdot \omega]$. A different choice of ω alters f_*^ω only by an inner automorphism. Therefore, we obtain a homomorphism

$$\gamma : \mathcal{E}(M) \to \mathrm{Out}(\Pi),$$

where $\Pi = \pi_1(M, x)$. **Suppose M is a $K(\Pi, 1)$ space**. Then $\mathcal{E}_0(M)$ is the kernel of γ so that γ factors through $\pi_0(\mathcal{E}(M)) = \mathcal{E}(M)/\mathcal{E}_0(M)$, where $\mathcal{E}_0(M)$ is the self homotopy equivalences homotopic to the identity. Moreover, γ is onto since every automorphism of Π can be realized by a self homotopy equivalence of M.

2.2.1 Definition. A homomorphism $\varphi : F \to \mathrm{Out}(\Pi) \cong \pi_0(\mathcal{E}(M))$ is called an *abstract kernel*. An injective abstract kernel is the same as a subgroup of homotopy classes of self-homotopy equivalences of M. A *lifting* of φ as a group of homeomorphisms is a homomorphism $\hat{\varphi} : F \to \mathrm{TOP}(M)$ which makes

$$\begin{array}{ccc} F & \xrightarrow{=} & F \\ \downarrow \hat{\varphi} & & \downarrow \varphi \\ \mathrm{TOP}(M) & \longrightarrow \mathcal{E}(M) \longrightarrow & \pi_0(\mathcal{E}(M)) \end{array}$$

commutative. The abstract kernel $F \to \mathrm{Out}(\Pi)$ is *geometrically realizable* if it can be realized as an action of F on M (i.e., a lifting as a group of homeomorphisms exists).

This problem became known as *Nielsen's realization problem*. J. Nielsen [**?**] had shown that every cyclic check this group of outer automorphisms on a closed surface could be geometrically realized. Others had shown, by sometimes different methods, that finite p-groups and solvable Lie groups could be geometrically realized on compact surfaces Zieschang–Macbeath—check up. In 1977, the first examples showing the failure of geometric realization on closed aspherical n-manifolds $n \geq 3$, were constructed [**?**]. These examples were nilmanifolds. Many other examples soon followed, e.g., [**?**], [**?**], Zie-Zimm. In 198?, S. Kerkhoff ref showed that all finite subgroups of $\mathrm{Out}(\pi_1(M))$, where M is a closed aspherical, can be geometrically realized.

Put in the original example or our example

In Corollary **??** and Theorem **??**, we have seen that the realization problem for certain solvmanifolds was solved. We study the problem in a more detail.

2.2.2 Definition. An extension $1 \to \Pi \to F^* \to F \to 1$, where the center $\mathcal{Z}(\Pi)$ is torsion-free, is called *admissible* [**?**] if $C_{F^*}(\Pi)$, the centralizer of Π in F^*, is torsion-free.

We have a necessary condition:

2.2.3 Lemma. *Let M be closed aspherical manifold with $\pi_1(M) = \Pi$; let $\widetilde{M}$ be the universal cover of M. Then the extension $1 \to \Pi \to N_{\mathrm{TOP}(\widetilde{M})}(\Pi) \xrightarrow{\eta} \mathrm{TOP}(M) \to 1$ is admissible where $N_{\mathrm{TOP}(\widetilde{M})}(\Pi)$ denotes the normalizer of Π in $\mathrm{TOP}(\widetilde{M})$.*

Proof. Suppose that there is $z \in N_{\mathrm{TOP}(\widetilde{M})}(\Pi)$ with finite order, and $\overline{\varphi}(z) = 1$, where $\overline{\varphi} : N_{\mathrm{TOP}(\widetilde{M})}(\Pi) \to \mathrm{Aut}(\Pi)$. Let F be the finite cyclic subgroup generated by z. Consider the induced

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \eta^{-1}(\eta(F)) & \xrightarrow{\eta} & \eta(F) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Pi) & \longrightarrow & \mathrm{Aut}(\Pi) & \longrightarrow & \mathrm{Out}(\Pi) & \longrightarrow & 1\,.
\end{array}$$

Since F is finite and Π is torsion-free, η is an isomorphism on F so that we have a semi-direct product structure on $\eta^{-1}(\eta(F))$. Now since $z \in \ker \overline{\varphi}$, conjugation by z yields $zxz^{-1} = x$ for all $x \in \Pi$. This implies $\eta^{-1}(\eta(F)) = \Pi \times F$. So the action $(\eta^{-1}(\eta(F)), \widetilde{M})$ contains a finite subgroup action $(F, \widetilde{M})$ which commutes with $(\Pi, \widetilde{M})$ so that $\overline{\varphi}(F) = 1$ in $\mathrm{Aut}(\Pi)$. Since M is an aspherical manifold, Coro p.48, just before 1.15.4 implies that F is trivial. Thus we have $z = 1$. □

2.2.4 Corollary. *Let (G, M) be an effective action of a finite group on a closed aspherical manifold M with $\pi_1(M) = \Pi$. Then the induced extension $1 \to \Pi \to G^* \to G \to 1$, where G^* denotes the group of all liftings of G to homeomorphisms of $\widetilde{M}$, is admissible.*

Notice that $G^* = \eta^{-1}(G)$ and

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & G^* & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi & \longrightarrow & N_{\mathrm{TOP}(\widetilde{M})})\Pi & \longrightarrow & \mathrm{TOP}(M) & \longrightarrow & 1
\end{array}$$

is a pullback diagram so that $G^* \subset N_{\mathrm{TOP}(\widetilde{M})})\Pi$ (see subsection **??** for pullback). Since the bottom sequence is admissible, so is the top one.

2.2.5 Remark. Let (G, M) be an action (not necessarily effective) of a finite group on a closed aspherical manifold M with $\pi_1(M) = \Pi$. Then there exists an extension (not necessarily admissible) $1 \to \Pi \to E \to G \to 1$ realizing the abstract kernel $\varphi : G \xrightarrow{\hat{\varphi}} \mathrm{TOP}(M) \xrightarrow{\varphi'} \mathrm{Out}(\pi_1(M))$.

Proof. Since $(\hat{\varphi}(G), M)$ is effective, there exists an admissible extension E' of Π by $\hat{\varphi}(G)$, $1 \to \Pi \to E' \to \hat{\varphi}(G) \to 1$. We can "pull-back" $G \xrightarrow{\hat{\varphi}} \hat{\varphi}(G)$ along finitely extendable

$E' \to \hat{\varphi}(G)$ to get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\hat{\varphi}} & & \\ 2 & \longrightarrow & \Pi & \longrightarrow & E' & \longrightarrow & \hat{\varphi}(G) & \longrightarrow & 1 \end{array}$$

Certainly the top row is an extension of Π by G realizing $(\Pi, G, \varphi = \varphi' \circ \hat{\varphi})$. □

Thus we have a necessary condition for the existence of a lifting of an abstract kernel $\varphi : F \to \mathrm{Out}(\Pi)$, as an (effective, resp.) group action: the existence of an (admissible, resp.) group extension of Π by F realizing the abstract kernel. For finite groups, this necessary condition is also sufficient for some tractable manifolds.

2.2.6(**Unsolved Problem**). Does there exist a closed aspherical manifold M such that there is an extension $1 \to \pi_1(M) \to E \to F \to 1$ with F finite, but F cannot be realized as a group action.

2.2.7 Definition. Let Q act properly on a space W and B be the quotient $Q\backslash W$. Suppose for each extension $1 \to Q \to E \to F \to 1$ by a finite group F, the action of Q extends to a proper action of E on W. Then we say that the Q action on W is *finitely extendable*. In particular, then F acts on B preserving the orbit structure.

If Γ is normal in Π, recall $\mathrm{Aut}(\Pi, \Gamma)$ denotes the automorphisms of Π that leave Γ invariant. Since $\mathrm{Inn}(\Pi)$ leaves Γ invariant, we can put $\mathrm{Aut}(\Pi, \Gamma)/\mathrm{Inn}(\Pi) = \mathrm{Out}(\Pi, \Gamma)$. It is a subgroup of $\mathrm{Out}(\Pi)$.

We are interested in realizing a finite abstract kernel $F \to \mathrm{Out}(\Pi)$ as a group action on a model Seifert fiber space $M(\Pi)$ with a principal fiber $\Gamma\backslash G$. In particular, we want the F action to be fiber-preserving maps; in fact, Seifert automorphisms. This means that, on the group level, the extension must leave the lattice Γ invariant. In other words, we consider only those abstract kernels which have images in $\mathrm{Out}(\Pi, \Gamma)$.

2.2.8 Theorem. *Let $M(\theta(\Pi)) = \theta(\Pi)\backslash(G \times W)$ be a Seifert manifold with principal fiber $\Gamma\backslash G$, where G is simply connected completely solvable Lie group. Suppose (Q, W) is finitely extendable, where $Q = \Pi/\Gamma$. Then each abstract kernel $\psi : F \to \mathrm{Out}(\Pi, \Gamma)$ of a finite group F can be geometrically realized as a group of Seifert automorphisms on $M(\theta(\Pi))$ if and only if the abstract kernel ψ admits some extension.* assuming θ is injective?

2.2.9 Remark. This theorem proves that if (Q, W) is finitely extendable, then $(\theta(\Pi), G \times W)$ becomes finitely extendable itself. Thus, we can enlarge the class of extendable pairs more and more. Here is a list of finitely extendable pairs:

(1) hyperbolic space and a cocompact lattice
(2) $\mathbb{R}^n$ and a crystallographic group

(3) connected, simply connected nilpotent Lie group and its almost crystallographic group
(4) connected, simply connected completely solvable Lie group and its almost crystallographic group

PROOF. Let $1 \to \mathit{\Pi} \to E \to F \to 1$ be an extension realizing the abstract kernel ψ. Since $\psi(F) \subset \mathrm{Out}(\mathit{\Pi}, \Gamma)$, Γ is normal in E. We have the commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \mathit{\Pi} & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & E & \longrightarrow & E/\Gamma & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & \xrightarrow{=} & F & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Consider the induced extension

$$1 \longrightarrow Q \longrightarrow E/\Gamma \longrightarrow F \longrightarrow 1.$$

Since ρ is finitely extendable, there exists $\rho' : E/\Gamma \to \mathrm{TOP}(W)$ extending $\rho : Q \to \mathrm{TOP}(W)$. Again by the existence theorem for special lattices, Theorem **??**, there exists $\theta' : E \to \mathrm{TOP}_G(G \times W)$, where $\theta'|_\Gamma = \theta|_\Gamma = i : \Gamma \hookrightarrow G$, and $\rho'|_Q = \rho$. Put $\theta'|_{\mathit{\Pi}} = \theta'$. Of course, θ' may be different from θ, but as θ and θ' agree on Γ and Q, we can apply Theorem **??** (2) to conjugate $\mathrm{TOP}_G(G \times W)$ by an element of $\mathrm{M}(W, G) \rtimes \mathrm{Inn}(G)$ which carries $\theta'|_{\mathit{\Pi}}$ to θ so that the new homomorphism $\theta' : E \to \mathrm{TOP}_G(G \times W)$ is an extension of $\theta : \mathit{\Pi} \to \mathrm{TOP}_G(G \times W)$. This yields an action of F on $\theta(\mathit{\Pi})\backslash(G \times W)$ as a group of Seifert automorphisms as desired. □

2.2.10 COROLLARY. *Let $M = \mathit{\Pi}\backslash G$ be an infra G-manifold, where G is simply connected completely solvable Lie group and $\mathit{\Pi} \subset G \rtimes \mathrm{Aut}(G)$. Then each abstract kernel $\psi : F \to \mathrm{Out}(\mathit{\Pi}, \Gamma)$ of a finite group F can be geometrically realized as an (effective, resp) group of affine diffeomorphisms on M if and only if the abstract kernel ψ admits an (admissible, resp) extension.*

2.2.11 EXERCISE. Let $\mathit{\Pi} = \mathbb{Z}^2$ and $M = \mathit{\Pi}\backslash\mathbb{R}^2 = T^2$. Realize the abstract kernel $\varphi : \mathbb{Z}_2 \to \mathrm{GL}(2, \mathbb{Z})$. How many distinct actions $(\mathbb{Z}_2, T^2)$ (up to equivalence) do you get?

PROOF. The infra-G-manifold $M(\mathit{\Pi})$ is modelled on $G \times \{p\}$ (p=point), ($G/K \times \{p\}$ for a convenient form of G in the semi-simple case), and $\mathrm{TOP}_G(G \times \{p\}) = \mathrm{Aff}(G)$ (resp. $\overline{\mathrm{Aff}}(G, K)$), see [**?**] for this notation. Trivially, every $Q \to \mathrm{TOP}(\{p\})$ extends to $E/\Gamma \to \mathrm{TOP}(\{p\})$. The above theorem then immediately applies, and F acts on $M(\mathit{\Pi})$ by Seifert automorphisms which are affine diffeomorphisms.

This proof talks about semi-simple, but no such thing in the statement!

□

2.2.12 Remark. 1. Since we may introduce a metric structure in the Corollary from a left invariant metric on G,$M(\mathit{\Pi})$ has the structure of a flat, almost flat, Riemannian infra-solvmanifold or a locally symmetric spaces. We may also further conjugate $\theta(\mathit{\Pi})$ in $\mathrm{Aff}(G)$ so that F now acts on the conjugated manifold by isometries preserving the flat, etc, structures.

2. The proper action of $\mathit{\Pi}$ in the Theorem is not necessarily free nor effective. Thus $M(\mathit{\Pi})$ could very well be a Seifert orbifold. The Corollary then works for such orbifolds, i.e., infra-G-spaces. In the Euclidean case, $M(\mathit{\Pi})$ would then be a Euclidean "crystal" and $\mathit{\Pi}$ a Euclidean crystallographic group. In theorem **??**, F sends fibers (which could be G-crystals instead of infra-G-spaces) to fibers.

For more about the realizations up to strict equivalences, and finding examples where F does not lift because there are no extensions realizing the abstract kernels, the reader is referred to [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] and [**?**].

For a torsion free poly{cyclic or finite} group $\mathit{\Pi}$, we can always find a (characteristic) predivisible subgroup Γ of finite index in $\mathit{\Pi}$. Let Q be the finite quotient $\mathit{\Pi}/\Gamma$, and choose W =point. Then the Seifert Construction of Theorem **??** produces an embedding $\theta(\mathit{\Pi}) \subset \overline{\mathrm{Aff}}(G,K)$ and the Seifert manifold $M(\mathit{\Pi}) = \theta(\mathit{\Pi})\backslash G/K$ is a closed smooth $K(\mathit{\Pi},1)$ manifold.

2.2.13 Corollary. *[Geometric realization of group actions from homotopy data] Under the conditions of Theorem* **??***, let $M(\mathit{\Pi}) = \theta(\mathit{\Pi})\backslash G/K$ be a Seifert manifold. Suppose now $\psi : F \to \mathrm{Out}(\mathit{\Pi}) = \pi_0\mathcal{E}(M(\mathit{\Pi}))$ is a homomorphism of a finite group F into the homotopy classes of self-homotopy equivalences of $M(\mathit{\Pi})$. Then, F acts on $M(\mathit{\Pi})$ if and only if there exists an extension,*

$$1 \to \mathit{\Pi} \to E \to F \to 1,$$

realizing the abstract kernel ψ. Moreover, the action can be chosen to be smooth, induced from smooth Seifert automorphisms contained in $\overline{\mathrm{Aff}}(G,K)$. The action of F is effective if and only if $C_E(\mathit{\Pi})$ is torsion free .

Proof. In order to have an action, we must have a lifting sequence and hence an extension, $1 \to \mathit{\Pi} \to E \to F \to 1$, that realizes the abstract kernel ψ. Since Γ is characteristic in $\mathit{\Pi}$, it is normal in E and and $1 \to Q = \mathit{\Pi}/\Gamma \to E/\Gamma \to F \to 1$ is exact. Because of the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \mathit{\Pi} & \longrightarrow & Q & \longrightarrow & 1\\
 & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & E & \longrightarrow & E/\Gamma & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & \xrightarrow{=} & F & &
\end{array}$$

we can find a Seifert construction $\theta' : E \to \overline{\mathrm{Aff}}(G,K)$ which extends $\theta : \mathit{\Pi} \to \overline{\mathrm{Aff}}(G,K)$. Therefore the group F acts on $M(\mathit{\Pi})$ smoothly as diffeomorphisms preserving the Seifert structure. The action of F, as mentioned in the previous corollary, is effective, if and only if, $C_E(\mathit{\Pi})$ is torsion free. In any case, we have a lift $\widetilde{\psi}$,

$$\begin{array}{ccc} F & \xrightarrow{\widetilde{\psi}} & \mathrm{Diff}(M(\mathit{\Pi})) \\ {\scriptstyle \psi}\big\downarrow & & {\scriptstyle j}\big\downarrow \\ \mathrm{Out}(\mathit{\Pi}) & \xrightarrow{=} & \mathcal{E}(M(\mathit{\Pi})) \end{array}$$

where j sends a self diffeomorphism to its homotopy class. In case there exists one extension realizing the abstract kernel ψ, then for each element of $H^2(F, \mathcal{Z}(\mathit{\Pi}))$ there is a congruence class of extensions E, realizing the abstract kernel ψ. Each of these extensions gives rise to a (not necessarily effective) action of F on $M(\mathit{\Pi})$. $\square$

If we combine this corollary with surgery results, we can get much stronger statements.

2.2.14 Theorem. *Let $1 \to \mathit{\Pi} \to E \to F \to 1$ be an extension of a torsion-free poly{cyclic or finite} group $\mathit{\Pi}$ by a finite group F with abstract kernel $\psi : F \to \mathrm{Out}(\mathit{\Pi})$. Let M be a closed aspherical n-manifold, $n \neq 3$, with $\pi_1(M) = \mathit{\Pi}$. Then there exists an action of F on M which realizes the abstract kernel $\psi : F \to \mathrm{Out}(\mathit{\Pi}) = \pi_0(\mathcal{E}(M))$, the group of homotopy classes of self-homotopy equivalences of M. The action is effective if and only if $C_E(\mathit{\Pi})$ is torsion-free, and is free if and only if E is torsion-free. In the latter case, any two such actions are weakly equivariant.*

Proof. Pick Γ a characteristic predivisible subgroup in $\mathit{\Pi}$. Then we have a commutative diagram as in the above corollary. And we get an action of E/Γ on $\Gamma\backslash S/K = M(\Gamma)$ and an action of F on $M' = \theta(\mathit{\Pi})\backslash S/K$, realizing ψ on M'. Since the torsion-free $\mathit{\Pi}$ is poly–$\mathbb{Z}$ (respectively, poly (cyclic or finite)), a theorem of Wall$^\clubsuit$ (respectively, Farrell-Jones)$^\clubsuit$ says that any homotopy equivalence between Wall:ref
M and M' is homotopic to a homeomorphism. Therefore, we need only pull back Farrell-Jones:ref
the action of F on M' to obtain the desired action on M. By uniqueness, we have that any two free actions on M' will be weakly equivariant. $\square$

2.2.15 Theorem. *Let G be a semi-simple centerless Lie group without any normal compact factors and if G contains any 3-dimensional factors (i.e., $\mathrm{PSL}(2, \mathbb{R})$), then the projection of the lattice to each of these factors is dense. Let $M(\theta(\mathit{\Pi})) = \theta(\mathit{\Pi})\backslash(G/K \times W)$ be a Seifert manifold with principal fiber $\Gamma\backslash G/K$. Suppose $(\mathit{\Pi}/\Gamma, W)$ is finitely extendable. Then each abstract kernel $\psi : F \to \mathrm{Out}(\mathit{\Pi}, \Gamma)$ of a finite group F can be geometrically realized as an (effective, resp) group of Seifert automorphisms on $M(\theta(\mathit{\Pi}))$ if and only if the abstract kernel ψ admits an (admissible, resp) extension.*

Proof. Same argument as Theorem **??**. $\square$

2.2.16 Corollary. *Let G be as in the above theorem, and let $M = \theta(\mathit{\Pi})\backslash G/K$. Then each abstract kernel $\psi : F \to \mathrm{Out}(\mathit{\Pi}, \Gamma)$ of a finite group F can be geometrically realized as an (effective, resp) group action on M if and only if the abstract kernel ψ admits an (admissible, resp) extension.*

affine structure on Γ
affine crystallographic group
ACG
polynomial structure on Γ
polynomial crystallographic group
PCG

Suppose F is a group, and $\varphi : F \to \mathrm{Out}(\Pi)$ a homomorphism. The obstruction to the existence of an extension with the given abstract kernel lies in $H^3(F; \mathcal{Z}(\Pi))$.

should have had in Lie group chapter? If F is finite, then the obstruction class has a finite order. In the case when G is solvable as in Theorem **??**, $\mathcal{Z}(\Pi) \cong \mathbb{Z}^k$ for some $k \geq 0$. By enlarging the group $\mathbb{Z}^k$ to $(\frac{1}{p}\mathbb{Z})^k$ for some p, one can kill the obstruction. This implies that, if we enlarge Π to $(\frac{1}{p}\mathbb{Z})^k \cdot \Pi$, then the abstract kernel has an extension E. should expand... H^3 Now if Π is normal inside E, then we would have an action of E/Π on M. Note that E/Π contains F. give reference to Zimmermann— for the term "inflation"?

For more examples and explicit calculations. Maybe orig or flat case.

2.3. Manifolds with few periodic maps

New section. Better one than #299?

2.4. Polynomial Structures for Solvmanifolds

The main reference for this section is [**?**]. In 1977 ([**?**]), John Milnor asked if every torsion-free polycyclic–by–finite group Γ occurs as the fundamental group of a compact, complete affinely flat manifold. This is equivalent to asking that if Γ can act on $\mathbb{R}^K$ properly as affine motions with $\Gamma\backslash\mathbb{R}^K$ compact.

Recently however, Y. Benoist ([**?**], [**?**]) produced an example of a 10–step nilpotent group Γ of Hirsch length 11 which does not admit an affine structure. This example was generalized to a family of examples by D. Burde and F. Grunewald ([**?**]). In ([**?**]) Burde constructs counter-examples of nilpotency class 9 and Hirsch length 10.

A polynomial diffeomorphism f of $\mathbb{R}^n$ is a bijective polynomial transformation of $\mathbb{R}^n$ for which the inverse mapping is again polynomial. Let us write $\mathrm{P}(\mathbb{R}^n)$ for the group consisting of all polynomial diffeomorphisms. Affine diffeomorphisms clearly are polynomial diffeomorphisms of degree ≤ 1; smooth actions could be considered as being "polynomial of infinite degree".

A representation $\theta : \Gamma \to \mathrm{Aff}(\mathbb{R}^K)$ which yields a proper action with $\theta(\Gamma)\backslash\mathbb{R}^K$ compact is called *affine structure on* Γ. It is also common to call $\theta(\Gamma)$ an *affine crystallographic group* (ACG) ([**?**], [**?**]). Analogously to the affine structure, a representation $\theta : \Gamma \to \mathrm{P}(\mathbb{R}^K)$ which yields a proper action with $\theta(\Gamma)\backslash\mathbb{R}^K$ compact is called *polynomial structure on* Γ; $\theta(\Gamma)$ is called a *polynomial crystallographic group* (PCG) . In this section, we sketch a proof of the following:

Theorem *Every polycyclic-by-finite group Γ admits a polynomial structure of bounded degree. That is, Γ can act on $\mathbb{R}^K$ properly as polynomial diffeomorphisms so that $\Gamma\backslash\mathbb{R}^K$ is compact. Moreover, all polynomials involved consist entirely of a bounded degree.*

The construction of this polynomial structure is a special case of an iterated Seifert Fiber Space construction, which can be achieved here because of a very strong and surprising cohomology vanishing theorem.

2.4.1(Polynomial Diffeomorphisms). Write $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ for the real vector space of polynomial mappings from $\mathbb{R}^K$ to $\mathbb{R}^k$. An element $p(x_1, \ldots, x_K)$ of $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ consists of k polynomials in K variables:

$$p(x_1, \ldots, x_K) = \begin{pmatrix} p_1(x_1, x_2, \ldots, x_K) \\ p_2(x_1, x_2, \ldots, x_K) \\ \vdots \\ p_k(x_1, x_2, \ldots, x_K) \end{pmatrix}, \text{ with } p_i(x_1, \ldots, x_K) \in \mathrm{P}(\mathbb{R}^K, \mathbb{R}).$$

By the degree of p, denoted by $\deg(p)$, we mean of course the maximum of the degrees of the p_i $(1 \leq i \leq k)$. Note in particular, that $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ contains $\mathbb{R}^k$ as the subgroup of constant mappings (degree-0 mappings).

We denote by $\mathrm{P}(\mathbb{R}^K)$ the group of polynomial diffeomorphisms of $\mathbb{R}^K$. Here, the group-law is composition of mappings (so $\mathrm{P}(\mathbb{R}^K)$ is a subset of $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^K)$, but not a subgroup). Elements of $\mathrm{P}(\mathbb{R}^K)$ are polynomial bijections whose inverse mappings are again polynomials.

Example. Let $p, q : \mathbb{R}^2 \to \mathbb{R}^2$ be such that

$$p(x, y) = (y + 1, x + y^2) \text{ and } q(x, y) = (y - x^2 + 2x - 1, x - 1).$$

Clearly, they are inverse to each other in $\mathrm{P}(\mathbb{R}^2)$.

The vector space $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ has $\mathrm{GL}(\mathbb{R}^k) \times \mathrm{P}(\mathbb{R}^K)$–module structure, via

$$\forall (g, h) \in \mathrm{GL}(\mathbb{R}^k) \times \mathrm{P}(\mathbb{R}^K),\ \forall p \in \mathrm{P}(\mathbb{R}^K, \mathbb{R}^k) :\ {}^{(g,h)}p = g \circ p \circ h^{-1}.$$

The resulting semi–direct product $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k) \rtimes (\mathrm{GL}(\mathbb{R}^k) \times \mathrm{P}(\mathbb{R}^K))$ embeds into $\mathrm{P}(\mathbb{R}^{k+K})$ as follows: $\forall p \in \mathrm{P}(\mathbb{R}^K, \mathbb{R}^k),\ \forall g \in \mathrm{GL}(\mathbb{R}^k),\ \forall h \in \mathrm{P}(\mathbb{R}^K)$:

$$\forall x \in \mathbb{R}^k,\ \forall y \in \mathbb{R}^K :\ \ (p, g, h)(x, y) = (g(x) + p(h(y)), h(y)).$$

The crux of the construction is the iteration of the following procedure. Let

$$1 \to \mathbb{Z}^k \to \Pi \to Q \to 1$$

be an exact sequence with abstract kernel $\varphi : Q \to \mathrm{GL}(k, \mathbb{R})$. Let

$$\rho : Q \to \mathrm{P}(\mathbb{R}^K)$$

be a representation which yields a proper action of Q on $\mathbb{R}^K$ with $Q\backslash\mathbb{R}^K$ compact. We try to find a homomorphism $\theta : \Pi \to \mathrm{P}(\mathbb{R}^K, \mathbb{R}^k) \rtimes (\mathrm{GL}(k, \mathbb{R}) \times \mathrm{P}(\mathbb{R}^K))$ so that the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & {\scriptstyle i}\downarrow & & {\scriptstyle \theta}\downarrow & & {\scriptstyle \varphi\times\rho}\downarrow & & \\
1 & \longrightarrow & \mathrm{P}(\mathbb{R}^K,\mathbb{R}^k) & \longrightarrow & \mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{GL}(k,\mathbb{R})\times\mathrm{P}(\mathbb{R}^K)) & \longrightarrow & \mathrm{GL}(k,\mathbb{R})\times\mathrm{P}(\mathbb{R}^K) & \longrightarrow & 1
\end{array}$$

where $i:\mathbb{Z}^k\to\mathbb{R}^k\subset\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)$ is the standard translations, is commutative.

Note that

$$\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\subset\mathrm{M}(\mathbb{R}^K,\mathbb{R}^k) \quad\text{and}\quad \mathrm{P}(\mathbb{R}^K)\subset\mathrm{TOP}(\mathbb{R}^K)$$

and therefore,

$$\begin{array}{ccc}
\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{GL}(k,\mathbb{R})\times\mathrm{P}(\mathbb{R}^K)) & \xrightarrow{\subset} & \mathrm{P}(\mathbb{R}^{K+k}) \\
{\scriptstyle\cap}\downarrow & & {\scriptstyle\cap}\downarrow \\
\mathrm{M}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{GL}(k,\mathbb{R})\times\mathrm{TOP}(\mathbb{R}^K)) & \xrightarrow{\subset} & \mathrm{TOP}(\mathbb{R}^{K+k})
\end{array}$$

See Corollary **??**.

2.4.2(Canonical type polynomial representations). It is well known ([**?**, lemma 6, pp. 16]) that, if Γ is a polycyclic–by–finite group, then there exists an ascending sequence (or filtration) of normal subgroups Γ_i $(0\le i\le c+1)$ of Γ

$$\Gamma_*:\quad \Gamma_0=1\subseteq\Gamma_1\subseteq\Gamma_2\subseteq\cdots\subseteq\Gamma_{c-1}\subseteq\Gamma_c\subseteq\Gamma_{c+1}=\Gamma \tag{2.4–1}$$

for which

$$\Gamma/\Gamma_i\cong\mathbb{Z}^{k_i}\ \text{for } 1\le i\le c \text{ and some } k_i\in\mathbb{N}_0 \text{ and } \Gamma/\Gamma_c \text{ is finite.}$$

Let us call such a filtration of Γ a **torsion–free filtration** (of length c). We will also use $K_i=k_i+k_{i+1}+\cdots+k_c$ and $K_{c+1}=0$.

Since Γ/Γ_c is a finite group, the trivial homomorphism $\rho_c:\Gamma/\Gamma_c\to\mathrm{P}(\mathbb{R}^0)$ exists. Therefore,

$$H^j(\Gamma/\Gamma_c;\mathrm{P}(\mathbb{R}^0,\mathbb{R}^{k_c}))=H^j(\Gamma/\Gamma_c;\mathbb{R}^{k_c})=0$$

for $j>0$. This implies that there exists a homomorphism

$$\rho_{c-1}:\Gamma/\Gamma_{c-1}\to\mathbf{P}_c=\mathrm{P}(\mathbb{R}^0,\mathbb{R}^{k_c})\rtimes(\mathrm{GL}(k_c,\mathbb{R})\times\mathrm{P}(\mathbb{R}^0))$$

which makes the following diagram commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^{k_c}\cong\Gamma/\Gamma_c & \longrightarrow & \Gamma/\Gamma_{c-1} & \longrightarrow & \Gamma/\Gamma_c & \longrightarrow & 1 \\
 & & {\scriptstyle j_c}\downarrow & & {\scriptstyle \rho_{c-1}}\downarrow & & {\scriptstyle \varphi_c\times\rho_c}\downarrow & & \\
1 & \longrightarrow & \mathrm{P}(\mathbb{R}^0,\mathbb{R}^k_c) & \longrightarrow & \mathbf{P}_c & \longrightarrow & \mathrm{GL}(k_c,\mathbb{R})\times\mathrm{P}(\mathbb{R}^0) & \longrightarrow & 1
\end{array}$$

Observe that $\mathbf{P}_c\subset\mathrm{P}(\mathbb{R}^{K_c})$. Now suppose we found

$$\rho_{i-1}:\Gamma/\Gamma_{i-1}\to\mathbf{P}_i=\mathrm{P}(\mathbb{R}^{K_{i+1}},\mathbb{R}^{k_i})\rtimes(\mathrm{GL}(k_i,\mathbb{R})\times\mathrm{P}(\mathbb{R}^{K_{i+1}}))$$

which fits the following commuting diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^{k_i}\cong\Gamma/\Gamma_i & \longrightarrow & \Gamma/\Gamma_{i-1} & \longrightarrow & \Gamma/\Gamma_i & \longrightarrow & 1 \\
 & & {\scriptstyle j_i}\downarrow & & {\scriptstyle \rho_{i-1}}\downarrow & & {\scriptstyle \varphi_i\times\rho_i}\downarrow & & \\
1 & \longrightarrow & \mathrm{P}(\mathbb{R}^{K_i+1},\mathbb{R}^k_i) & \longrightarrow & \mathbf{P}_i & \longrightarrow & \mathrm{GL}(k_i,\mathbb{R})\times\mathrm{P}(\mathbb{R}^{K_i+1}) & \longrightarrow & 1
\end{array}$$

Iterating this procedure, we will have found a desired homomorphism $\Gamma \to \mathrm{P}(\mathbb{R}_K)$. The existence of ρ_{i-1} is guaranteed by

$$H^2(\Gamma/\Gamma_i; \mathrm{P}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^k_i)) = 0$$

as the proof of the general construction shows. See Theorem **??**. Also,

$$H^1(\Gamma/\Gamma_i; \mathrm{P}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^k_i)) = 0$$

guarantees the uniqueness of such ρ_{i-1} (with fixed j_i and $\psi_i \times \rho_i$). These are achieved by the following

Lemma *If Γ is a polycyclic-by-finite group admitting a canonical type polynomial representation $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^m)$, then, for every representation $\varphi : \Gamma \to \mathrm{GL}(\mathbb{R}^n)$ and for all $i > 0$ $H^i_{\varphi\times\rho}(\Gamma, \mathrm{P}(\mathbb{R}^m, \mathbb{R}^n)) = 0$.*

The major work of the paper [**?**] is proving this Lemma. We refer the readers to that paper.

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(1) Kamishima's Conformal stuff into Appl.
(2) Take out "realization" from double coset and put in Appl.
(3) Few periodic maps
(4) maximal torus action into where?

2.5. Applications to Fixed-point Theory

We show that Bieberbach's rigidity theorem for flat manifolds still holds true for any continuous maps on infra-nilmanifolds. Namely, every endomorphism of an almost crystallographic group is semi-conjugate to an affine endomorphism. Applying this result to Fixed-point theory, we obtain a criterion for the Lefschetz number and Nielsen number for a map on infra-nilmanifolds to be equal.

2.5.1. Let G be a connected Lie group. Consider the semi-group $\mathrm{Endo}(G)$, the set of all endomorphisms of G, under the composition as operation. We form the semi-direct product $G \rtimes \mathrm{Endo}(G)$ and call it $\mathrm{aff}(G)$. With the binary operation

$$(a, A)(b, B) = (a \cdot Ab, AB),$$

the set $\mathrm{aff}(G)$ forms a semi-group with identity $(e, 1)$, where $e \in G$ and $1 \in \mathrm{Endo}(G)$ are the identity elements. The semi-group $\mathrm{aff}(G)$ "acts" on G by

$$(a, A) \cdot x = a \cdot Ax$$

Note that (a, A) is not a homeomorphism unless $A \in \mathrm{Aut}(G)$. Clearly, $\mathrm{aff}(G)$ is a subsemi-group of the semi-group of all continuous maps of G into itself, for $((a, A)(b, B))x = (a, A)((b, B)x)$ for all $x \in G$. We call elements of $\mathrm{aff}(G)$ *affine endomorphisms*.

2.5.2(Generalization of Bieberbach's second Theorem). Let G be a connected and simply connected nilpotent Lie group. In Chapter **??**, we have seen: *Let $\pi, \pi' \subset \mathrm{Aff}(G)$ be two almost crystallographic groups. Then for any isomorphism $\theta : \pi \to \pi'$, there exists $g = (a, A) \in \mathrm{Aff}(G)$ such that $\theta(\alpha) = g \cdot \alpha g^{-1}$ for all $\alpha \in \pi$.*

We shall generalize this result to all homomorphisms (not necessarily isomorphisms). Topologically, this implies that every continuous map on an infra-nilmanifold is homotopic to a map induced by an affine endomorphism on the Lie group level. It can be stated as: every endomorphism of an almost crystallographic group is semi-conjugate to an affine endomorphism.

2.5.3 Theorem. *Let $\pi, \pi' \subset \mathrm{Aff}(G)$ be two almost crystallographic groups. Then for any homomorphism $\theta : \pi \to \pi'$, there exists $g = (d, D) \in \mathrm{aff}(G)$ such that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$.*

2.5.4 Example. The subgroup $\Gamma = \pi \cap G$ of of an almost crystallographic group π is characteristic, but not fully invariant. The homomorphism θ in the theorem 1.1 may not map the maximal normal nilpotent subgroup Γ of π into that of π'. This causes a lot of trouble. Let π be an orientable 4-dimensional Bieberbach group with holonomy group $\mathbb{Z}_2$. More precisely, $\pi \subset \mathbb{R}^4 \rtimes O(4) = E(4) \subset \mathrm{Aff}(\mathbb{R}^4)$ is generated by $(e_1, I), (e_2, I)$, (e_3, I), (e_4, I) and (a, A), where $a = (1/2, 0, 0, 0)^t$, and A is diagonal matrix with diagonal entries $1, -1, -1$ and 1. Note that $(a, A)^2 = (e_1, I)$. The subgroup generated by (e_1, I), (e_2, I), (e_3, I), and (a, A) forms a 3-dimensional Bieberbach group $\mathcal{G}_2$, and $\pi = \mathcal{G}_2 \times \mathbb{Z}$. Consider the endomorphism $\theta : \pi \to \pi$ which is the composite $\pi \to \mathbb{Z} \to \pi$, where the first map is the projection onto $\mathbb{Z} = \langle (e_4, I) \rangle$ and the second map sends (e_4, I) to (a, A). Thus the homomorphism θ does not map the maximal normal abelian subgroup $\mathbb{Z}^4$ (generated by the 4 translations) into itself. Such a $\mathbb{Z}^4$ is characteristic but not fully invariant in π. Let

$$d = \begin{bmatrix} x \\ 0 \\ 0 \\ y \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and let $g = (d, D)$. Then it is easy to see $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$.

According to the proposition 1.4, the element $g = (d, D)$ is the most general form. The matrix D is uniquely determined and the translation part d can vary only in two dimensions.

Proof of Theorem. Let $\Gamma = \pi \cap G$, $\Gamma' = \pi' \cap G$. As the example shows, the characteristic subgroup Γ may not go into Γ' by the homomorphism θ. Let $\Lambda = \Gamma \cap \theta^{-1}(\Gamma')$. Then Λ is a normal subgroup of π and has a finite index. Let $Q = \pi/\Lambda$.

Consider the homomorphism $\Lambda \to \Gamma' \hookrightarrow G$, where the first map is the restriction of θ. Since Λ is a lattice of G, by Malćev's work, any such a homomorphism extends uniquely to a continuous homomorphism $C : G \to G$, cf. [**?**, 2.11]. Thus, $\theta|_\Lambda = C|_\Lambda$, where $C \in \mathrm{Endo}(G)$; and hence, $\theta(z, 1) = (Cz, 1)$ for all $z \in \Lambda$ (more precisely, $(z, 1) \in \Lambda$).

Let us denote the composite homomorphism $\pi \to \pi' \to G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ by $\overline{\theta}$; and define a map $f : \pi \to G$ by

$$\theta(w, K) = (Cw \cdot f(w, K), \overline{\theta}(w, K)) \tag{1}$$

For any $(z, 1) \in \Lambda$ and $(w, K) \in \pi$, apply θ to both sides of $(w, K)(z, 1)(w, K)^{-1} = (w \cdot Kz \cdot w^{-1}, 1)$ to get $Cw \cdot f(w, K) \cdot \overline{\theta}(w, K)(Cz) \cdot f(w, K)^{-1} \cdot (Cw)^{-1} = \theta(w \cdot Kz \cdot w^{-1})$. However, $w \cdot Kz \cdot w^{-1} \in \Lambda$ since Λ is normal in π, and the latter term equals to $C(w \cdot Kz \cdot w^{-1}) = Cw \cdot CKz \cdot (Cw)^{-1}$ since $C : G \to G$ is a homomorphism. From this we have

$$\overline{\theta}(w, K)(Cz) = f(w, K)^{-1} \cdot CKz \cdot f(w, K) \tag{2}$$

This is true for all $z \in \Lambda$. Note that $\overline{\theta}(w, K)$ and K are automorphisms of the Lie group G; and $C : G \to G$ is an endomorphism. By the uniqueness of extension of a homomorphism $\Lambda \to G$ to an endomorphism $G \to G$, as mentioned above, *the equality (2) holds true for all* $z \in G$. It is also easy to see that $f(zw, K) = f(w, K)$ for all $z \in \Lambda$ so that $f : \pi \to G$ does not depend on Λ. Thus, f factors through $Q = \pi/\Lambda$. Moreover, $\overline{\theta} : \pi \to \mathrm{Aut}(G)$ also factors through Q since Λ maps trivially into $\mathrm{Aut}(G)$. We still use the notation (w, K) to denote elements of Q and $\overline{\theta}$ to denote the induced map $Q \to \mathrm{Aut}(G)$. ???? □

<u>Claim</u> With the Q-structure on G via $\overline{\theta} : Q \to \mathrm{Aut}(G)$, $f \in Z^1(Q; G)$; i.e., $f : Q \to G$ is a crossed homomorphism.

Proof. We shall show $f((w, K) \cdot (w', K')) = f(w, K) \cdot \overline{\theta}(w, K) f(w', K')$ for all (w, K), $(w', K') \in \pi$. (Note that we are using the elements of π to denote the elements of Q). Apply θ to both sides of $(w, K)(w', K') = (w \cdot Kw', KK')$ to get $Cw \cdot f(w, K) \cdot \overline{\theta}(w, K)[Cw' \cdot f(w', K')] = C(w \cdot Kw') \cdot f((w, K)(w', K'))$. From this it follows that

$$f((w, K)(w', K')) = (CKw')^{-1} \cdot f(w, K) \cdot \overline{\theta}(w, K)(Cw') \cdot \overline{\theta}(w, K) f(w', K')$$

From (2) we have $\overline{\theta}(w, K)Cw' = f(w, K)^{-1} \cdot CKw' \cdot f(w, K)$ so that $f((w, K) \cdot (w', K')) = f(w, K) \cdot \overline{\theta}(w, K) f(w', K')$.

In [?], it was proved that $H^1(Q; G) = 0$ whenever Q is a finite group and G is a connected and simply connected nilpotent Lie group. The proof uses induction on the nilpotency of G together with the fact that $H^1(Q; G) = 0$ for a finite group Q and a real vector group G. This means that any crossed homomorphism is "principal". In other words, there exists $d \in G$ such that

$$f(w, K) = d \cdot \overline{\theta}(w, K)(d^{-1}) \tag{3}$$

Let $D = \mu(d^{-1}) \circ C$ and $g = (d, D) \in \mathrm{aff}(G)$, and we check that θ is "conjugation" by g. Using (1), (2) and (3), one can show $\overline{\theta}(w, K) \circ \mu(d^{-1}) \circ C = \mu(d^{-1}) \circ C \circ K$. Thus, for any $(w, K) \in \pi$,

$$\begin{aligned}
\theta(w, K) \cdot (d, D) \quad &= (Cw \cdot f(w, K),\ \overline{\theta}(w, K)) \cdot (d,\ \mu(d^{-1}) \circ C) \\
&= (Cw \cdot f(w, K) \cdot \overline{\theta}(w, K)(d),\ \overline{\theta}(w, K) \circ \mu(d^{-1}) \circ C) \\
&= (Cw \cdot d \cdot \overline{\theta}(w, K)(d^{-1}) \cdot \overline{\theta}(w, K)(d),\ \overline{\theta}(w, K) \circ \mu(d^{-1}) \circ C) \\
&= (Cw \cdot d,\ \mu(d^{-1}) \circ C \circ K) \\
&= (d, D) \cdot (w, K).
\end{aligned}$$

This finishes the proof of theorem. □

2.5.5 Corollary. *Let $M = \pi\backslash G$ be an infra-nilmanifold, and $h : M \to M$ be any map. Then h is homotopic to a map induced from an affine endomorphism $G \to G$.*

Proof. We start with the homomorphism $h_\# : \pi_1(M) \to \pi_1(M)$, induced from h, as our θ in the Theorem **??** and obtain $\widetilde{g} = (d, D)$ satisfying

$$h_\#(\alpha) \circ \widetilde{g} = \widetilde{g} \circ \alpha.$$

Let $g : M \to M$ be the induced map. Then $h_\# = g_\#$. Since any two continuous maps on a closed aspherical manifold inducing the same homomorphism on the fundamental group (up to conjugation by an element of the fundamental group) are homotopic to each other, h is homotopic to g. This completes the proof of the corollary. □

2.5.6 Corollary ([**?**],[**?**]). *Homotopy equivalent infra-nilmanifolds are affinely diffeomorphic.*

Now we consider the uniqueness problem: How many g's are there? Let $\Phi = \pi/(G \cap \pi) \subset \mathrm{Aut}(G)$ and $\Phi' = \pi'/(G \cap \pi') \subset \mathrm{Aut}(G)$ be the holonomy groups of π and π'. Let Ψ' be the image of $\theta(\pi)$ in Φ'. So $\Phi' \subset \mathrm{Aut}(G)$. Let $G^{\Psi'}$ denote the fixed point set of the action. For $c \in G$, $\mu(c)$ denotes conjugation by c. Therefore, $\mu(c)(x) = cxc^{-1}$ for all $x \in G$. The group of all inner automorphisms is denoted by $\mathrm{Inn}(G)$.

2.5.7 Proposition (Uniqueness). *With the same notation as above, suppose $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$. Then $\theta(\alpha) \cdot \gamma = \gamma \cdot \alpha$ for all $\alpha \in \pi$ if and only if $\gamma = \xi \cdot g$, where $\xi = (c, \mu(c^{-1}))$, for $c \in G^{\Psi'}$. Therefore, D is unique up to* $\mathrm{Inn}(G)$. *If θ is an isomorphism, then $c \in G^{\Phi'}$. In particular, if π is a Bieberbach group with $H^1(\pi; \mathbb{R}) = 0$ and θ is an isomorphism, then such a g is unique.*

Proof. Let $g = (d, D)$, $\gamma = (c, C)$. Since $\theta(\alpha) \cdot g = g \cdot \alpha$ holds when $\alpha = (z, 1) \in \Lambda$, we have $Dz = d^{-1}z'd$, where $\theta(z, 1) = (z', 1)$. Similarly, $Cz = c^{-1}z'c$. Thus $Cz = \mu(c^{-1}d)Dz$ for all $z \in \Lambda$. Since Λ is a lattice, this equality holds on G. Consequently, $C = \mu(c^{-1}d)D$. Now $\gamma = (c, C) = (c,\ \mu(c^{-1}d)D) = (d^{-1}c,\ \mu(c^{-1}d))(d, D) = (h, \mu(h^{-1}))(d, D)$, if we let $h = d^{-1}c$. Set $\xi = (h, \mu(h^{-1}))$. Then $\gamma = \xi \cdot g$. Now we shall observe that $h \in G^{\Psi'}$. Let $\theta(\alpha) = (b, B)$. Then $\theta(\alpha)\xi g = \theta(\alpha)\gamma = \gamma\alpha = \xi g\alpha = \xi\theta(\alpha)g$ yields $Bh = h$ for all $(b, B) = \theta(\alpha)$. Clearly then $B \in \Psi'$ by definition. For a Bieberbach group π, note that rank $H^1(\pi; \mathbb{Z}) = \dim G^{\Phi}$. □

2.5.8(Application to Fixed-point theory). Let M be a closed manifold and let $f : M \to M$ be a continuous map. The *Lefschetz number* $L(f)$ of f is defined by

$$L(f) := \underset{k}{\longrightarrow} \Sigma \quad \mathrm{trace}\{(f_*)_k : H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})\}$$

To define the *Nielsen number* $N(f)$ of f, we define an equivalence relation on $\mathrm{Fix}(f)$ as follows: For $x_0, x_1 \in \mathrm{Fix}(f)$, $x_0 \sim x_1$ if and only if there exists a path c from x_0 to x_1 such that c is homotopic to $f \circ c$ relative to the end points. An equivalence class of this relation is called a *fixed point class* (=FPC) of f. To each FPC F, one can assign an integer $\mathrm{ind}(f,F)$. A FPC F is called *essential* if $\mathrm{ind}(f,F)\neq 0$. Now,

$$N(f) := \text{the number of essential fixed point classes.}$$

These two numbers give information on the existence of fixed point sets. If $L(f)\neq 0$, every self-map g of M homotopic to f has a non-empty fixed point set. The Nielsen number is a lower bound for the number of components of the fixed point set of all maps homotopic to f. Even though $N(f)$ gives more information than $L(f)$ does, it is harder to calculate. If M is an infra-nilmanifold, and f is homotopically periodic, then it is known that $L(f)=N(f)$.

2.5.9 LEMMA. *Let $B\in \mathrm{GL}(n,\mathbb{R})$ with a finite order. Then* $\det(I-B)\geq 0$.

PROOF. Since B has finite order, it can be conjugated into the orthogonal group $O(n)$. Since all eigenvalues are roots of unity, there exists $P\in \mathrm{GL}(n,\mathbb{R})$ such that PBP^{-1} is a block diagonal matrix, with each block being a 1×1, or, a 2×2-matrix. All 1×1 blocks must be $D=[\pm 1]$, and hence $\det(I-D)=0$ or 2. For a 2×2 block, it is of the form $\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$. Consequently, each 2×2-block D has the property that $\det(I-D)=(1-\cos t)^2+\sin^2 t = 2(1-\cos t)\geq 0$. □

2.5.10 THEOREM. *Let $f: M\to M$ be a continuous map on an infra-nilmanifold $M=\pi\backslash G$. Let $g=(d,D)\in \mathrm{aff}(G)$ be a homotopy lift of f by Corollary 1.2. Then $L(f)=N(f)$ (resp., $L(f)=-N(f)$) if and only if* $\det(I-D_*A_*)\geq 0$ *(resp.,* $\det(I-D_*A_*)\leq 0$*) for all $A\in\Phi$, the holonomy group of M.*

PROOF. Since $L(f)$ and $N(f)$ are homotopy invariants, we may assume that $f=g$. Let $\Gamma=\pi\cap G$, and let $\Lambda=\Gamma\cap f_{\#}^{-1}f_{\#}(\Gamma\cap f_{\#}^{-1}(\Gamma))$. Then Γ is a normal subgroup of π, of finite index. Moreover, $f_{\#}:\pi\to\pi$ maps Λ into itself. Therefore, f induces a map on the finite-sheeted regular covering space $\Lambda\backslash G$ of $\pi\backslash G$.

Let $\widetilde{f}$ be a lift of f to $\Gamma\backslash G$. Then

$$L(f)=\frac{1}{[\pi:\Lambda]}\Sigma\ \mathrm{ind}(f,p_{_\Lambda}\mathrm{Fix}(\alpha\widetilde{f}))$$

$$N(f)=\frac{1}{[\pi:\Lambda]}\Sigma\ |\mathrm{ind}(f,p_{_\Lambda}\mathrm{Fix}(\alpha\widetilde{f}))|$$

where the sum ranges over all $\alpha\in\pi/\Lambda$. See, [**?**, III 2.12]. However, each $\alpha\widetilde{f}$ is a map on the nilmanifold $\Lambda\backslash G$, and hence $\mathrm{ind}(f,p_{_\Lambda}\mathrm{Fix}(\alpha\widetilde{f}))$ is determined by $\det(I-(\alpha f)_*)$. It is not hard to see that, for any $\alpha\in\mathrm{Inn}(G)$, α_* has eigenvalue only 1. Therefore, it is enough to look at elements with non-trivial holonomy. Now the hypothesis guarantees that $\det(I-(\alpha f)_*)=\det(I-D_*A_*)\geq 0$ or ≤ 0 always. Consequently, $L(f)=N(f)$ or $L(f)=-N(f)$.

Conversely, suppose $L(f)=N(f)$ (resp. $L(f)=-N(f)$). Let $\alpha=(a,A)\in\pi$. If $\mathrm{Fix}(g\circ\alpha)=\emptyset$, then clearly $\det(I-D_*A_*)=0$. Otherwise, $\mathrm{Fix}(g\circ\alpha)$ is isolated, and the indices at these fixed points are $\det(I-D_*A_*)$. By the formula above relating $L(f)$, $N(f)$ with the ones on covering spaces, all $\det(I-D_*A_*)$ must have the same sign. This proves the theorem. □

2.5.11 Corollary ([**?**]). *Let $f : M \to M$ be a homotopically periodic map on an infra-nilmanifold. Then $N(f) = L(f)$.*

Proof. Here is an argument which is completely different from the one in [**?**]. Let $\Gamma = \pi \cap G$, and $\Phi = \pi/\Gamma$, the holonomy group. Let $g = (d, D) \in G \rtimes \text{Aut}(G)$ be a homotopy lift of f to G. Let E be the lifting group of the action of $\langle g \rangle$ to G. That is, E is generated by π and g. Then E/Γ is a finite group generated by Φ and D. For every $A \in \Phi$, DA lies in E/Γ, and has a finite order. By Lemma 2.1, $\det(I - DA) \geq 0$ for all $A \in \Phi$. By Theorem 2.2, $L(f) = N(f)$. □

Let S be a connected, simply connected solvable Lie group and H be a closed subgroup of S. The coset space $H\backslash S$ is called a solvmanifold.

2.5.12 Corollary ([**?**]). *Let $f : M \to M$ be a homotopically periodic map on an infra-solvmanifold. Then $N(f) = L(f)$.*

Proof. In [**?**], the statement for solvmanifolds was proved. We needed a subgroup invariant under $f_{\#}$. To achieve this, a new model space M' which is homotopy equivalent to M, together with a map $f' : M' \to M'$ corresponding to f was constructed. The new space M' is a fiber bundle over a torus with fiber a nilmanifold; and f' is fiber-preserving. Moreover, we found a fully invariant subgroup Λ of π of finite index (so, is invariant under $f'_{\#}$). Now we can apply the same argument as in the proof of Theorem 2.2. □

2.5.13 Example. Let π be an orientable 3-dimensional Bieberbach group with holonomy group $\mathbb{Z}_2$. More precisely, $\pi \subset \mathbb{R}^3 \rtimes O(3) = E(3)$ is generated by $(e_1, I), (e_2, I)$, (e_3, I) and (a, A), where $a = (1/2, 0, 0)^t$, A is a diagonal matrix with diagonal entries $1, -1$ and -1. Note that $(a, A)^2 = (e_1, I)$. Let $M = \mathbb{R}^3/\pi$ be the flat manifold. Consider the endomorphism $\theta : \pi \to \pi$ which is defined by the conjugation by $g = (d, D)$, where

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

Let $f : M \to M$ be the map induced from g. There are only two conjugacy classes of g; namely, g and αg. $\text{Fix}(g) = (0, 0, 0)^t$ and $\text{Fix}(\alpha g) = (1/4, 0, 0)^t$. Since $\det(I - D) = \det(I - AD) = +2$, $L(f) = N(f) = 2$.

The Lefschetz number can be calculated from homology groups also.

(1) $H_0(M; \mathbb{R}) = \mathbb{R}$; f_* is the identity map.
(2) $H_1(M; \mathbb{R}) = \mathbb{R}$, which is generated by the element (e_1, I).
    f_* is multiplication by 3 (the (1,1)-entry of D).
(3) $H_2(M; \mathbb{R}) = \mathbb{R}$; f_* is multiplication by $\det \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = -2$.
(4) $H_3(M; \mathbb{R}) = \mathbb{R}$; f_* is multiplication by $\det(D) = -6$.

Therefore, $L(f) = \Sigma(-1)^i \text{trace} f_* = 1 - 3 + (-2) - (-6) = 2$. Note that f has infinite period, and this example is not covered by Corollary 2.3.

2.5.14 EXAMPLE. Let π be same as in Example 2.5. This time $g = (d, D)$, is given by

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Let $f : M \to M$ be the map induced from g. There are six conjugacy classes of g; namely, g and αg, $\alpha t_1 g$, $\alpha t_1^2 g$, $\alpha t_1^3 g$, and $\alpha t_1^4 g$, Each class has exactly one fixed point. Clearly, $\det(I - D) = +2$ and $\det(I - AD) = -10$. Therefore, the first fixed point has index $+1$ and the rest have index -1. Consequently, $L(f) = -4$, while $N(f) = 6$.

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2.6. Maximal torus actions on solvmanifolds and double coset spaces

2.6.1(Maximal torus action on infra-nilmanifolds). Let M be an infra-nil (infra-toral) manifold. Then we claim the following sequence is exact:

$$1 \to \mathrm{Aff}_0(M) \to \mathrm{Aff}(M) \to \mathrm{Out}(\pi_1(M)) \to 1,$$

Explain the meaning of $\mathrm{Aff}(M)$ where $\mathrm{Aff}(M)$ is the group of affine self diffeomorphisms of M. Since $\widetilde{M} = G$ is a simply connected nilpotent Lie group, $M = \Pi\backslash G$ with $\Pi \subset \mathrm{Aff}(G)$,

$$N_{\mathrm{Aff}(G)}(\Pi)/\Pi = \mathrm{Aff}(M).$$

This is a Lie group. Now every self homotopy equivalence induces an automorphism of Π, unique up to an inner automorphism. The homotopy classes of self homotopy equivalences is in 1-1 correspondence with $\mathrm{Out}(\Pi)$. Then every isomorphism $\Pi \to \Pi$ is given by conjugation by a homeomorphism in $\mathrm{Aff}(G)$ and an automorphism of Π will have a realization in $N_{\mathrm{Aff}(G)}(\Pi)$. Thus $\mathrm{Aff}(M) = N_{\mathrm{Aff}(G)}(\Pi)/\Pi \to \mathrm{Out}(\Pi)$ is onto. We are interested in the kernel of this homomorphism. Look at the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{Z}(\Pi) & \longrightarrow & C_{\mathrm{Aff}(G)}(\Pi) & \longrightarrow & C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi & \longrightarrow & N_{\mathrm{Aff}(G)}(\Pi) & \longrightarrow & \mathrm{Aff}(M) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Pi) & \longrightarrow & \mathrm{Aut}(\Pi) & \longrightarrow & \mathrm{Out}(\Pi) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

We see the kernel of $\mathrm{Aff}(M) \to \mathrm{Out}(\Pi)$ is $C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi)$. There are two different ways of expressing $\mathrm{Aff}(G)$. Namely,

$$\begin{aligned} \mathrm{Aff}(G) &= r(G) \rtimes \mathrm{Aut}(G) \\ &= \ell(G) \rtimes \mathrm{Aut}(G). \end{aligned}$$

maximal torus action

The actions on $x \in G$ are as follows:

$$(r(a), \alpha) \cdot x = \alpha(x) \cdot a^{-1}, \qquad (\ell(a), \alpha) \cdot x = a \cdot \alpha(x).$$

Clearly, the correspondence

$$(a, \alpha) \longleftrightarrow (a^{-1}, \mu(a) \circ \alpha)$$

is a bijective map between $r(G) \rtimes \mathrm{Aut}(G)$ and $\ell(G) \rtimes \mathrm{Aut}(G)$. The first expression is more natural with respect to the Seifert Construction because

$$\begin{aligned} \mathrm{TOP}_G(G \times \{w\}) &= \mathrm{M}(\{w\}, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(\{w\})) \\ &= r(G) \rtimes \mathrm{Aut}(G). \end{aligned}$$

But, in this section, we shall use $\mathrm{Aff}(G) = \ell(G) \rtimes \mathrm{Aut}(G)$.

Do we need to remind the definitions of infranil, almost B?

2.6.2 Theorem. *Let $M = \Pi \backslash G$ be an infra-nilmanifold, where G be a simply connected nilpotent Lie group, $\Pi \subset G \rtimes \mathrm{Aut}(G)$ an almost Bieberbach group. Then $\mathrm{Aff}_0(M) = C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi) = G^Q/\mathcal{Z}(\Pi)$ (Q the holonomy group), and it contains a* maximal torus action (see section 1.15.12) $\mathcal{Z}(G^Q)/\mathcal{Z}(\Pi)$.

Proof. We want to show $C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi)$ is connected (as a Lie subgroup). Let $(a, \alpha) \in C_{\mathrm{Aff}(G)}(\Pi)$. Then (a, α) must centralize $\Gamma \subset \Pi$ and so also all of G. Thus,

$$(a, \alpha)(g, 1) = (g, 1)(a, \alpha)$$

which implies $\alpha(g) = a^{-1} g a$ for all $g \in G$. Thus $\alpha = \mu(a^{-1})$. Therefore each element of $C_{\mathrm{Aff}(G)}(\Pi)$ must be of the form $(a, \mu(a^{-1})) = r(a)$. Now it must also centralize Π. Let $\Pi \cap G = \Gamma$ and $\Pi/\Gamma = Q$. Then the holonomy group Q injects into $\mathrm{Aut}(G)$ naturally and we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G & \longrightarrow & G \rtimes \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & 1 \end{array}$$

Let $(y, \beta) \in \Pi$. Then $(a, u(a^{-1}))(y, \beta) = (y, \beta)(a, \mu^{-1}(a))$ implies $ya = y\beta(a)$ for all $y \in \Gamma$. Thus $a = \beta(a)$. That is, as we run through all the $\beta \in Q$, $\beta(a) = a$ so that $a \in G^Q$, a closed subgroup of G. Clearly G^Q is a simply connected nilpotent subgroup, best seen from the Lie algebra. We have

$$\begin{aligned} \mathrm{Aff}_0(M) &= C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi) \\ &= \{(a, u(a^{-1})) : \ a \in G^Q\}/\mathcal{Z}(\Pi) \\ &\cong r(G^Q)/\mathcal{Z}(\Pi) \end{aligned}$$

Since $\mathcal{Z}(\Pi) \subset \mathcal{Z}(G)$, we have $\mathcal{Z}(\Pi) \subset \mathcal{Z}(G)^Q$. It is easy to see that $\mathcal{Z}(\Pi)$ is a uniform lattice of $\mathcal{Z}(G)^Q$, and so $\mathcal{Z}(G^G)/\mathcal{Z}(\Pi)$ is a torus, acting effectively on M. Since $\Pi_1(\mathcal{Z}(G^G)/\mathcal{Z}(\Pi)) = \mathcal{Z}(\Pi)$, this torus action is a maximal torus action by Corollary 1.15.13. Note $G^Q/\mathcal{Z}(\Pi)$ may not be compact.♣ □

Related to Tondeor-Kamber. Aff is not ordinary metric connection.

2.6.3 Example (Klein bottle). The fundamental group has presentation $\Pi = \{a, b \mid a^2 b^2 = 1\}$. $\quad \mathcal{Z}(\Pi) = \{a^2\}$; the maximal normal abelian subgroup is

$$\Gamma = \mathbb{R}^2 \cap \Pi = \{a^2, b | [a^2, b] = 1\};$$

and the *holonomy group* is $Q = \mathbb{Z}/2$. The universal covering space is the abelian Lie group $G = \mathbb{R}^2$, and $G^Q = \mathbb{R}^1$ with $G^Q \cap \mathit{\Pi} = \{a^2\} \approx \mathbb{Z}$. Therefore, $\mathrm{Aff}_0(M) \cong G^Q/\mathcal{Z}(\mathit{\Pi}) = \mathbb{R}/\mathbb{Z} = S^1$, a circle. This is the maximal torus action by affine diffeomorphisms (in fact, isometries).

holonomy group

2.6.4 Exercise. Find $\mathrm{Out}(\pi_1(\text{Kleinbottle}))$ (Answer: $\mathbb{Z}_2 \times \mathbb{Z}_2$). Find $\mathrm{Aut}(\pi_1(\text{Kleinbottle}))$.

2.6.5 Example. Let G be a simply connected nilpotent Lie group, $\mathit{\Pi} \subset G \rtimes \mathrm{Aut}(G)$ an almost Bieberbach group. Suppose $\Gamma = \mathit{\Pi}$.$^{\clubsuit}$ That is, $\mathit{\Pi}$ has trivial holonomy group Q. Then $G^Q = G$ since $Q = 1$. Therefore,

Why did you circle it?

$$\mathrm{Aff}_0(M) = G^Q/\mathcal{Z}(\Gamma) = G/\mathcal{Z}(\Gamma).$$

Topologically, this is a product $\mathcal{Z}(G)/\mathcal{Z}(\Gamma) \times G/\mathcal{Z}(G)$, of a torus with a simply connected nilpotent Lie group. This is the covering space of M corresponding to the image of the evaluation homomorphism of the maximal torus action. Note that $\Gamma/\mathcal{Z}(\Gamma)$ is a lattice in $G/\mathcal{Z}(G)$. Consequently, the maximal torus action on a nilmanifold is free.

2.6.6 Corollary. *Let G be a simply connected nilpotent Lie group, $\mathit{\Pi} \subset G \rtimes \mathrm{Aut}(G)$ an almost Bieberbach group. Then*

$$\mathrm{Aff}_0(M) = r(G^Q)/\mathcal{Z}(\mathit{\Pi})$$

contains a toral subgroup $\mathcal{Z}(G)^Q/\mathcal{Z}(\mathit{\Pi})$, and quotient group a simply connected nilpotent Lie group $G^Q/\mathcal{Z}(G)^Q$. Therefore, if $G = \mathbb{R}^n$ (i.e., $\mathit{\Pi}$ is a Bieberbach group), Then then $\mathrm{Aff}_0(M)$ is a torus $G^Q/\mathcal{Z}(\mathit{\Pi})$.$^{\clubsuit}$ □

This coro is the same as the Theorem except the last sentence.

2.6.7 Remark. The torus is a maximal torus action and represents also the connected component of the full isometry group. This works also for the case of infra-solvmanifolds where G is of type (R), the (S-3)-case. The same argument applies. See subsection **??**.

NEW

2.6.8 Proposition. *Let G be a finite group acting freely on a space M such that $\pi_1(M)$ and $\pi_1(G\backslash M)$ are torsion-free. Assume $\mathcal{Z}(\pi_1(M))$ is finitely generated and that every element of G is homotopic to the identity. Then*

(1) *$\pi_1(M)$ and $\pi_1(G\backslash M)$ have non-trivial centers of rank $k \geq 1$,*
(2) *G is an abelian group of rank $\leq k$ (i.e., can be embedded in a k-torus),*
(3) *the center of $\pi_1(G\backslash M)$ is the centralizer of $\pi_1(M)$ in $\pi_1(G\backslash M)$ and is an extension of $\mathcal{Z}(\pi_1(M))$ by G.*

Proof. The lifting sequence (see subsection 1.8.1)$^{\clubsuit}$ for the G action is given by

Do we put these sort of things everywhere?

$$1 \to \pi_1(M) \to \pi_1(G\backslash M) \to G \to 1.$$

Let us abbreviate $\pi_1(M)$ by $\mathit{\Pi}$ and $\pi_1(G\backslash M)$ by E. Since G acts homotopically trivially, we have that G goes trivially into $\mathrm{Out}(\mathit{\Pi})$ and there is induced the exact sequence

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{Z}(\varPi) & \longrightarrow & C_E(\varPi) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \varPi & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\varPi) & \longrightarrow & \mathrm{Aut}(\varPi) & \longrightarrow & \mathrm{Out}(\varPi) & \longrightarrow & 1
\end{array}$$

We need only observe that $C_E(\varPi)$ is mapped onto G. Since E is torsion free, $C_E(\varPi)$ is also torsion free. Since $G \neq 1$, $E = \pi_1(G\backslash M)$ is not finite and so, $\pi_1(M) = \varPi$ is infinite. Therefore, $\mathcal{Z}(\varPi)$ has rank k for some $k \geq 1$. (Otherwise, $C_E(\varPi)$ which is torsion free could not map onto G). The top horizontal row is a central extension. Since $C_E(\varPi)$ is torsion free and central, $C_E(\varPi) \cong \mathbb{Z}^k$ and $\mathcal{Z}(\varPi) = \mathbb{Z}^k$ is a sub-lattice of $C_E(\varPi)$. The quotient G is isomorphic to a subgroup of T^k.

Now, for $e \in E$, $\theta(e)$ goes trivially into $\mathrm{Out}(\varPi)$. Consequently, $\theta(e)$ is conjugation $\mu(\gamma)$ by some element $\gamma \in \varPi$. Therefore, $ece^{-1} = \gamma c \gamma^{-1} = c$, for $c \in \mathcal{Z}(\varPi)$. So, e acts trivially on $\mathcal{Z}(\varPi)$ and because $\mathcal{Z}(\varPi)$ is a sub-lattice of $C_E(\varPi)$, E also acts trivially on $C_E(\varPi)$. This implies that $C_E(\varPi)$ is the full center of E. □

2.6.9 Corollary. *Let M be closed aspherical manifold with $\mathcal{Z}(\pi_1(M))$ finitely generated. Let G, a finite group, act effectively, and homotopically trivially on M. Then*

(1) *$\pi_1(M)$ has anon-trivial center of rank $k \geq 1$.*
(2) *G is an abelian group of rank $\leq k$ (i.e., can be embedded in to k-torus).*
(3) *In the lifting sequence $1 \to \pi_1(M) \to E \to G \to 1$ for (G, M), the center of E is $C_E(\varPi)$ and is central extension of $\mathcal{Z}(\pi_1(M))$ by G.*

Is this before? Put it in Ch 1

Proof. Effectiveness of the G action guarantees that the lifting sequence is an admissible extension (see subsection **??**)♣ That is, $C_E(\varPi)$ is torsion free. Now, in the diagram of the proof of the proposition, we observe that $\mathcal{Z}(\pi_1(M)) \neq 1$, for otherwise G would have to inject into $\mathrm{Out}(\varPi)$. The rest of the argument proceeds as in the Proposition. □

2.6.10 Corollary. *Let M and E be as above except that we assume that M is an admissible manifold (see subsection* **??***) instead of a closed aspherical manifold. In addition, assume that $\pi_1(M)$ is torsion free. Then, the same conclusions hold.*

Proof. Here we need to check that $C_E(\varPi)$ is torsion free. If not, there exists a $\mathbb{Z}_p$ subgroup, p prime, commuting with $\pi_1(M)$. Since M is an admissible space, $\mathbb{Z}_p$ will have to be in the center of $\pi_1(M)$ contradicting that $\varPi$ is torsion free. □

NEW–END

2.6.11 THEOREM (need air-tight argument). *Suppose H is a compact Lie group of homeomorphisms acting freely and locally smoothly (see subsection XXX)*♣ *on an infra-nilmanifold M so that every $h \in H$ is homotopic to the identity. Then H is contained in some maximal torus action on M and is topologically equivalent to a restriction of the standard maximal toral action on M, provided that* dimension$(H\backslash M) \neq 3$.

No def of "locally smoothly". Put one in Chapter 1. After cohomology manifold

PROOF. The connected component of id in H is a torus of dimension, say s, (because M is aspherical). Let $\pi_1(M) = \varPi$. Then by Corollary 1.15.10, $\pi_1(H_0) \cong \mathbb{Z}^s$ is a direct summand of $\mathcal{Z}(\pi_1(M))$. Then $\varPi/\mathbb{Z}^s$ is again virtually nilpotent and torsion free since the action of H_0 is free and effective.

Now we have a standard maximal torus action on M. Say $G^k \cong T^k$. We can re-parametrize $G^k = G^s \times G^{s-k}$ so that the standard action of G^s has the same evaluation homomorphism as T^s. Put $G^s\backslash M = N_1$, and $T^s\backslash M = N_2$. They are topological manifolds because the actions are free and locally smooth. The action of G^s is free because $\pi_1(M)/\mathrm{ev}_*^x(\pi_1(T^s)) = Q$ is torsion free, for otherwise the T^s action could not be free, see Exercise NEW(1.11.10-11). Both N_1 and N_2 are aspherical with $\pi_1(N_i) = \pi_1(M)/\mathbb{Z}^s,\ i = 1, 2$.

[Modified from here] We may lift the T^s and G^s actions to $M_{\mathrm{Im}(\mathrm{ev}_*^x)} = M_{\mathbb{Z}^s}$ and we get two splittings

$$(T^s, T^s \times W_i) \longrightarrow (Q_i = \pi_1(M)/\mathbb{Z}^s, W_i)$$

and 2 actions of Q on the projections W_i. The projections W_i are both contractible manifolds (in fact, they are homeomorphic to $\mathbb{R}^{n-s}$, where n is the dimension of M). N_1 is an infra-nilmanifold because G is a free direct summand of the maximal torus action, N_2 is homeomorphic to N_1, by virtue of the theorem of Farrell and Hsiang for dimension $N_i \geq 5$ and for dimension 4 by Freedman, Farrell and Jones. Therefore, the free Q_i actions on W_i, $i = 1, 2$, are weakly equivalent. That is, there is a homeomorphism $\widetilde{h} : W_1 \to W_2$ and an isomorphism $\alpha : Q_1 \to Q_2$ so that $\widetilde{h}(q_1 w_1) = \alpha(q_1)\widetilde{h}(w_2)$, where $\widetilde{h}$ is a lift of a homeomorphism $h : N_1 \to N_2$.

This means that on M, the two torus actions are equivalent by a Seifert automorphism by virtue of the uniqueness and rigidity of Seifert Constructions.

Now we wish to study the finite part. We can assume that $H_0 = G^s$ is in the standard toral action. Let $F = H/G^s$. We claim that F is a finite abelian group and that H splits as $G^s \oplus F$ with $G^s \oplus F$ inside the standard maximal toral action.

Let E be the group of all lifts of the action of H on M so that the following is exact:

$$1 \to \varPi \to E \to H \to 1.$$

Since $H \to \mathrm{Out}(\varPi)$ is trivial we have induced

$$1 \to \mathcal{Z}(\varPi) \to C_E(\varPi) \to H \to 1$$

with $C_E(\varPi)$ being the kernel of $E \to \mathrm{Aut}(\varPi)$. Thus it is a central extension. Furthermore it is torsion free. For if not, then we can find a prime p-subgroup P of $C_E(\varPi)$ which acts on $\widetilde{M}$ and has a fixed point set. Then this $P \subset \mathrm{TOP}(\widetilde{M})$ projects to $P \subset \mathrm{TOP}(M)$ which is acting freely, a contradiction. We see that $\mathcal{Z}(\varPi) = \mathbb{Z}^s \oplus \mathbb{Z}^{k-s}$. The inverse image of $H_0 = G^s \approx T^s$ lies in $C_E(\varPi)_0 = \mathbb{R}^s$ and so we have, by dividing out by $\mathbb{R}^s$,

$$0 \to \mathbb{Z}^{k-s} \to C_E(\varPi)/\mathbb{R}^s \to F \to 1$$

is exact and $C_E(\mathit{\Pi})/\mathbb{R}^s$ is a subgroup of $\pi_1(M/T^s)$ which is also aspherical. Again $C_E(\mathit{\Pi})/\mathbb{R}^s$ is torsion free since F acts freely and the extension is central.

Note that $E/\mathbb{R}^s$ is $\pi_1(F\backslash N_1) = \pi_1(F\backslash(G^s\backslash M))$ which is torsion free. So therefore $C_E(\mathit{\Pi})/\mathbb{R}^s$ is the same as $C_{\pi_1(F\backslash N_1)}(\pi_1(N_1))$ and by the proposition, it is isomorphic to $\mathbb{Z}^{k-s}$ in $\mathcal{Z}(\mathit{\Pi})$ as a sublattice. Thus, $C_E(\mathit{\Pi})$ is free abelian and F is isomorphic to a subgroup of T^{k-s}. Now

$$1 \to \mathbb{R}^s \to C_E(\mathit{\Pi}) \to' \mathbb{Z}^{k-s} \to 1$$

is exact. Since $C_E(\mathit{\Pi})$ is 2-step nilpotent, it is easily seen that $'\mathbb{Z}^{k-s} = C_E(\mathit{\Pi})/\mathbb{R}^s$ splits back to $C_E(\mathit{\Pi})$. Hence, $C_E(\mathit{\Pi}) = \mathbb{R}^s \oplus' \mathbb{Z}^{k-s}$ and, $T^s \oplus F \subset T^s \oplus T^{k-s}$. Therefore on $G^s\backslash M = T^s\backslash M = N_1$, we have 2 free F-actions. One we have just described coming from H/H_0 and the other, F' coming from $F' \subset G^{k-s} \subset G^k$ in the standard maximal torus action on M. This second F-action $'F$ is induced from $'\mathbb{Z}^{k-s}/\mathbb{Z}^{k-s}$, with $'\mathbb{Z}^{k-s} \subset \mathbb{R}^{k-s}$. Since F and F' are free on N_1, an infra-nilmanifold, and $\pi_1(F\backslash N_1) \cong \pi_1(F'\backslash N_1)$, the orbit spaces are homeomorphic by virtue of [**?**].♣ Let $k : F\backslash N_1 \to F'\backslash N_1$ be this homeomorphism and $\widehat{k} : N_1 \to N_1$ a lift to N_1. Then $\widehat{k} \circ h \circ \widehat{k}^{-1}$ conjugates the F action into the F' action which is contained in the projection of the maximal torus action. □

Right ref?

page $7\frac{1}{2}$–1

2.6.12 Theorem. *Suppose a finite group F acts freely and homotopically trivally on an infra-nilmanifold M. Then the action of F is topologically equivalent to an action of a subgroup of the standard maximal torus action on M, and $F\backslash M$ is homotopic to an infra-nilmanifold.*

Proof. We claim that $E = F\backslash\pi_1(M)$ is a torsion free abstract almost crystallographic group. Let G be the simply connected nilpotent Lie group where $\pi_1(M) = \mathit{\Pi} \subset G \times \mathrm{Aut}(G)$. Let $\Gamma = \mathit{\Pi} \cap G$, and $Q = \mathit{\Pi}/\Gamma$, the holonomy. Since F acts freely, we have that $0 \to \mathcal{Z}(\mathit{\Pi}) \to C_E(\mathit{\Pi}) \to F \to 0$ is exact and $C_E(\mathit{\Pi})$ is free abelian of the same rank as $\mathcal{Z}(\mathit{\Pi})$, say k (see subsection **??**). Therefore, the maximal normal nilpotent subgroup in E is $\Gamma \cdot C_E(\mathit{\Pi})$. Note, $C_E(\mathit{\Pi}) = \mathcal{Z}(E)$.

Since E is an almost abstract Bieberbach group containing Γ, we can embed E via the monomorphism θ into $G \rtimes C \subset G \rtimes \mathrm{Aut}(G) = \mathrm{Aff}(G)$. θ carries $\mathit{\Pi}$ into $\mathrm{Aff}(G)$ and so there exists $\rho \in \mathrm{Aff}(G)$ such that $\mu(\rho) \circ \theta|_{\mathit{\Pi}} : \mathit{\Pi} \to \pi_1(M)$ is an isomorphism.

Thus $\mu(\rho) \circ \theta(E)$ is an almost Bieberbach group containing $\pi_1(M)$. The action of $\theta(E)/\theta(\mathit{\Pi}) \cong F$, is part of the maximal torus action on $\pi_1(M)$. Since $\theta(E)$ and $\pi_1(M)/F$ are isomorphic, these manifolds are homeomorphic by Farrell-Hsaing (dimension $\neq 3, 4$) and Farrell-Jones-Freedman for dimension 4. Consequently, the action of F on M is topologically equaivleant to a subgroup of the maximal torus action. □

Ideal statement if Theorem **??** doesn't work

2.6.13 Theorem. *Suppose H is a connected, compact group (resp. a finite group) acting freely and locally smoothly (see subsection XXX) (resp. topoogically)*♣ *on an infra-nilmanifold M so that every $h \in H$ is homotopic to the identity. Then H is a torus (resp. an abelian group) and the action is topologically equivalent to the*

No def of "locally smoothly". Put one in Chapter 1. After cohomology manifold *action of a subtorus (resp. a finite subgroup) of the standard maximal toral action on M, provided that* $\dim(H\backslash M) \neq 3$.

2.6.14. Replacement of 0.1.10

In our analysis of infra-nilmanifolds (and by the Remark of infra-solvmanifolds of type (R)), we saw that much of the information about the manifolds is encoded in the Lie group $\mathrm{Aff}(M) = N_{\mathrm{Aff}(M)}(\Pi)/\Pi$. (This is a special case of the geometric information found in $N_{\mathrm{TOP}_G(G\times W)}(\Pi)/\Pi$ for an injective Seifert fibering $M = \Pi\backslash G \times W$). We know that $\mathrm{Out}(\Pi) = \pi_0(\mathrm{Aff}(M))$, but $\pi_0(\mathrm{Diff}(M))$ and $\pi_0(\mathrm{TOP}(M))$ are much larger. That is, if we take $\mathrm{Diff}(M)$/(homotopic to the identity in the space of homotopy equivalences), we get $\mathrm{Out}(\Pi)$. In the following commuting diagram, $\mathcal{E}(M)$ is the H-space of all self-homotopy equivalences of M.

$$\begin{array}{ccccc}
\mathrm{Aff}(M) & \xrightarrow{\text{inc}} & \mathrm{Diff}(M) & \xrightarrow{\text{inc}} & \mathcal{E}(M) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0(\mathrm{Aff}(M)) & \xrightarrow{\text{inj}} & \pi_0(\mathrm{Diff}(M)) & \xrightarrow{\text{surj}} & \pi_0(\mathcal{E}(M))
\end{array}$$

Note that the composite homomorphism on the bottom $\pi_0(\mathrm{Aff}(M)) \longrightarrow \pi_0(\mathcal{E}(M))$ is an isomorphism.

The kernel, $\pi_0(\mathrm{TOP}(M) \to \mathrm{Out}(\Pi) = \pi_0(\mathcal{E}(M))$, denoted by K is "exotic". It is known that when M is a closed aspherical manifold that the element of K have order 2 (at least when dimension of M is bigger than 10)♣ . In fact, K is an infinite citeHatcher
quotient of $\mathbb{Z}_2^\infty$ =countable direct sum of $\mathbb{Z}_2$'s when $\dim(M) > 10$. We have the following

2.6.15 Proposition. *If M is a closed flat manifold, then no nontrivial finite subgroup of K can lift to act freey on M, provided* $\dim(M) > 10$.

Proof. Let G be a finite subgroup of K and suppose it has a geometric realization as a group of homeomorphisms acting effectively on M. Each elelement $g \in G$ is homotopic to the identity but not isotopic to the identity, $g \neq 1$. Moreover, $g^2 = \mathrm{id}$. Let $1 \to \Pi \to E \to G \to 1$ be the lifting sequence for (G, M). Becasue G acts effectively, $C_E(\mathbb{Z}^n)$ is torsion free, normal and maximal abelian and of finite index in E. (See [**?**, Proposiition 2]). Thus E is an abstract crystallographic group, and the rank of G is $\leq$ rank of the center of $\pi_1(M)$. Now assume G acts freely on M. Then $G\backslash M$ is homeomorphic to a flat manifold (Farrell-Hsaing) and up to conjugation by a homeomorphism, the G action lies inside the connected component T^k of the isometry group of M. Thus each term of G is isotopic to the identity which is a contradiction. See cite[Theorem 4.1]L-R II for more details. □

2.6.16 Remark. We may guarantee that any finite group acting homotopically trivially on M must act freely if

(1) $\mathcal{Z}(\pi_1(M))$ is a direct summand of $\pi_1(M)$
(2) The holonomy of M is of odd order (this depends upon G being a 2-group). Yes. This is true!

infra-solvmanifold

Since $G = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, if it does not act freely, there is an element g such that $g^2 = \text{id}$ and g is homotopic but not isotopic to the identity. So g fixes some point in M. So, $\widetilde{M}^{\mathbb{Z}_2} \neq \emptyset$ and $E = \pi_1(M) \rtimes \mathbb{Z}_2$. However, because of pathologies with fixed points (non-freeness), we cannot necessarily put $\mathbb{Z}_2$ into some torus action.

Why is infra-solv here?

As we have seen in Chapter 1, any compact, connected Lie group which acts effectively on a closed aspherical manifold is a torus T^k with $k \leq$ rank of $\mathcal{Z}(\pi_1(M))$, the center of $\pi_1(M)$. In this section, a smooth maximal torus action is constructed on each solvmanifold. We also construct smooth maximal torus actions on some double coset spaces of general Lie groups as applications. The main reference for this section is [**?**]. Remove this much ?

2.6.17(Do we need this? Where is it used?). Let S be a connected, simply connected solvable Lie group, H be a closed subgroup of S. The coset space $H\backslash S$ is called a *solvmanifold*. In this section, we consider only the case when $H = \Gamma$ is discrete so that Γ is a uniform lattice of G. More generally, let Π be a subgroup of $\text{Aff}(S) = S \rtimes \text{Aut}(S)$ acting freely on S such that $\Gamma = \Pi \cap S$ is a lattice of S and Π/Γ is finite. We call the orbit space $\Pi\backslash S$ an *infra-solvmanifold*. Therefore, a compact infra-solvmanifold is finitely covered by a solvmanifold.

It is a theorem of Mostow that two compact solvmanifolds of the same fundamental group are diffeomorphic. The significance of this statement is seen from the fact, differently from the nilpotent theory, that the group Γ does not determine the Lie group S. In other words, given a group Γ, there may exist two distinct connected, simply connected solvable Lie groups S_1, S_2 both containing a copy of Γ as a lattice. Mostow's theorem says that S_1 is diffeomorphic to S_2; and $\Gamma\backslash S_1$ is diffeomorphic to $\Gamma\backslash S_2$.

For many closed $K(\Pi, 1)$-manifolds, it is verified in [**?**] that $\mathcal{Z}(\Pi)$ is finitely generated and the manifold admits a maximal torus action. In fact, one may find a topological version of Theorem **??** (see below) in [**?**]. It uses surgery results of Wall and does not explain how the torus action arises explicitly from the solvable group S. The following theorem does not rely on surgery theory, and the solution is given explicitly in terms of Lie theory.

2.6.18 Theorem. *Let S be a simply connected solvable Lie group and Π be a lattice of S. Then the solvmanifold $\Pi\backslash S$ admits a smooth maximal torus action.*

Relation with section **??**?

Proof. The plan based upon Lie theorey is to construct a new connected and simply connected solvable Lie group $S(\Gamma)$ and an embedding of Π into $\text{Aff}(S(\Gamma)) = S(\Gamma) \rtimes \text{Aut}(S(\Gamma))$ with the following properties: (a) the infra-solvmanifold $\Pi\backslash S(\Gamma)$ is diffeomorphic to the solvmanifold $\Pi\backslash S$; and (b) $\Pi\backslash S(\Gamma)$ admits a smooth maximal torus action. The torus action constructed on $\Pi\backslash S(\Gamma)$ descends from a central vector group of $S(\Gamma)$ which commutes with the affine action of Π on $S(\Gamma)$. Then we can pull back the torus action on $\Pi\backslash S(\Gamma)$ to $\Pi\backslash S$ obtaining a smooth maximal torus action on the solvmanifold $\Pi\backslash S$. The reader should find the elementary example **??** instructive. It illustrates, in an explicit fashion, some of steps to be taken in the proof of the theorem. The proof will be given in the following 4 sections.♣

4 sections

strongly torsion-free $\mathcal{S}$ group
maximal normal nilpotent subgroup
strongly torsion-free $\mathcal{S}$ group of type I
predivisible group
Mal'cev completion
nilradical

2.6.19. Since Π is a lattice of S, it is a *strongly torsion-free $\mathcal{S}$ group*; that is, Π contains a finitely generated, torsion-free nilpotent normal subgroup D with the quotient Π/D free abelian of finite rank. Such a group Π contains a unique *maximal normal nilpotent subgroup* M which automatically contains $[\Pi, \Pi]$. The group Π also contains a characteristic subgroup Γ of finite index such that Γ is *strongly torsion-free $\mathcal{S}$ group of type I* (=*predivisible group*), and $\Gamma \supset M$. This means that

(1) Γ/M is torsion free,
(2) Let $\mu(\gamma)$ be the automorphism of the real nilpotent Lie group $M_{\mathbb{R}}$ (see below for notation) containing M as a lattice, induced from the conjugation by an element $\gamma \in \Gamma$. If θ is the eigen-value of the derivative of $\mu(\gamma)$, then
$$\theta|\theta|^{-1} = \cos 2\pi\rho + i \sin 2\pi\rho$$
where ρ is either 0 or irrational.

2.6.20 Notation. For a finitely generated, torsion free nilpotent group D, the unique connected and simply connected nilpotent Lie group is denoted by $D_{\mathbb{R}}$. This is the *Mal'cev completion*.

2.6.21. The short exact sequence of groups $1 \to M \to \Gamma \to \mathbb{Z}^k \to 1$ induces an exact sequence $1 \to M_{\mathbb{R}} \to \Gamma M_{\mathbb{R}} \to \mathbb{Z}^k \to 1$. One may think $\Gamma M_{\mathbb{R}}$ as the pushout of $M \to \Gamma$ with $M \hookrightarrow M_{\mathbb{R}}$ since $(M, M_{\mathbb{R}})$ has the unique automorphism extension property. See Definition **??**. In other words, $\Gamma M_{\mathbb{R}}$ is the unique group fitting the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & M_{\mathbb{R}} & \longrightarrow & \Gamma M_{\mathbb{R}} & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 1
\end{array}$$

Does there exist a connected and simply connected solvable Lie group $S(\Gamma)$ containing $\Gamma M_{\mathbb{R}}$? Using Wang's construction, Auslander constructed such a group $S(\Gamma)$ which fits the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_{\mathbb{R}} & \longrightarrow & \Gamma M_{\mathbb{R}} & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 1 \\
 & & {=}\downarrow & & \downarrow & & \cap\downarrow & & \\
1 & \longrightarrow & M_{\mathbb{R}} & \longrightarrow & S(\Gamma) & \longrightarrow & \mathbb{R}^k & \longrightarrow & 1
\end{array}$$

where $\mathbb{Z}^k \subset \mathbb{R}^k$ as a lattice. See [**?**] and [**?**]. Moreover, $S(\Gamma)$ has the property that there exists $\gamma_1, \gamma_2, \cdots, \gamma_k$ whose images form a set of generators for Γ/M which lie on 1-parameter groups in $S(\Gamma)$.

2.6.22(More properties of $S(\Gamma)$). (1) $\Gamma \subset S(\Gamma)$ as a lattice.
(2) There exists a toral subgroup T^* of $\mathrm{Aut}(S(\Gamma))$ such that $S \subset S(\Gamma) \rtimes T^*$. Moreover, $S \hookrightarrow S(\Gamma) \rtimes T^* \to T^*$ is surjective.
(3) Let N be the *nilradical* of S; that is, the maximal normal nilpotent connected Lie subgroup of S. Then ΓN can be naturally identified with

$\Gamma M_{\mathbb{R}}$. With this identification, we have $[S(\Gamma), S(\Gamma)] \subset \Gamma N$, and hence ΓN is normal in $S(\Gamma)$.

(4) Any automorphism θ of $\Gamma M_{\mathbb{R}}$ which is trivial on $\Gamma M_{\mathbb{R}}/M_{\mathbb{R}}$ can be uniquely extended to an automorphism of $S(\Gamma)$.

(5) N is normal in $S(\Gamma) \rtimes T^*$.

We shall study Seifert fiberings of infra-solvmanifolds. Suppose our model space P itself is a connected, simply connected Lie group; G a connected closed normal subgroup and $W = P/G$. We shall consider the short exact sequence of groups $1 \to G \to P \to W \to 1$ as a principal G-bundle. The group $\mathrm{Diff}_G(P)$ of all weakly G-equivariant smooth diffeomorphisms of P onto itself is exactly the normalizer of $G = \ell(G)$ in $\mathrm{Diff}(P)$, and is equal to $\mathrm{TOP}_G(P) \cap \mathrm{Diff}(P)$. Let $C(W, G^*)$ be the group of all smooth maps from W to G. Suppose $P \to W$ has a *smooth cross section.* Then we have a short exact sequence

$$1 \to C(W, G) \rtimes \mathrm{Inn}(G) \to \mathrm{Diff}_G(P) \to \mathrm{Out}(G) \times \mathrm{Diff}(W) \to 1.$$

The "affine group" $\mathrm{Aff}(P) = P \rtimes \mathrm{Aut}(P)$ acts on P by: $(p, \gamma) \cdot u = p \cdot \gamma(u)$ for $(p, \gamma) \in \mathrm{Aff}(P)$ and $u \in P$. Note that P acts as left translations. For $g \in G$, we have $(p, \gamma)(g, 1)(p, \gamma)^{-1} = (p\gamma(g)p^{-1}, 1)$. Let us denote the subgroup of $\mathrm{Aut}(P)$ which leave G invariant by $\mathrm{Aut}(P, G)$. An important fact for us is

$$P \rtimes \mathrm{Aut}(P, G) \subset \mathrm{Diff}_G(P) \tag{$*$}$$

This is true because $\mathrm{Diff}_G(P)$ is the normalizer of $\ell(G)$ in $\mathrm{Diff}(P)$, and $\ell(G)$ is normal in $P \rtimes \mathrm{Aut}(P, G)$.

We go back to our solvable Lie groups. Since N is the nilradical of S, S/N is commutative, say of dimension s. Therefore, we have an exact sequence of groups

$$1 \to N \to S \to S/N = \mathbb{R}^s \to 1$$

On the other hand, since $[S(\Gamma), S(\Gamma)] \subset \Gamma N$ from Property **??** (3) of $S(\Gamma)$ and $[S(\Gamma), S(\Gamma)]$ is connected, we have $[S(\Gamma), S(\Gamma)] \subset N$. Therefore N is normal in $S(\Gamma)$ and $S(\Gamma)/N$ is a commutative Lie group, $\mathbb{R}^s$. Therefore

$$1 \to N \to S(\Gamma) \to S(\Gamma)/N = \mathbb{R}^s \to 1$$

is exact.

Since N is normal in both S and $S(\Gamma)$, the inclusion maps $\Gamma N \hookrightarrow S$ and $\Gamma N \hookrightarrow S(\Gamma)$ induce $\Gamma/\Gamma \cap N \hookrightarrow S/N$ and $\Gamma/\Gamma \cap N \hookrightarrow S(\Gamma)/N$. By these homomorphisms we identify $S/N = \mathbb{R}^s$ with $S(\Gamma)/N = \mathbb{R}^s$.

The group $\Pi \subset S$ acts on S as left multiplications. Therefore, from $(*)$ we have $\Pi \subset \mathrm{Diff}_N(S)$. In fact, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi \cap N & \longrightarrow & \Pi & \longrightarrow & \Pi/(\Pi \cap N) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C(\mathbb{R}^s, N) \rtimes \mathrm{Inn}(N) & \longrightarrow & \mathrm{Diff}_N(S) & \longrightarrow & \mathrm{Out}(N) \times \mathrm{Diff}(\mathbb{R}^s) & \longrightarrow & 1
\end{array}$$

Similarly, $\Pi \subset S(\Gamma) \rtimes T^* \subset S(\Gamma) \rtimes \mathrm{Aut}(S(\Gamma), N)$, because N is normal in $S(\Gamma) \rtimes T^*$. $S(\Gamma) \rtimes T^*$ acts on $S(\Gamma)$ as affine maps which implies that $\Pi \subset \mathrm{Diff}_N(S(\Gamma))$ by $(*)$. We have

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi \cap N & \longrightarrow & \Pi & \longrightarrow & \Pi/(\Pi \cap N) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C(\mathbb{R}^s, N) \rtimes \mathrm{Inn}(N) & \longrightarrow & \mathrm{Diff}_N(S(\Gamma)) & \longrightarrow & \mathrm{Out}(N) \times \mathrm{Diff}(\mathbb{R}^s) & \longrightarrow & 1
\end{array}$$

Let us denote $\Pi/(\Pi\cap N)$ simply by Q. Then Q is a free abelian group of rank s, where $s=\dim(S/N)$. Clearly, $\Gamma/(\Gamma\cap N)$ is a subgroup of Q of finite index, because $M\subset\Gamma$ (so, $\Pi\cap N=\Gamma\cap N$). We shall examine the two actions of Q on S/N and $S(\Gamma)/N$.

The action of Q on S/N is induced by the left translation by Π on S. Therefore, $Q=\mathbb{Z}^s$ acts on $S/N=\mathbb{R}^s$ also as left translations. Moreover, Q is a lattice in S/N.

Now the action of Q on $S(\Gamma)/N$ is induced by the affine action of Π on $S(\Gamma)$. The projection $S(\Gamma)\to S(\Gamma)/N$ yields a homomorphism $S(\Gamma)\rtimes\mathrm{Aut}(S(\Gamma),N)\to(S(\Gamma)/N)\rtimes\mathrm{Aut}(S(\Gamma)/N)$ naturally. We recall how $S\subset S(\Gamma)\cdot T^*$ of Property **??** (2) was constructed in [**?**]. S acts on ΓN by conjugation, which extends to an automorphism of $\Gamma M_{\mathbb{R}}$. The latter is trivial on $\Gamma M_{\mathbb{R}}/M_{\mathbb{R}}$, and hence it can be extended to an automorphism of $S(\Gamma)$ by Property **??** (4). Since the S action on $\Gamma N/N$ is trivial, and $\Gamma N/N=\mathbb{Z}^s$ sits in $\mathbb{R}^s=S(\Gamma)/N$ as a uniform lattice, the action of S on $S(\Gamma)/N$ must be trivial as well. This implies that $S\subset S(\Gamma)\cdot T^*\subset S(\Gamma)\rtimes\mathrm{Aut}(S(\Gamma),N)\to(S(\Gamma)/N)\rtimes\mathrm{Aut}(S(\Gamma)/N)$ has image in $S(\Gamma)/N\times\{1\}$. Therefore, $Q=\mathbb{Z}^s$ acts on $S(\Gamma)/N=\mathbb{R}^s$ as left translations. Moreover, Q is a lattice $S(\Gamma)/N$. We conclude that both actions of $Q=\Pi/\Pi\cap N$ on S/N and $S(\Gamma)/N$ are as left translations.

Furthermore, $z\in\Pi\cap N$ goes into $C(S/N,N)\rtimes\mathrm{Inn}(N)$ and $C(S(\Gamma)/N,N)\rtimes\mathrm{Inn}(N)$ as $(z,\mu(z))$, as left translations, where $\mu(z)$ is the conjugation by z so that $\mu(z)(a)=zaz^{-1}$. Actually, $\Pi\cap N\subset N$ sits in $C(\mathbb{R}^s,N)\rtimes\mathrm{Inn}(N)$ as constant maps.

Choose an N-equivariant diffeomorphism $\tau:S\to S(\Gamma)$. This can be done as follows: Take smooth sections (not homomorphisms) $s_1:\mathbb{R}^s\to S$ and $s_2:\mathbb{R}^s\to S(\Gamma)$. With these sections, we define an N-bundle equivalence $\tau:S\to S(\Gamma)$ by $\tau(x\cdot s_1(w))=x\cdot s_2(w)$ for all $x\in N$ and $w\in\mathbb{R}^s$. Let us denote the representations of Π into $\mathrm{Diff}_N(S)$ and $\mathrm{Diff}_N(S(\Gamma))$ by ψ_1,ψ_2, respectively. More precisely, $\psi_1:\Pi\to S\subset\mathrm{Diff}_N(S)$; and $\psi_2:\Pi\to S\subset S(\Gamma)\rtimes T^*\subset S(\Gamma)\rtimes\mathrm{Aut}(S(\Gamma),N)\subset\mathrm{Diff}_N(S(\Gamma))$. Since τ is N-equivariant, $\mu(\tau)\circ\psi_1$ is a representation of Π into $\mathrm{Diff}_N(S(\Gamma))$. This bundle map $\tau:S\to S(\Gamma)$ induces an isomorphism $f\mapsto\tau\cdot f\cdot\tau^{-1}$ of $\mathrm{Diff}_N(S)$ onto $\mathrm{Diff}_N(S(\Gamma))$.

Consider the two representations $\mu(\tau)\circ\psi_1,\psi_2:\Pi\to\mathrm{Diff}_N(S(\Gamma))$. Since they induce the same maps of the kernel $\Pi\cap N$ into $C(\mathbb{R}^s,N)\rtimes\mathrm{Inn}(N)$, and of the quotient $\Pi/(\Pi\cap N)$ into $\mathrm{Out}(N)\times\mathrm{Diff}(\mathbb{R}^s)$, we can now apply the uniqueness of the Seifert Construction. We have commutative diagrams make "da" bigger

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi\cap N & \longrightarrow & \Pi & \longrightarrow & \Pi/(\Pi\cap N) & \longrightarrow & 1\\
 & & \downarrow & & \psi_2\downarrow\big\downarrow\mu(\tau)\circ\psi_1 & & \downarrow & & \\
1 & \longrightarrow & C(\mathbb{R}^s,N)\rtimes\mathrm{Inn}(N) & \longrightarrow & \mathrm{Diff}_N(S(\Gamma)) & \longrightarrow & \mathrm{Out}(N)\times\mathrm{Diff}(\mathbb{R}^s) & \longrightarrow & 1
\end{array}$$

By [**?**, Theorem 2.3], there exists an element $\lambda\in C(\mathbb{R}^s,N)$ which conjugates ψ_2 to $\mu(\tau)\circ\psi_1$. Thus

$$\begin{array}{ccc}
\Pi & \xrightarrow{\psi_1} & \mathrm{Diff}_N(S)\\
\psi_2\big\downarrow & & \big\downarrow\mu(\tau)\\
\mathrm{Diff}_N(S(\Gamma)) & \xrightarrow{\mu(\lambda)} & \mathrm{Diff}_N(S(\Gamma))
\end{array}$$

is commutative. The map $\mu(\tau^{-1}\circ\lambda)$ sends $\psi_2(\mathit{\Pi})$ to $\psi_1(\mathit{\Pi})$ yielding a diffeomorphism from $\mathit{\Pi}\backslash S(\Gamma)$ onto $\mathit{\Pi}\backslash S$. In this argument, the fact that N is a connected, simply connected nilpotent Lie group is essential.

Now we show the space $\mathit{\Pi}\backslash S(\Gamma)$ admits a smooth maximal torus action. Let $\mathcal{Z}(\mathit{\Pi})=\mathbb{Z}^k$ be the center of $\mathit{\Pi}$. Since M is the maximal normal nilpotent subgroup of $\mathit{\Pi}$, $\mathbb{Z}^k\subset M$. Let $\mathbb{R}^k$ be the smallest connected subgroup of $M_{\mathbb{R}}$ containing $\mathbb{Z}^k$. Since $\mathit{\Pi}$ commutes with $\mathbb{Z}^k$ and $\mathit{\Pi}\subset \mathrm{Aff}(S(\Gamma))$, $\mathit{\Pi}$ commutes with $(\mathbb{Z}^k)_{\mathbb{R}}=\mathbb{R}^k$. This means that $\mathbb{R}^k$ lies in the centralizer of $\mathit{\Pi}$ in $\mathrm{Diff}_N(S(\Gamma))$. Of course, $\mathbb{R}^k\cap\mathit{\Pi}=\mathcal{Z}(\mathit{\Pi})$. Thus we obtain an action of torus $\mathbb{R}^k/\mathbb{Z}^k$ on the model space $\mathit{\Pi}\backslash S(\Gamma)$. This action is smooth, (actually, it is a group of isometries if we give a left invariant metric on $S(\Gamma)$), and is a maximal torus action on $\mathit{\Pi}\backslash S(\Gamma)$. Now one can pull back this action to a smooth action on $\mathit{\Pi}\backslash S$. This completes the proof of Theorem. □

2.6.23 COROLLARY (Mostow). *Let S_1, S_2 be two connected, simply connected solvable Lie groups. Let Γ_i be a lattice in S_i, $i=1,2$. Suppose Γ_1 is isomorphic to Γ_2. Then S_1/Γ_1 is diffeomorphic to S_2/Γ_2.*

For Mostow's argument, see [**?**, Theorem 3.6]. We give a different proof. Since $\Gamma_1\cong\Gamma_2(=\mathit{\Pi})$, construct a connected, simply connected solvable Lie group $S(\Gamma)$ on which these groups act. By Theorem **??**, S_i/Γ_i is diffeomorphic to $S(\Gamma)/\mathit{\Pi}$, $i=1,2$. Therefore, S_1/Γ_1 is diffeomorphic to S_2/Γ_2.

The following example illustrates the construction employed in the proof of the theorem. Moreover the example serves to illustrate why one is compelled to look for a larger group than S if one wishes to construct a maximal torus action from the descent of a vector subgroup.

2.6.24 EXAMPLE. Let $S=\widetilde{E_0(2)}=\mathbb{R}^2\rtimes\mathbb{R}$ be the universal covering group of the 2-dimensional Euclidean group, where $(0,t)$ acts on $\mathbb{R}^2$ by $x\mapsto e^{2\pi i t}x$, x seen as a complex number. Let $\mathit{\Pi}$ be the lattice generated by

$$t_1=\left(\begin{bmatrix}1\\0\end{bmatrix},0\right),\quad t_2=\left(\begin{bmatrix}0\\1\end{bmatrix},0\right),\quad \alpha=\left(\begin{bmatrix}0\\0\end{bmatrix},\frac{1}{2}\right)$$

The subgroup Γ generated by t_1, t_2 and α^2 is a characteristic subgroup of $\mathit{\Pi}$, isomorphic to $\mathbb{Z}^3$. Then $S(\Gamma)=\mathbb{R}^3$ and we get an embedding of S into $S(\Gamma)\rtimes S^1=\mathbb{R}^3\rtimes\mathrm{SO}(2)\subset\mathbb{R}^3\rtimes\mathrm{O}(3)=E(3)$. The homomorphism is obvious:

$$\left(\begin{bmatrix}x_1\\x_2\end{bmatrix},t\right)\mapsto\left(\begin{bmatrix}x_1\\x_2\\t\end{bmatrix},\begin{bmatrix}\cos 2\pi t & \sin 2\pi t & 0\\ -\sin 2\pi t & \cos 2\pi t & 0\\ 0 & 0 & 1\end{bmatrix}\right)$$

The image of $\mathit{\Pi}$ in $E(3)$ is the orientable Bieberbach group of dimension 3 with holonomy group $\mathbb{Z}_2$. Clearly the manifold $\mathit{\Pi}\backslash\mathbb{R}^2\rtimes\mathbb{R}$ is diffeomorphic to the flat manifold $\mathit{\Pi}\backslash\mathbb{R}^3$. On $\mathit{\Pi}\backslash\mathbb{R}^3$, there is a maximal torus action by S^1, generated by the left translation by $\mathbb{R}=\{[0\ 0\ s]^t : s\in\mathbb{R}\}$. Note that this subgroup $\mathbb{R}$ of $S(\Gamma)$ is not in the image of S. This means that there is no S^1-action on $\mathit{\Pi}\backslash S$ coming from the left translation. In fact, it comes from the right translation by the $\mathbb{R}$-factor of $S=\mathbb{R}^2\rtimes\mathbb{R}$.

If we consider just the subgroup Γ, it is even clearer what the theorem says. The solvmanifold $\Gamma\backslash\mathbb{R}^2\rtimes\mathbb{R}$ is diffeomorphic to the torus $\Gamma\backslash\mathbb{R}^3$. On the latter torus, there is a standard T^3-action as translations. However, no vector subgroup in S descends to give a maximal torus action on $\Gamma\backslash\mathbb{R}^2\rtimes\mathbb{R}$.

We now turn to general Lie groups. Little is known for the existence of a maximal torus action on general double coset spaces. Under some strong conditions, we can show a double coset space of a Lie group which is aspherical admits a maximal torus action.

2.6.25 THEOREM. *Let G be a connected simply connected Lie group, R its radical. Suppose $S = G/R$ does not contain any normal compact factor. Let K be a maximal compact subgroup of G and Γ a torsion free cocompact lattice in G such that $(\Gamma\cap R, R)$ has the unique automorphism extension property. If* $\exp : \mathcal{R} \to R$ *is surjective, then the double coset space $\Gamma\backslash G/K$ admits a smooth maximal torus action.*

PROOF. Let $G = R\rtimes S$ be the Levi-decomposition of G. Let $A = \{a \in R | (a, u) \in \mathcal{Z}(\Gamma)$ for some $u \in S\}$. Let $(a, u) \in \mathcal{Z}(\Gamma)$. Then for any $(z, 1) \in \Gamma_R = \Gamma \cap R$, $(z, 1)(a, u) = (a, u)(z, 1)$. This implies that ${}^{u}z = a^{-1}za$. Since $(\Gamma\cap R, R)$ has the unique automorphism extension property, the two automorphisms u and $\mu(a^{-1})$ induce the same automorphisms on R. Therefore, ${}^{u}x = a^{-1}xa$ for all $x \in R$. Moreover, for any $(b, v) \in \Gamma$, we have ${}^{v}a = a$. Now it is easy to see that A is a commutative subgroup of R.

Choose generators $(a_i, u_i), i = 1, 2, \cdots, k$ for $\mathcal{Z}(\Gamma)$. We define a homomorphism $\phi_R : \mathbb{R}^k \to R$ as follows: Since $\exp : \mathcal{R} \to R$ is onto, log is defined on R. Let $A_i = \log a_i$. Then ϕ_R is the composite $\mathbb{R}^k \to \mathcal{R} \xrightarrow{\exp} R$, where the first map is the linear transformation from $\mathbb{R}^k$ to $\mathcal{R}$ sending the standard basis to $A_1, A_2, \cdots, A_k$. Since $[A_i, A_j] = 0$, the image of $\mathbb{R}^k$ in $\mathcal{R}$ is a commutative Lie subalgebra, and hence the exponential map restricted to this subalgebra is a homomorphism. Consequently, ϕ_R is a homomorphism.

Next, we define $\phi_S : \mathbb{R}^k \to S$ as follows: Let $S = S_1 \times S_2 \times \cdots \times S_r$, where each S_i is a simple group. For each i, let S_i^* denote the adjoint form of S_i, and choose a maximal compact subgroup of S_i^*. This maximal compact subgroup is of the form either $S^1 \times H$ or H, where H does not have a circle factor, depending on whether S_i has infinite center or not. This determines a subgroup $\mathbb{R}^{\epsilon_i} \times \widetilde{H}_i \subset S_i$, where $\widetilde{H}_i$ is compact, and $\epsilon_i = 1$ or 0, depending on whether S_i has infinite center or not. In the former case, $\mathbb{R}$ contains the infinite summand of the center of S_i. Then $K = \Pi\widetilde{H}_i$ is a maximal compact subgroup of S.

Consider the map $\mathcal{Z}(\Gamma) \to \Pi(\mathbb{R}^{\epsilon_i} \times \widetilde{H}_i) \to \Pi\mathbb{R}^{\epsilon_i} \subset \Pi S_i$, where $\Pi(\mathbb{R}^{\epsilon_i} \times \widetilde{H}_i) \to \Pi\mathbb{R}^{\epsilon_i}$ is a projection. We extend this to a homomorphism $\phi_S : \mathbb{R}^k \to \Pi\mathbb{R}^{\epsilon_i} \subset S$. Note that $\phi_S(\mathcal{Z}(\Gamma))$ differs from $p(\mathcal{Z}(\Gamma))$ by elements in $\Pi\widetilde{H}_i \subset K$.

Note that K commutes with $\mathbb{R}^{\epsilon_1} \times \mathbb{R}^{\epsilon_2} \times \cdots \times \mathbb{R}^{\epsilon_r}$. Thus we have an induced action of $\mathbb{R}^k$ on G/K. The action of $\mathbb{R}^k$ on G/K will not be effective in general, because $\mathcal{Z}(\Gamma) \to \Pi\mathbb{R}^{\epsilon_i} \subset S$ may have a non-trivial kernel. Even though the actions by $\mathbb{Z}^k \subset \mathbb{R}^k$ and by $\mathcal{Z}(\Gamma)$ are different on S, they induce the same one over S/K.

A desired action of $\mathbb{R}^k$ on $G/K = R \cdot S/K$ is then given by

$$\phi(t)(x, w) = (x \cdot \phi_R(t), w \cdot \phi_S(t)).$$

Since Γ acts on G as left multiplications, it commutes with the $\mathbb{R}^k$ action defined above. Moreover, we have $\mathbb{R}^k \cap \Gamma = \mathcal{Z}(\Gamma)$ on G/K. Consequently, we have obtained a smooth action of $T^k = \mathcal{Z}(\Gamma)\backslash\mathbb{R}^k$ on $\Gamma\backslash G/K$. □

2.6.26 Theorem ([**?**]). *Let G be a connected, simply connected Lie group without any normal compact factors in its semi-simple part. Let Γ be a torsion free cocompact lattice and K be a maximal compact subgroup of G. Then there is a smooth manifold M, which is homotopy equivalent to the double coset space $\Gamma\backslash G/K$, admitting a smooth maximal torus action.*

Proof. We may assume that $\Gamma = \pi_1(\Gamma\backslash G/K)$. Let R be the radical of G. Then $G = R \rtimes S$. Let $p : G \to S$ be the projection; and let $\mathcal{Z}(\Gamma)$ denote the center of Γ. Let $\widetilde{\Gamma} = \Gamma_R \cdot \mathcal{Z}(\Gamma)$, where $\Gamma_R = \Gamma \cap R$. It is poly {cyclic or finite} since $1 \to \Gamma_R \to \Gamma_R \cdot \mathcal{Z}(\Gamma) \to p(\mathcal{Z}(\Gamma)) \to 1$ is exact, Γ_R is a lattice of R and $p(\mathcal{Z}(\Gamma))$ is a finitely generated abelian group. Such a group $\widetilde{\Gamma}$ contains a characteristic subgroup Γ' of finite index which is a Mostow-Wang group (see Definition **??**), with ${}^n\Gamma' = {}^n\widetilde{\Gamma}$, where ${}^n()$ denotes the discrete nilradical of (). Now $\widetilde{\Gamma}$ contains a characteristic subgroup $\widehat{\Gamma}$ of finite index which is predivisible and ${}^n\widehat{\Gamma} = {}^n\Gamma'$. This implies that $\widehat{\Gamma}/{}^n\widehat{\Gamma}$ is free abelian, say $\mathbb{Z}^m$.

Let $Q = \Gamma/\widehat{\Gamma}$ and $S^* = S/p(\mathcal{Z}(\Gamma))$. Note that S^* is not necessarily the adjoint form of S. Let K^* be a maximal compact subgroup of S^*. Note that K^* is a finite quotient of $T \times K$, where T is a torus generated by free abelian factors of $p(\mathcal{Z}(\Gamma))$. Now $\Gamma/\widetilde{\Gamma} = \Gamma/\Gamma_R \cdot \mathcal{Z}(\Gamma)$ acts on S^*/K^* with compact quotient. Therefore $Q = \Gamma/\widehat{\Gamma}$ acts on S^*/K^* with compact quotient via the homomorphism $\Gamma/\widehat{\Gamma} \to \Gamma/\widetilde{\Gamma}$. Let us denote ${}^n\widehat{\Gamma}$ by Δ. Since ${}^n\widehat{\Gamma}$ is characteristic in $\widehat{\Gamma}$, it is normal in Γ. Consider the exact sequences $1 \to \Delta \to \Gamma \to \Gamma/\Delta \to 1$ and $1 \to \mathbb{Z}^m \to \Gamma/\Delta \to Q \to 1$. We get the latter exact sequence from the fact that $\widehat{\Gamma}$ is predivisible. We do the Seifert space construction with the latter exact sequence and the action of Q on the space S^*/K^* to obtain an action of Γ/Δ on $\mathbb{R}^m \times S^*/K^*$. Now we do a Seifert space construction with the first exact sequence and the action of Γ/Δ on $\mathbb{R}^m \times S^*/K^*$. Consequently we obtain an action of Γ on $\Delta_{\mathbb{R}} \times \mathbb{R}^m \times S^*/K^*$. Let $\mathbb{Z}^k$ be the center of Γ. It lies in the center of Δ. Since the center of Δ lies in the center of $\Delta_{\mathbb{R}}$, there is a unique subgroup $\mathbb{R}^k$ in the center of $\Delta_{\mathbb{R}}$ containing $\mathbb{Z}^k$ as a uniform lattice. The action of $\mathbb{R}^k$ on $\Delta_{\mathbb{R}} \times \mathbb{R}^m \times S^*/K^*$, by left multiplication on the first factor, commutes with the action of Γ. Therefore, it induces an action of torus $\mathbb{R}^k/\mathbb{Z}^k$ on M. Clearly, this is a smooth maximal torus action. □

2.6.27 Remark. In [**?**], Farrell and Jones show that any closed manifold M^n homotopically equivalent to $\Gamma\backslash G/K$, is homeomorphic to it provided G has a faithful representation into $\mathrm{GL}(m, \mathbb{R})$ for some m, $n \neq 3, 4$ Thus when G ,in the theorem above, has a faithful linear representation then $\Gamma\backslash G/K$ has a maximal torus action. Of course there are simply connected G without faithful representations.

Check $\Gamma \backslash G/K$ section

CHAPTER 3

b-hattori: Hattori

Hattori-Yoshida's theorem for Q connected: 5-17-99, revised 6-1-99, 8-22-2000

Comment from Frank to Kyung

I have repeatedly said that H-Y works for Q locally compact Lie group. For this to truely work, one needs to have

$$H^*(Q; \mathrm{M}(W, \mathbb{R}^k)) = 0$$

in lower dimensions.

Mostow explicitly proves this for Q compact. I can't figure out if it holds for Q locally compact. Graham Segal has a very categorical approach and I can't penetrate enough of it to see if the vanishing result there works for Q locally compact. [Of course it works for Q discrete and we have proofs when W/Q is compact, or paracompact $Q\backslash W$ finite dimensional (and we will give the general case when $Q\backslash W$ is paracompact when we introduce a bit of sheaf theory. Segal uses sheaf theory too)].

So I have restricted the H-Y stuff to Q compact, connected Lie group and W being a connected, locally compact Hausdorff space.

The use of W being locally compact is to guarantee that $\mathrm{M}(W, T^k)$ is an abelian topological subgroup of $\mathrm{TOP}^0_{T^k}(P)$. I guess they want it that way because the vanishing theorem is proved with the C-topology. (It may be also true for W say, paracompact, and using the point-open topology [$\mathrm{TOP}(W)$ is a topological group in the point-open topology without W being locally compact]. In fact, Mostow's first part is for general topological groups and using point-open topology–it is only in the later sections that he restricts to locally compact and CO-topology).

I can't figure out why they want to use CW-structure of W. It never comes in *explicitly* and may be hidden in the parts that they leave out– such as their claim (without proof) of naturality of the obstruction class– but is seems that naturality of the obstruction class never uses CW-structure– the argument I give does not need CW-structure of W. Also, perhaps they are using it implicitly in $H^2(W; \mathbb{Z}^k) = [W, B_{T^k}]$; i.e., in the classification of principal bundles– which in the 1970's many topologists still didn't understand that the classification works for paracompact spaces – using Cech cohomology.

3.1. Lifting Q actions — Q discrete

3.1.1. Let G be a Lie group, and $P \to W$ be a principal G-bundle. If Q acts on P as a group of principal G-bundle automorphisms, then the projection onto W introduces a Q-action on W. Conversely, we can take a Q-action on W and search for conditions that will allow us to lift Q to a group of bundle automorphisms of

Q-lifting
weak Q-liftings

P. That is, we seek a Q-action on P that commutes with the left G-action and projects onto the given Q-action on W. We shall call such a Q-action a *Q-lifting* of (Q, W).

3.1.2. We have discussed previously this lifting problem when Q is a discrete group acting properly on W and $G = T^k$, a k-torus. In this case, the problem was solved even for more general lifting Q-actions to weak bundle automorphisms. This work was put into a final form in [?] where it was discussed in conjunction with holomorphic Seifert fiberings. The topological Q-lifting problem for general compact groups Q and toral G, has had a distinguished history, Stewart, J.C. Su, Hattori-Yoshida, Gottlieb, Lashof, May and G.B. Segal, among others. P. E. Conner (Lectures on the action of a finite group, SLN 73(1968)) and [?] and [?].♣

References

We shall review the discrete case first so that the analogy between the discrete and connected case becomes more clear.

3.1.3. Let $\psi : Q \to \mathrm{Aut}(\mathbb{Z}^k) = \mathrm{Aut}(T^k) \subset \mathrm{Aut}(\mathbb{R}^k)$, and let Q be discrete and act properly on W. As seen in section $\boxed{\text{XXX}}$,♣ the $E_2^{p,q}$ spectral sequence gave rise to an exact sequence:

section number

$$0 \to H^2_\psi(Q; \mathbb{Z}^k) \xrightarrow{\ i\ } H^2(Q; \mathfrak{Z}^k) \xrightarrow{\ j\ } H^2(W; \mathbb{Z}^k)^Q \xrightarrow{\ \delta\ } H^3_\psi(Q; \mathbb{Z}^k) \to H^3(Q; \mathfrak{Z}^k),$$

if $H^1(W; \mathbb{Z}^k) = 0$. This exact sequence encodes the information on liftings and Seifert fiberings.

coordinate principal bundle?
unclear

$H^2(Q; \mathfrak{Z}^k)$ is naturally isomorphic to $H^1(Q; \mathfrak{T}^k)$, which is the group of equivalence classes of coordinate principal T^k-bundles♣ over W admitting a Q-action normalizing the translational T^k-action according to ψ. ♣ The homomorphism j assigns to a coordinate T^k-bundle with Q-action, the characteristic class of the T^k-bundle over W. The elements not in the kernel of δ are the equivalence classes of the weakly Q-invariant T^k-bundles over W which do not have a lifting of Q to a group of weak bundle equivalences, called *weak Q-liftings*. The kernel of j, which is equal to the image of i, is the Q-actions on $T^k \times W$ up to $T^k \rtimes_\psi Q$ equivalence. An action of Q on $T^k \times W$ lifts to an action of Π on $\mathbb{R}^k \times W$ and normalizes the $\mathbb{R}^k$-action. This group Π is an extension of $\mathbb{Z}^k$ by Q and is represented by its cohomology class in $H^2_\psi(Q; \mathbb{Z}^k)$. [Later we see that this is also the equivalence class of different Q-liftings to $T^k \times W$. It will be isomorphic to $H^1(Q; \mathrm{M}(W, T^k))$.]

It was also shown that $H^2(Q; \mathfrak{Z}^k)$ is naturally isomorphic to $H^2_\psi(W_Q; \mathbb{Z}^k)$ where W_Q is the Borel space $EQ \times_Q W$ of the Q-action on W. The subscript ψ denotes twisted coefficients $\mathbb{Z}^k$ if ψ is not trivial.

If ψ is trivial, $H^2(W_Q; \mathbb{Z}^k)$ is just the characteristic classes representing the equivalence classes of principal T^k-bundles over W_Q. The homomorphism j is $(\pi \circ i)^*$, where $i : W \hookrightarrow E_Q \times W$ is given by $i(w) = (e_0, w)$, for some fixed $e_0 \in EQ$ and $\pi : EQ \times W \to EQ \times_Q W = W_Q$ is the orbit mapping under the diagonal Q-action. Therefore those principal T^k-bundles over W which have a Q-lifting are identical with those bundles over W which are equivalent to pullback from W_Q via $(\pi \circ i)^*$. Note these bundles must be invariant under Q. That is, $q^*(P) = P$, for $q \in Q$. This is the same as saying that $[P] \in H^2(W; \mathbb{Z}^k)^Q$. The non-zero elements of the image of δ can be regarded as the obstruction to a Q-lifting on a Q-invariant T^k-bundle over W.

$“call˙Q(W)$
$“cale˙G(W˙Q)$

3.2. Lifting Q actions — Q connected

3.2.1. It is of interest to obtain results similar to those in section **??** for connected Lie groups instead of discrete groups. We will present the results of Hattori-Yoshida for connected Lie groups Q and principal toral bundles. First, we shall make some general comments on Q-liftings.

Let Q be a Lie group, and $\varphi : Q \times W \to W$ be a *proper* action. Since Q has a topology, we shall assume, for convenience, that W is locally compact, Hausdorff, connected and locally connected. Then $\mathrm{TOP}(W)$ is a topological group under the compact-open topology and $\widetilde{\varphi} : Q \to \mathrm{TOP}(W)$ is a homomorphism of topological groups and $\widetilde{\varphi}(Q)$ is a closed (since the action is proper) subgroup of $\mathrm{TOP}(W)$. The topology of $\widetilde{\varphi}(Q)$ is the Lie group topology of Q/Q_0, where Q_0 is the kernel of $\widetilde{\varphi}$. (See section 1.2.6).$^{\clubsuit}$ We shall repeatedly use the following

Ref number?

3.2.2 Lemma. *Let $f : X \to Y$ and $p : Z \to Y$ be Q-maps between Q-spaces. Then the pullback $f^*(Z)$ has an induced Q-action and the maps in the induced commutative diagram*

$$\begin{array}{ccc} f^*(Z) & \xrightarrow{\widetilde{f}} & Z \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

are Q-maps. Moreover, if f and p are also G-maps with the action of G and Q commuting, then the induced maps are also G-maps and the induced actions commute.

Proof. Recall that

$$f^*(Z) = \{(x, z) \in X \times Z \mid f(x) = p(z)\}.$$

Let $(x, z) \in f^*(Z)$. Define $q \cdot (x, z) = (qx, qz)$ for all $q \in Q$ and $(x, z) \in f^*(Z)$. This gives a well-defined Q-action and makes the induced maps Q-equivariant. The rest of the lemma follows easily. □

3.2.3 Exercise. If $f : P' \to P$ is a G-bundle equivalence and if P' has a Q-lifting (resp., weak Q-lifting), then so does P.

3.2.4. Let Q and G be Lie groups, Q acting on W properly, let $\mathcal{L}_Q(W)^{\clubsuit}$ denote the equivalence classes of principal G-bundles over W which admit Q-liftings. Denote by $\mathcal{E}_G(W_Q)$ be the equivalence classes of principal G-bundles over the Borel space $W_Q = EQ \times_Q W$. Embed W into W_Q via the map $\pi \circ i : W \to W_Q$, where

No G in this notation!

$$\begin{aligned} i :&\quad W \to EQ \times W, \quad i(w) = (e_0, w), \text{ for some fixed } e_0 \in EQ, \\ \pi :&\quad EQ \times W \to EQ \times_Q W, \text{ the natural projection.} \end{aligned}$$

This embeds W as the fiber over the image of e_0 in BQ, the classifying space for the principal Q-bundles, and $W \to W_Q \to BQ$ is a bundle.

$“cale˙G(W;W˙Q)$

Let $\mathcal{E}_G(W;W_Q)$ be the equivalence classes of principal G-bundles over W that have a representative which is the pullback of a principal G-bundle over W_Q. In other words, $\mathcal{E}_G(W;W_Q) = i^* \circ \pi^*(\mathcal{E}_G(W_Q))$.

3.2.5 Proposition. *$\mathcal{L}_Q(W) \subseteq \mathcal{E}_G(W;W_Q)$.*

Proof. Suppose the principal G-bundle P has a Q-lifting over W. Then $\pi_2 : EQ \times W \to W$ is a Q-map, and if we take the trivial G-actions on $EQ \times W$ and W, π_2 is also a G-map and commutes with the Q-actions. The projection $p : P \to W$ is also a G- and Q-map with the G and Q actions commuting. Then by Lemma **??**, $\pi_2^*(P)$, a principal G-bundle over $EQ \times W$, has induced commuting G- and Q-actions. Both actions are free and proper. Form the commuting diagram of orbit mappings

$$\begin{array}{ccc} \pi_2^*(P) & \xrightarrow{Q\backslash} & Q\backslash\pi_2^*(P) \\ {\scriptstyle G\backslash}\big\downarrow & & \big\downarrow{\scriptstyle G\backslash} \\ EQ \times W & \xrightarrow[Q\backslash]{\pi} & EQ \times_Q W \end{array}$$

yielding $Q\backslash\pi_2^*(P)$ a principal G-bundle over W_Q. Clearly, $\pi^*(Q\backslash\pi_2^*(P)) = \pi_2^*(P)$. Now $i^*(\pi_2^*(P)) \cong P$, hence $i^* \circ \pi^*(Q\backslash\pi_2^*(P)) \cong P$. □

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3.2.6. Under what conditions is $\mathcal{L}_Q(W) = \mathcal{E}_G(W;W_Q)$? When $G = T^k$ and Q is compact, this was first shown to hold by Hattori-Yoshida, and generalized earlier results of Stewart and Su. However, Gottlieb gave an example, attributed to Bredon, that equality fails for $G = \mathrm{Sp}(1)$, citeXXX.♣ The following is also observed by Gottlieb.

3.2.7 Proposition. *If Q acts freely on W, then $\mathcal{L}_Q(W) = \mathcal{E}_G(W;W_Q)$.*

Two backward arrows needed.

Consider the diagram of maps♣

$$\begin{array}{ccc} W & \xrightarrow[\pi_2]{i} & EQ \times W \\ {\scriptstyle \overline{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Q\backslash W & \xrightarrow[\overline{\pi}_2]{\overline{i}} & EQ \times_Q W \end{array}$$

We have $\pi \circ i = \overline{i} \circ \overline{\pi}$ and $\overline{\pi}_2 \circ \pi = \overline{\pi} \circ \pi_2$ and all the horizontal maps $i, \pi_2, \overline{i}, \overline{\pi}_2$ are homotopy equivalences. [Since Q acts freely, $\overline{\pi}_2$ is a fiber bundle map with fiber EQ so that $\overline{\pi}_2$ is a homotopy equivalence]. For a map $f : W_Q = EQ \times_Q W \to BG$, define $\overline{f} : Q\backslash W \to BG$ by $\overline{f} = f \circ \overline{i}$. Then we have $\overline{f} \circ \overline{\pi} \simeq f \circ \pi \circ i$. Thus if $P \in i^* \circ \pi^*(\mathcal{E}_G(W_Q)) = \mathcal{E}_G(W;W_Q)$, then $P \cong R = \pi^* \circ \overline{f}^*(\xi)$, with ξ being the universal G-bundle over BG. Since $\overline{f} \circ \overline{\pi}$ is a Q-map, the pullback R of the universal G-bundle has $G \times Q$ action lifting the Q action on W. Since $R \cong P$, we have shown what we wanted to prove. [We can define a Q-lifting to P via $h : R \to P$, the G-bundle equivalence by setting $q \circ h(r) = h(q(r))$. One can also use Exercise **??**.]

3.3. Lifting Q actions — $Q = T^k$

pseudo-lifting

3.3.1 . We will now explain the argument of Hattori-Yoshida in the case when $G = T^k$, and Q and W are connected and Q is compact. Their argument treats Q compact and not necessarily connected, citeXXX. By different methods, Lashof, May and Segal [?]♣ treat the same Q with G compact abelian. ref.

In the discrete case, the proof relied on the cohomology of Q. In the continuous case, the *continuous cohomology* of Q will be used. The cohomology groups are defined analogously as in the discrete case but maps

$$Q^q = Q \times \cdots \times Q \longrightarrow A,$$

where A is a topological abelian Q-module, are taken continuously. Both methods, discrete and continuous, rely on the vanishing of certain cohomology groups when the coefficients are taken in the maps of W into $\mathbb{R}^k$. In both cases, this is a non-trivial fact and, in the continuous case, was first proved by G.D. Mostow citexxx.

Since we are only considering lifting Q actions to bundle automorphisms, we can simplify our universal group $\mathrm{TOP}_{T^k}(P)$ to

$$\mathrm{TOP}^0_{T^k}(P) = \{f \in \mathrm{TOP}_{T^k}(P) : \ f(au) = af(u)\}.$$

Then the sequence

$$0 \to \mathrm{M}(W, T^k) \to \mathrm{TOP}^0_{T^k}(P) \xrightarrow{\rho} \mathrm{TOP}(W)$$

is exact. If $\widetilde{\varphi}(Q) = Q'$ denotes the image of Q in $\mathrm{TOP}(W)$, we may consider the induced exact sequence

$$0 \longrightarrow \mathrm{M}(W, T^k) \longrightarrow \rho^{-1}(\widetilde{\varphi}(Q) \xrightarrow{\rho} \widetilde{\varphi}(Q) \longrightarrow 1.$$

That the image of ρ contains $\widetilde{\varphi}(Q)$ follows from the fact that $\widetilde{\varphi}(Q)$ is in the connected path component of the identity of $\mathrm{TOP}(W)$, see (chapter 4).♣ We want to describe the extension in terms of factor sets. This is, in general, not possible, for to do so we need a *continuous* map $\psi : Q \to \mathrm{TOP}^0_{T^k}(P)$ so that $\rho \circ \psi = \widetilde{\varphi}$. So we search for a condition that guarantees that such a ψ exists (In [?], ψ is called a *pseudo-lifting*). Ref.

3.3.2 Lemma ([?, Lemma 2.3]). *If $P \in \mathcal{E}_{T^k}(W; W_Q)$, then there is a continuous $\psi : Q \to \mathrm{TOP}^0_{T^k}(P)$ with $\widetilde{\varphi} = \rho \circ \psi$.*

Proof. Since the equivalence class of P is an element of $\mathcal{E}_{T^k}(W; W_Q)$, there is a principal T^k-bundle S over W_Q so that $P \cong i^*\pi^*(S)$. In particular, $\pi_2^*(P) \cong \pi^*(S)$ over $EQ \times W$, and we may identify P with $\pi^*S|_{e_0 \times W}$. There is a contraction $r_t : EQ \to EQ$ such that $r_0 = e_0$, and r_1 is the identity. Consequently there is a covering homotopy

$$\overline{r}_t : \pi^*S \longrightarrow \pi^*S$$

of $r_t \times 1 : EQ \times W \to EQ \times W$ so that $\overline{r}_1$ is the identity and $\overline{r}_0$ maps π^*S into $\pi^*S|_{e_0 \times W} = P$.

Consider the composite map

$$Q \times P \ \subset \ Q \times \pi^*S \longrightarrow \pi^*(S) \xrightarrow{\overline{r}_0} P,$$

which covers the $\varphi : Q \times W \to W$ action. (The Q action on $\pi^*(S)$ is induced from the fact that the classifying map of $\pi^*(S)$ is a Q-map and where the action of Q on B_{T^k} is taken as trivial). Thus each $q \in Q$ is an automorphism of P and we get a continuous map $\psi : Q \to \mathrm{TOP}^0_{T^k}(P)$ with $\widetilde{\varphi} = \rho \circ \psi$. □

3.3.3. 08-22-2000 We are now, with the help of Lemma **??**, ready to characterize our group extension

$$0 \longrightarrow \mathrm{M}(W, T^k) \longrightarrow \rho^*(\rho^{-1}(\widetilde{\varphi}(Q))) \longrightarrow Q \longrightarrow 1 \tag{3.3–1}$$

which is the pullback of the group extension

$$0 \longrightarrow \mathrm{M}(W, T^k) \longrightarrow \rho^{-1}(\widetilde{\varphi}(Q)) \xrightarrow{\rho} \widetilde{\varphi}(Q) \longrightarrow 1$$

in terms of a factor set or 2-cocycle. We assume that the Q-action on W has a pseudo-lifting ψ.

S. T. Hu, Cohomology theory in topological groups, Mich. Math. J. We define

$$f(\alpha, \beta) = \psi(\beta)\psi(\alpha)\psi(\alpha\beta)^{-1},$$

as our 2-cocycle. Its value is in $\mathrm{M}(W, T^k)$ with $\mathrm{M}(W, T^k)$ a left Q-module in the usual sense; namely,

$$\alpha \cdot \lambda = \lambda \circ \psi(\alpha)^{-1}$$

for $\lambda \in \mathrm{M}(W, T^k)$ and $\alpha \in Q$. Then our cocycle is in $\mathrm{M}(Q \times Q; \mathrm{M}(W, T^k)) = \mathrm{M}(Q^2; \mathrm{M}(W, T^k)) = C^2(Q; \mathrm{M}(W, T^k))$. Its cohomology class is denoted by $o(P)$ and it is the obstruction for splitting. The class $o(P)$ vanishes if and only if the sequence (**??**) splits; that is, whenever P has a Q-lifting. What will be shown is that

3.3.4 Theorem ([**?**]). *$H^2(Q; \mathrm{M}(W, T^k)) = 0.$*

3.3.5 Corollary. *$\mathcal{L}_Q(W) = \mathcal{E}_{T^k}(W; W_Q)$.*

3.3.6. 08-22-2000 First let us recall some facts about the cohomology of topological groups with coefficients in the abelian topological group, (see **Hu**, **Mostow**).

Let Q and A be tiopological groups with A abelian. Define the Q-module of continuous p-cochains of Q into A by $C^p(Q; A) = \mathrm{M}(Q^p, A)$ for $q > 0$. That is, continuous maps of $Q^p = Q \times Q \times \cdots \times Q$, the cartesian product of p copies of Q, into the abelian topological group A. These maps forms a group where addition of maps is given by addition of functional values. We assume that Q acts on A as a group of transformations and so this extends to an action on $C^p(Q; A)$. We define $C^0(Q; A) = A$ For each $p \geq 0$, we define the inhomogeneous coboundary operation

$$\delta^p : C^p(Q; A) \longrightarrow C^{p+1}(Q; A)$$

which will satisfy $\delta^{p+1}\delta^p = 0$. For $p = 0$, $\delta^0 : C^0(Q; A) = A \longrightarrow C^1(Q; A) = \mathrm{M}(Q, A)$ is given by

$$\delta^0 a(x) = x \cdot a - a$$

for $x \in Q$, $a \in A$. For $p > 0$,

$$\begin{aligned} \delta^p f(x_1, \cdots, x_{p+1}) = \quad & x_1 \cdot f(x_2, \cdots, x_{p+1}) + \cdots \\ & + \textstyle\sum_{i=0}^{i=p} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_{p+1}) \\ & + (-1)^{p+1} f(x_1, \cdots, x_p). \end{aligned}$$

Define $H^p(Q; A) = \ker(\delta^p)/\text{image}(\delta^{p-1})$. Then H^p has the obvious functorial properties. Clearly, if Q is discrete, this continuous cohomology is the same as the ordinary cohomology of the discrete group Q with coefficients in the group A because every map from Q is continuous.

Our definition of coboundary differes slightly from that of Hattori-Yoshida because we require Q to act on $\mathrm{M}(Q, A)$ as a left action instead of their right action. This is consitent with our preceeding chapters.

If $0 \to A' \xrightarrow{i} A \xrightarrow{i} A'' \to 0$ is an exact sequence of abelian topological Q-modules, then

$$0 \longrightarrow C^p(Q; A') \xrightarrow{i_*} C^p(Q; A) \xrightarrow{j_*} C^p(Q; A'')$$

is exact but, unfortunately, the last homomorphism is not necessarily onto. owever, it A'' admits a continuous cross-section in A, then $j_* : C^p(Q; A) \to C^p(Q; A'')$ is surjective as a cochain map. Consequently, under this restriction, we get a long exact sequence of cohomology. This differs from the discrete case since there a cross-section always exists because all maps from a discrete Q are continuous. This point is serious one and the general lack of exactness makse calculations of continuous cohomology difficult.

If W is lcoally compact Hausdorff space, $\mathrm{M}(W, \mathbb{R}^k)$ and $\mathrm{M}(W, T^k)$ are abelian topological groups. We will eventually need the vanishing theorem of Mostow:

3.3.7 Proposition (Mostow). *If Q is a compact Lie group, and W is a locally compact Hausdorff space, $H^p(Q; \mathrm{M}(W, \mathbb{R}^k)) = 0$ for all $p \geq 1$.*

3.3.8(Proof of the Theorem **??**). From the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{M}(W, \mathbb{Z}^k) \longrightarrow \mathrm{M}(W, \mathbb{R}^k) \xrightarrow{\exp} \mathrm{M}(W, T^k) \longrightarrow H^1(W; \mathbb{Z}^k) \longrightarrow 0,$$

we obtain the following exact sequences

$$0 \longrightarrow \mathrm{M}(W, \mathbb{Z}^k) \longrightarrow \mathrm{M}(W, \mathbb{R}^k) \xrightarrow{\exp} \mathrm{M}_0(W, T^k) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{M}_0(W, T^k) \longrightarrow \mathrm{M}(W, T^k) \longrightarrow H^1(W; \mathbb{Z}^k) \longrightarrow 0,$$

where $\mathrm{M}_0(W, T^k)$ is the subgroup of maps of W into T^k which are homotopic to a constant. Since W is connected, $\mathrm{M}(W, \mathbb{Z}^k) \cong \mathbb{Z}^k$ and $\mathrm{M}(W, \mathbb{R}^k)$ is the universal covering of $\mathrm{M}_0(W, T^k)$. Also $\mathrm{M}(W, \mathbb{R}^k)$ is contractible and $\mathrm{M}_0(W, T^k)$ is a $K(\mathbb{Z}^k, 1)$. These are all Q-modules and so we have resulting cochain complexes:

$$0 \longrightarrow C^*(Q; \mathrm{M}_0(W, T^k)) \longrightarrow C^*(Q; \mathrm{M}(W, T^k)) \longrightarrow C^*(Q; H^1(W; \mathbb{Z}^k)).$$

In the last term, $H^1(W; \mathbb{Z}^k)$ is discrete and therefore the last homomorphism is onto. In particular, we have

$$H^p(Q; \mathrm{M}_0(W, T^k)) \cong H^p(Q; \mathrm{M}(W, T^k)), \quad \text{for } q \geq 2$$

[we are using that $H^p(Q;\text{discrete}) = 0$ for $q \geq 1$, when Q is connected] and

$$\begin{aligned}0 \longrightarrow \mathrm{M}_0(W,T^k)^Q \longrightarrow \mathrm{M}(W,T^k)^Q \longrightarrow H^1(W;\mathbb{Z}^k)\\ \longrightarrow H^1(Q;\mathrm{M}_0(W,T^k)) \longrightarrow H^1(Q;\mathrm{M}(W,T^k)) \longrightarrow 0\end{aligned} \tag{3.3–2}$$

is exact.

It remains to show $H^2(Q;\mathrm{M}_0(W,T^k)) = 0$. The cochain complex

$$0 \longrightarrow C^*(Q;\mathbb{Z}^k) \longrightarrow C^*(Q;\mathrm{M}(W,\mathbb{R}^k)) \longrightarrow C^*(Q;\mathrm{M}_0(W,T^k))$$

is exact but the last homomorphism is not onto. If

$$f \in C^p(Q;\mathrm{M}_0(W,T^k)) \overset{\text{def.}}{=} M(Q^p,\mathrm{M}_0(W,T^k)),$$

the space of all continuous maps of Q^p into $\mathrm{M}_0(W,T^k)$, then f is the image of an element of $C(Q^p;\mathrm{M}(W,\mathbb{R}^k))$ if and only if $f : Q^p \to \mathrm{M}_0(W,T^k)$ is trivial on π_1. [$\mathrm{M}(W,\mathbb{R}^k)$ is the contractible universal covering of $\mathrm{M}_0(W,T^k)$, and $\mathrm{M}_0(W,T^k)$ is a $K(\mathbb{Z}^k,1)$.] Therefore, we can assign to f the homomorphism $f_* : \pi_1(Q^p) \to \mathbb{Z}^k$. We need not concern ourselves with base points as all our groups are abelian and Q^p is simple. Thus,

$$M(Q^p,\mathrm{M}_0(W,T^k)) \longrightarrow \mathrm{Hom}(\pi_1(G^p),\mathbb{Z}^k)$$

is onto. This particular map is a cochain mapping:

Recall

$$\partial_j : Q^{p+1} \longrightarrow Q^p, \quad j = 0,1,\cdots,p+1$$

is defined by

$$\begin{aligned}\partial_0(u_1,\cdots,u_{p+1}) &= (u_2,u_3,\cdots,u_{p+1})\\ \partial_i(u_1,\cdots,u_{p+1}) &= (u_1,\cdots,u_i\cdot u_{i+1},\cdots,u_{p+1}),\ 1 \leq i \leq p\\ \partial_{p+1}(u_1,\cdots,u_{p+1}) &= (u_1,u_2,\cdots,u_p).\end{aligned}$$

Induced is $\partial_{i*} : \pi_1(G^{p+1}) \to \pi_1(G^p)$. The coboundary map is given by

$$\delta = \Sigma_{i=0}^{p+1}(-1)^i\delta_i$$

where δ_i is the transpose of ∂_i, $0 < i \leq p+1$ and where δ_0' is the transpose of ∂_0 together with the composition of the operation by the element of Q in the first factor of Q^{p+1}. This operation becomes trivial on the homotopy level and so we have $(\delta f)^* = \delta f^*$. Thus we get the following exact sequence of cochain complexes

$$0 \longrightarrow C^*(Q;\mathbb{Z}^k) \longrightarrow C^*(Q;\mathrm{M}(W,\mathbb{R}^k)) \longrightarrow C^*(Q;\mathrm{M}_0(W,T^k)) \longrightarrow \mathrm{Hom}^*(\pi_1(Q),\mathbb{Z}^k) \longrightarrow 0,$$

where the p-cochains of $\mathrm{Hom}^*(\pi_1(Q),\mathbb{Z}^k)$ are given by $\mathrm{Hom}^p(\pi_1(Q),\mathbb{Z}^k) \overset{\text{defn}}{=} \mathrm{Hom}(\pi_1(Q^p),\mathbb{Z}^k)$. We define $\mathrm{Hom}^0(\pi_1(Q),\mathbb{Z}^k) = 0$. Then,

$$\begin{aligned}C^0(Q;\mathrm{M}_0(W,T^k)) = \mathrm{M}_0(W,T^k) \quad &\xrightarrow{\delta^0} C^1(Q;\mathrm{M}_0(W,T^k))\\ &\longrightarrow \mathrm{Hom}^1(\pi_1(Q),\mathbb{Z}^k) = \mathrm{Hom}(\pi_1(Q),\mathbb{Z}^k)\end{aligned}$$

is the trivial homomorphism ensuring that $C^*(Q;\mathrm{M}_0(W,T^k)) \longrightarrow \mathrm{Hom}^*(\pi_1(Q),\mathbb{Z}^k)$ is a cochain mapping at the 0 and 1-level. HY does not have H^0 and H^1.

Let K^* denote the image of $C^*(Q; \mathrm{M}(W, \mathbb{R}^k)) \stackrel{\exp_*}{\longrightarrow} C^*(Q; \mathrm{M}_0(W, T^k))$. Then we have two short exact sequences

$$0 \longrightarrow C^*(Q; \mathbb{Z}^k) \longrightarrow C^*(Q; \mathrm{M}(W, \mathbb{R}^k)) \longrightarrow K^* \longrightarrow 0$$

$$0 \longrightarrow K^* \longrightarrow C^*(Q; \mathrm{M}_0(W, T^k)) \longrightarrow \mathrm{Hom}^*(\pi_1(Q), \mathbb{Z}^k) \longrightarrow 0.$$

Passing the first exact sequence to cohomology, we get the long exact sequence

$$\longrightarrow H^p(Q; \mathbb{Z}^k) \longrightarrow H^p(Q; \mathrm{M}(W, \mathbb{R}^k)) \longrightarrow H^p(Q; K^*) \longrightarrow H^{p+1}(Q; \mathbb{Z}^k) \longrightarrow,$$

Now $H^p(Q; \mathbb{Z}^k) = 0$ for $p > 0$ since Q is connected, so $H^p(Q; \mathrm{M}(W, \mathbb{R}^k)) \cong H^p(Q; K^*)$, for $p \geq 1$. But $H^p(Q; \mathrm{M}(W, \mathbb{R}^k)) = 0$ for all $p \geq 1$ [Mostow]♣ so that $H^p(Q; K^*) = 0$ for all $p \geq 1$. ref

Passing the second exact sequence to cohomology, we get the long exact sequence

$$\longrightarrow H^p(Q; K^*) \longrightarrow H^p(Q; \mathrm{M}_0(W, T^k)) \longrightarrow H^p(\mathrm{Hom}^*(\pi_1(Q), \mathbb{Z}^k)) \longrightarrow H^{p+1}(Q; K^*) \longrightarrow .$$

Since $H^p(Q; K^*) = 0$ for all $p \geq 1$,

$$H^p(Q; \mathrm{M}_0(W, T^k)) \cong H^p(\mathrm{Hom}^*(\pi_1(Q), \mathbb{Z}^k))$$

for $p \geq 1$.

We now calculate $H^p(Q; \mathrm{M}_0(W, T^k))$ for $p \leq 2$. This reduces to computing $H^p(\mathrm{Hom}^*(\pi_1(Q), \mathbb{Z}^k))$ for $p = 1$ and 2. Now $\mathrm{Hom}^1(\pi_1(Q), \mathbb{Z}^k) = \mathrm{Hom}(\pi_1(Q), \mathbb{Z}^k)$.

If f is a 1-cochain, then

$$\begin{aligned} \delta f(\alpha, \beta) &= f\partial_{0*}(\alpha, \beta) - f\partial_{1*}(\alpha, \beta) + f\partial_{2*}(\alpha, \beta) \\ &= f(\beta) - f(\alpha + \beta) + f(\alpha) \end{aligned}$$

since $\partial_1(\alpha, \beta) = \alpha\beta$ induces the addition in $\pi_1(Q)$. However, $f \in \mathrm{Hom}(\pi_1(Q), \mathbb{Z}^k)$, hence $\delta f(\alpha, \beta) = 0$. Since $\mathrm{Hom}^0(\pi_1(Q), \mathbb{Z}^k) = 0$,

$$H^1(\mathrm{Hom}^*(\pi_1(Q), \mathbb{Z}^k)) = \mathrm{Hom}(\pi_1(Q), \mathbb{Z}^k).$$

Similarly, for $f \in \mathrm{Hom}^2(\pi_1(Q), \mathbb{Z}^k) = \mathrm{Hom}(\pi_1(Q^2), \mathbb{Z}^k)$, we have

$$\begin{aligned} \delta f(\alpha, \beta, \gamma) &= f(\beta, \gamma) - f(\alpha + \beta, \gamma) + f(\alpha, \beta + \gamma) - f(\alpha, \beta) \\ &= f(\beta, \gamma) - f(\beta, \gamma) - f(\alpha, 0) + f(\alpha, \beta) + f(0, \gamma) - f(\alpha, \beta) \\ &= f(0, \gamma) - f(\alpha, 0) \\ &= f(-\alpha, \gamma). \end{aligned}$$

Thus if $\delta f = 0$, then $f = 0$. Therefore,

$$H^2(\mathrm{Hom}^*(\pi_1(Q), \mathbb{Z}^k) = 0.$$

This completes the proof of the theorem. The corollary also follows since any element of $\mathcal{E}_{T^k}(W; W_Q)$ has a pseudo Q-lifting and its obstruction $o(P)$ will now vanish.

3.3.9. Because $\mathcal{L}_Q(W) = i^* \circ \pi^* \mathcal{E}_G(W_Q)$, the characteristic class of P, $c_1(P)$, lies in the image of $i^* \circ \pi^* : H^2(W_Q; \mathbb{Z}^k) \to H^2(W; \mathbb{Z}^k)$, if and only if, P has a Q-lifting. Therefore we may take the spectral sequence of the fibering $W_Q \to BQ$, where $E_2^{p,q} = H^p(BQ; H^p(W; \mathbb{Z}^k))$. Consequently,

3.3.10 Proposition. *When $H^1(W; \mathbb{Z}^k) = 0$, we get the complete analogue of the exact sequence of section* **??***; namely,*

$$0 \to H^2(BQ; \mathbb{Z}^k) \xrightarrow{\ell} H^2(W_Q; \mathbb{Z}^k) \xrightarrow{j} H^2(W; \mathbb{Z}^k)^Q \longrightarrow H^3(BQ; \mathbb{Z}^k) \to H^3(W_Q; \mathbb{Z}^k).$$

Notice that $H^2(W; \mathbb{Z}^k)^Q = H^2(W; \mathbb{Z}^k)$ as Q is connected. We may interpret each of these terms analogous to those in section **??**.

Proof. Using Corollary **??**, $\mathcal{L}_Q(W) = \mathcal{E}_{T^k}(W; W_Q)$, we consider the fibering $EQ \times_Q W = W_Q \to BQ$, where W is the fiber and structure group is Q. Since Q is connected, the coefficient system in the $E_2^{p,q} = H^p(BQ; H^p(W; \mathbb{Z}^k))$ is simple (i.e., not twisted) and the sequence converges to $H^*(W_Q; \mathbb{Z}^k)$. In dimension 2, we have $H^2(W_Q; \mathbb{Z}^k) = [W_Q, B_{T^k}]$ and the edge homomorphism $H^2(W_Q; \mathbb{Z}^k) \xrightarrow{j} H^2(W; \mathbb{Z}^k)$ coincides with $i^* \circ \pi^* : H^2(W_Q; \mathbb{Z}^k) \to H^2(W; \mathbb{Z}^k)$. We obtain the terms of low degree for this spectral sequence

$$0 \to H^1(BQ; \mathbb{Z}^k) \longrightarrow H^1(W_Q; \mathbb{Z}^k) \longrightarrow H^1(W; \mathbb{Z}^k) \longrightarrow H^2(BQ; \mathbb{Z}^k) \to H^2(W_Q; \mathbb{Z}^k).$$

If $H^1(W; \mathbb{Z}^k) = 0$, the exact sequence continues with

$$0 \to H^2(BQ; \mathbb{Z}^k) \longrightarrow H^2(W_Q; \mathbb{Z}^k) \longrightarrow H^2(W; \mathbb{Z}^k) \longrightarrow H^3(BQ; \mathbb{Z}^k) \to H^3(W_Q; \mathbb{Z}^k).$$

Therefore the elements $c_1(P) \in H^2(W; \mathbb{Z}^k)$ which are the characteristic classes of the bundles in $\mathcal{L}_Q(W)$ are precisely the image of j. For each such P, the group $H^2(BQ; \mathbb{Z}^k)$ classifies the distinct Q-liftings to P. In particular, it classifies the Q-liftings for the product bundle. □

If $H^1(W; \mathbb{Z}^k) \neq 0$, then using the spectral sequence again, we may characterize the image of j (and hence the elements of $\mathcal{L}_Q(W)$) as those elements $c \in H^2(W; \mathbb{Z}^k) = E_2^{0,2}$ such that $d^2(c) = 0$ and $d^3(c) = 0$. [Here, $d^3 : E_3^{0,2}(=$ kernel $d^2) \longrightarrow E_3^{3,0} = H^3(BQ; \mathbb{Z}^k)$ since BQ is simply connected.]

Also, note that $H^2(BQ; \mathbb{Z}^k)$ is naturally isomorphic to $\mathrm{Hom}(\pi_1(Q), \mathbb{Z}^k)$. For $H^2(BQ; \mathbb{Z}^k) \cong \mathrm{Hom}(H_2(BQ), \mathbb{Z}^k)$ since BQ is simply connected and $H_2(BQ) \cong \pi_2(BQ) \cong \pi_1(Q)$.

3.3.11. We have the following corollaries.

Corollary 1(cf [**?**, **?**]). *If Q is a simply connected, compact Lie group acting properly on W, then any principal T^k-bundle over W has a Q-lifting. Furthermore, this lifting is unique up to conjugation by elements of $\mathrm{M}(W, T^k)$.*

PROOF. The map $H^2(W_Q;\mathbb{Z}^k) \xrightarrow{j} H_2(W;\mathbb{Z}^k)$ arises from the edge homomorphism in the spectral sequence associated with the fibering $W_Q \to BQ$. But as $\pi_1(Q) = \pi_2(Q) = 0$, BQ is 3-connected and so $H^p(BQ, H^q(W;\mathbb{Z}^k)) = E_2^{p,q}$ is 0 for $p = 1, 2, 3$. This implies that $H^1(W;\mathbb{Z}^k) \cong H^1(W_Q;\mathbb{Z}^k)$ and $H^2(W_Q;\mathbb{Z}^k) \xrightarrow{j} H^2(W;\mathbb{Z}^k)$ is an isomorphism and so each principal T^k-bundle has a Q-lifting

Just as in the discrete case, the Q-liftings are in $1-1$ correspondence with the group $H^1(Q, \mathrm{M}(W, T^k))$ up to conjugation by elements of $\mathrm{M}(W, T^k)$). From the proof of the theorem, we found that

$$H^1(Q; \mathrm{M}_0(W, T^k)) \cong \mathrm{Hom}(\pi_1(Q), \mathbb{Z}^k) = 0$$

and $H^1(Q; \mathrm{M}_0(W, T^k))$ mapped onto $H^1(Q; \mathrm{M}(W, T^k))$. □

Corollary 2 (cf [**?**, **?**]). *If $H^1(W;\mathbb{Z}^k) = 0$ and T^n acts on W, then every principal T^k-bundle over W has a T^n lifting.*

PROOF. From the exact sequence arising from the spectral sequence, we have $H^3(BT^n, \mathbb{Z}^k) = 0$ and so j is onto. □

If Q is a compact connected Lie group, then there exists a finite central covering $\widetilde{Q}$ of Q with

$$\widetilde{Q} = T^n \times \widetilde{Q}_1 \times \cdots \times \widetilde{Q}_m$$

for some $m \geq 0$, $n \geq 0$, and where each $\widetilde{Q}_i$ is a simply connected simple Lie group. If Q acts on W, then the homomorphism $\widetilde{Q} \to Q$ defines an action of $\widetilde{Q}$ on W.

Corollary 3. *If Q acts on W with $H^1(W;\mathbb{Z}^k) = 0$, then the action of $\widetilde{Q}$ is $\widetilde{Q}$-liftable to every principal T^k-bundle over W.*

PROOF. $B\widetilde{Q} = BT^n \times BQ_1 \times \cdots \times BQ_m$. Therefore, $H^3(BQ;\mathbb{Z}^k) = 0$, and consequently, $H^2(W_{\widetilde{Q}};\mathbb{Z}^k) \xrightarrow{j} H^2(W;\mathbb{Z}^k)$ is onto. □

Corollary 4. *If Q is connected and $H^1(W;\mathbb{Z}^k) = 0$, then the Q-liftings, up to conjugation by elements of $\mathrm{M}(W, T^k)$, are in $1-1$ correspondence with $\mathrm{Hom}(\pi_1(Q), \mathbb{Z}^k)$.*

PROOF. By the exact sequence **??** in section **??**, we have $H^1(Q; \mathrm{M}_0(W, T^k))$ is isomorphic to $H^1(Q; \mathrm{M}(W, T^k))$ because $H^1(W;\mathbb{Z}^k) = 0$. Moreover, $H^1(Q; \mathrm{M}_0(W, T^k))$ was seen to be isomorphic to $\mathrm{Hom}(\pi_1(Q), \mathbb{Z}^k)$. □

8-22-00 The Q-lifting theorem **??** is proved in [**?**] for Q a compact Lie group, not necessarily connected. In this case, $H^2(Q; \mathrm{M}(W, T^k))$ is not necessarily 0. So instead of the vanishing of the entire second cohomology group, they show $o(P) = 0$. The reader is referred to their paper for a complete argument. Here, we will adopt their idea to a case they have not treated and which gives a different perspective to the discrete case treated in earlier chapters and subsection **??**.

Let Q be discrete and act properly on W which is paracompact, admitting covering space theory with $Q\backslash W$ also paracompact.

3.3.12 THEOREM. $\mathcal{L}_Q(W) = \mathcal{E}_{T^k}(W; W_Q)$.

PROOF. Since Q is discrete, we do not need to topologize $\mathrm{M}(W, T^k)$ etc. Since maps from Q to $\mathrm{M}(W, T^k)$ will always be continuous whatever the topology on $\mathrm{M}(W, T^k)$. But *if you wish* it may be convenient to think of $\mathrm{M}(W, T^k)$ with the CO-topology. First, we need the image of $\mathrm{TOP}^0_{T^k} \xrightarrow{\ p\ } \mathrm{TOP}(W)$ to include $\widetilde{\varphi}(Q)$. Without this, there is no chance of a Q-lifting. We also need that P is Q-invariant.

It is clear if P has a Q-lifting, then as in Proposition **??**, P is the pullback from a bundle over W_Q. Suppose P is a pullback of a bundle L over W_Q. Let

$$\begin{aligned} i:&\quad W \to e_0 \times W \subset EQ \times W,\\ \pi:&\quad EQ \times W \to EQ \times_Q W, \text{ a } Q\text{-map},\\ \pi_2:&\quad EQ \times W \to W, \text{ projection onto the second factor.}\end{aligned}$$

Let $j = \pi \circ i$. Then we have $j^*(L) = i^* \circ \pi^*(L) \cong P$. Thus, $\pi_2^*(P) = \pi^*(L)$ as bundles over $EQ \times W$. Then $\pi^*(L)$ is a principal T^k-bundle and has a Q-lifting by subsection **??**. Note, P will then be Q-invariant. By the argument in Lemma **??**, we know that $\mathrm{TOP}^0_{T^k}(P) \to \mathrm{TOP}(W)$ has $\widetilde{\varphi}(Q)$ in its image. Therefore, we have a pseudo-lifting for the bundle P, and so the obstruction cocycle is defined. That is, for each $\alpha \in Q$, there is $s(\alpha) \in \mathrm{TOP}_{T^k}(P)$ which covers $\rho(\alpha) = \widetilde{\varphi}(\alpha) \in \mathrm{TOP}(W)$. Now π_2 is a Q-map and $\pi_2^*(P) = \pi^*(L)$. So we may assume $\pi_2^*(P) = \pi^*(L)$. The Q-bundle automorphism of $s(\alpha)$ pulls back a Q-bundle automorphism of P to $\pi_2^*(P)$ as follows. Let $((e, w), u)$ be an element of $\pi_2^*(P)$, where $u \in P$, u projects to w in P, and $e \in EQ$. Then, define the Q-bundle automorphism $\widetilde{s}(\alpha)$ of $\pi_2^*(P)$ by $\widetilde{s}(\alpha)((e, w), u) = (\alpha e, \alpha w), s(\alpha)u)$. This defines the obstruction cocycle $\widetilde{f}(\alpha, \beta) = \widetilde{s}(\alpha)\widetilde{s}(\beta)\widetilde{s}(\alpha\beta)^{-1}$ for splitting over $EQ \times W$. Since the extension splits (i.e., there is a Q-lifting on $\pi^*(L)$), the obstruction cocycle is cohomologous to 0. That is, $o(\pi_2^*(P)) = 0$. From the construction of this cocycle, it is easily seen that $f(\alpha, \beta)$ maps to $\widetilde{f}(\alpha, \beta)$ and $o(\pi_2^*(P)) = \pi_2^*(o(P))$. It remains to show π_2^* is an isomorphism. We have two exact sequences of coefficient modules:

$$0 \longrightarrow \mathrm{M}(W, \mathbb{Z}^k) \longrightarrow \mathrm{M}(W, \mathbb{R}^k) \longrightarrow \mathrm{M}_0(W, T^k) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{M}_0(W, T^k) \longrightarrow \mathrm{M}(W, T^k) \longrightarrow H^1(W, \mathbb{Z}^k) \longrightarrow 0.$$

They give rise to two long exact sequence for the group Q. Now $H^p(Q; \mathrm{M}(W, \mathbb{R}^k)) = 0$, for $p > 0$, see section citeXXX. So $H^p(Q; \mathrm{M}_0(W, T^k)) \cong H^{p+1}(Q; \mathbb{Z}^k)$ for $p \geq 1$. Similarly, replacing W by the homotopy equivalent $EQ \times W$, we have $\pi_2^* : H^p(Q; \mathrm{M}_0(W, T^k)) \to H^p(Q; \mathrm{M}_0(EQ \times W, T^k))$ is an isomorphism as well as $H^p(Q; \mathrm{M}_0(EQ \times W, T^k)) \to H^{p+1}(Q; \mathbb{Z}^k))$. Now apply the 5 lemma to

$$\begin{array}{ccccccc} \to & H^2(Q; \mathrm{M}_0(W, T^k)) & \to & H^2(Q; \mathrm{M}(W, T^k)) & \to & H^1(Q; H^1(W; \mathbb{Z}^k)) & \to \\ & \downarrow \approx & & \downarrow \pi_2^* & & \downarrow \approx & \\ \to & H^2(Q; \mathrm{M}_0(EQ \times W, T^k)) & \to & H^2(Q; \mathrm{M}(EQ \times W, T^k)) & \to & H^1(Q; H^1(EQ \times W; \mathbb{Z}^k)) & \to \end{array}$$

and get π_2^* is an isomorphism. Therefore, $o(P) = 0$ and we have a Q-lifting. □

comfortable now??

3.4

3.4.1. Let's work out the S^1-liftings on S^1 bundles over S^2. There is a unique, up to S^1-equivalence, effective S^1-action on S^2; namely, rotating about the poles N and S. Since S^2 is simply connected, we have the exact sequence in Proposition **??**:

$$0 \to H^2(BS^1;\mathbb{Z}) \xrightarrow{\ell} H^2({S^2}_{S^1};\mathbb{Z}) \xrightarrow{j} H^2(S^2;\mathbb{Z}) \longrightarrow H^3(BS^1;\mathbb{Z}) \to H^3({S^2}_{S^1};\mathbb{Z}).$$

But, $H^2(BS^1;\mathbb{Z}) = \mathbb{Z}$, $H^2(S^2;\mathbb{Z}) = \mathbb{Z}$, $H^3(BS^1;\mathbb{Z}) = 0$, so we get

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^2({S^2}_{S^1};\mathbb{Z}) \xrightarrow{j} \mathbb{Z} \longrightarrow 0$$

So, $H^2({S^2}_{S^1};\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

For S^2, i.e., $c_1(P) = +1$, we can take the following S^1-liftings, a different lifting for each $n \in \mathbb{Z}$:

$$z \times (z_1, z_2) \mapsto (z^n z_1, z^{n-1} z_2).$$

These are S^1-liftings because of the following commutative diagram:

$$\begin{array}{ccccc}
(z_1, z_2) & \xrightarrow{S^1\backslash} & z_1/z_2 & \in S^2 = \mathbb{C}P_1 & z_1\overline{z}_1 + z_2\overline{z}_2 = 1 \\
{\scriptstyle \cdot z}\downarrow & & {\scriptstyle \cdot z}\downarrow & & \\
(z^n z_1, z^{n-1} z_2) & \xrightarrow{S^1\backslash} & z \cdot \left(\frac{z_1}{z_2}\right) & & \text{rotation about the poles of } S^2
\end{array}$$

For $L(m,1)$, i.e., $c_1(P) = m$, we define $\langle z_1, z_2\rangle \in L(m,1)$ by taking the orbit space of the diagonal $\mathbb{Z}_m$-action on S^3 given by

$$e^{2\pi i \frac{k}{m}} \times (z_1, z_2) \mapsto (e^{2\pi i \frac{k}{m}} z_1, e^{2\pi i \frac{k}{m}} z_2), \quad k = 0, 1, \cdots, m-1.$$

Thus we get a different S^1-lifting for each $n \in \mathbb{Z}$:

$$z \times \langle z_1, z_2\rangle \mapsto \langle z^n z_1, z^{n-1} z_2\rangle.$$

3.4.2 Exercise. If we take the ineffective S^1-action on S^2 given by $z \times \frac{z_1}{z_2} \mapsto z^p \frac{z_1}{z_2}$, then determine the S^1-liftings to the Hopf fibering $S^3 \to S^2$ up to equivalence. Which of these S^1-liftings will be effective?

What about the product case, i.e., on $S^1 \times S^2$? We can take

$$\begin{array}{ccc}
e^{2\pi i\theta} \times (z, w) & \longrightarrow & (z, e^{2\pi i\theta} w) \\
\downarrow & & \downarrow \\
(e^{2\pi i\theta}, w) & \longrightarrow & e^{2\pi i\theta} w
\end{array}$$

Then

$$\begin{array}{ccc}
e^{2\pi i\theta} \times (z, w) & \longrightarrow & (e^{2\pi i n\theta} z, e^{2\pi i\theta} w) \\
\downarrow & & \downarrow \\
(e^{2\pi i\theta}, w) & \longrightarrow & e^{2\pi i\theta} w
\end{array}$$

would give us an infinite number of spaces covering the original? These are all inequivalent S^1-liftings.

3.4.3 Exercise. Points of $\mathbb{C}P_n$ are represented by $[z_1 : z_2 : \cdots : z_{n+1}]$, the homogeneous coordinates which is the S^1-orbit of $(z_1, z_2, \cdots, z_{n+1}) \in S^{2n+1}$ under the Hopf map. We can take an S^1-action, say

$$z \times [z_1 : z_2 : z_3 : \cdots : z_{n+1}] \mapsto [z^n z_1 : z^{n-1} z_2 : z_3 : \cdots : z_{n+1}]$$

for definiteness. Describe the S^1-liftings to S^{2n+1}.

3.4.4 Exercise. Let Q be a connected Lie group acting properly on W and P be a principal G-bundle over W with G connected. Let

$$H = \mathrm{Im}(\mathrm{ev}^w_* : \pi_1(Q, 1) \to \pi_1(W, w)),$$

and K be the kernel of $\pi_1(Q, 1) \to H$. Corresponding to K, there is a unique connected covering group Q_K of Q (whose fundamental group is K). Let E be the image of $\mathrm{ev}^u_* : \pi_1(G, 1) \to \pi_1(P, u)$, where $u \mapsto w$ under the bundle projection map. Then the action of G lifts to P', the covering space of P corresponding to the subgroup E. This is a principal G-bundle over $\widetilde{W}$, the universal covering of W. Show that the action of Q is liftable to P over W, if and only if, the action of Q_K is liftable to P' over $\widetilde{W}$.
Hint: Use the following exercise and Lemma **??**.

3.4.5 Exercise (cf. Lemma **??**). Let G and Q be Lie groups and act on a space Z. Suppose there exists a continuous homomorphism $\varphi : Q \to \mathrm{Aut}(G)$ such that

$$h(az) = \varphi(a)h(z), \quad \forall a \in G, h \in Q.$$

(Assume that the image of Q is closed). Form the Lie group $G \rtimes Q$ by defining

$$(a, h)(b, k) = (a \cdot \varphi(h)b, hk).$$

Let $G \rtimes Q$ act on Z by $(a, h) \cdot z = a(h(z))$. Show that this is well defined and encodes the actions of G and Q on Z.

3.4.6 Exercise. Let Q be discrete and act properly on W. Let $\pi_1(W) \to Q' \to Q \to 1$ be the lifting sequence to $\widetilde{W}$, the universal covering of W. Let P be a principal G-bundle over W, G connected and E be the image of $\mathrm{ev}^u_* : \pi_1(G, 1) \to \pi_1(P, u)$. Then G lifts to P', the covering space of P corresponding to the subgroup $E \subset \pi_1(P)$. Show this is a principal G-bundle over $\widetilde{W}$ and that the group Q is liftable to a group of weak bundle automorphism of P over W, compatible with the homomorphism $\varphi : Q \to \mathrm{Aut}(G)$ if and only if Q' is liftable to a group of weak bundle automorphism $Q' \to Q \to \mathrm{Aut}(G)$.

3.4.7 Exercise. Formulate Exercise **??** for $\varphi : Q \to \mathrm{Aut}(G)$ and principal G-bundle over W, where G and Q are connected and in terms of Q-liftings to weak bundle automorphisms compatible with φ. Again use **??**

These 4 exercises, in some sense, reduces the problems of liftable over W to simply connected W's.

Question to answer if we can

3.5

In our stuff, if we take $Q \hookrightarrow \mathrm{TOP}(W)$ and it is a very nice subgroup of $\mathrm{TOP}(W)$, say $\mathrm{Isom}(W)$, then any extension Π that we can stick into $\mathrm{TOP}_G(P)$ should not be very weird. In fact, as we vary a possible embedding (or a homomorphism into $\mathrm{TOP}_G(P)$) keeping $\varphi \times \rho : Q \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$ fixed, we can really only alter by elements of $\ell(G) \times_{\mathbb{Z}G} \mathrm{M}(P, G)$, along the fibers. So what chance do we have in taking some embedding of Π into $\mathrm{TOP}_G(P)$ and conjugating it into a nice subgroup of $\mathrm{TOP}_G(P)$? Of course, we discussed this in the product case several times in the past. But I am wondering specifically if we can say more. Let's say for the case Π is a spherical space-form group.

Recall that if Π is a free irreducible unitary representation, then $\Pi \hookrightarrow U(n+1) \subset O(2n+2)$ and Π acts isometrically on S^{2n+1} and S^{2n+1}/Π is a spherical space form. In this case, $\Pi \cap \mathcal{Z}(U(n+1)) = \Pi \cap U(1) = \mathcal{Z}(\Pi)$, the center of Π, and

$$0 \to \mathcal{Z}(\Pi) \to \Pi \to Q \to 1$$

is exact. Then Q embeds in $\mathrm{PSU}(n+1) = \mathrm{SU}(n+1)/\mathcal{Z}(\mathrm{SU}(n+1))$. Now this group, $\mathrm{PSU}(n+1)$ should be a natural subgroup of $\mathrm{Isom}(\mathbb{C}P_n)$. In fact, either by taking $\mathrm{Isom}(\mathbb{C}P_n)$ ($\mathrm{Isom}(W)$, resp.) and a principal S^1-bundle (principal G-bundle P over W, resp.) over $\mathbb{C}P_n$, we should be able to create a subgroup of $\mathrm{Isom}(S^{2n+1})$ and vice versa. So what I'd like to see is if we start with Q in $\mathrm{PSU}(n+1)$ and take a Π as a central extension

$$0 \to C \to \Pi \to Q \to 1,$$

$C \subset S^1$ ($C \subset G$, resp), and if Π embeds in $\mathrm{TOP}(S^{2n+1})$ ($\mathrm{TOP}_G(P)$, resp) via a Seifert construction, then automatically we can conjugate this embedding into our specially constructed subgroup of $\mathrm{Isom}(S^{2n+1})$.

Do you get my point? Since our construction should be "linear" (or "nice" model manifold, resp), we should be able to modify a construction (i. e., by conjugation along the fibers) so that it is nice; i.e., belongs to a group constructed from the special structure that Q belongs to on W and the group W.

In particular, taking any Q on $\mathbb{C}P_n$, say it belongs to $\mathrm{PSU}(n+1) \subset \mathrm{Isom}(\mathbb{C}P_n)$ and $0 \to C \to \Pi \to Q \to 1$ an extension for which there exists $\Phi : \Pi \to \mathrm{TOP}^0_G(P)$, where P is a principal S^1-bundle over $\mathbb{C}P_n$ (Π not necessarily acting freely) but $C \subset$ diagonal $U(1) \subset U(n+1)$), then can we conjugate this so that Π is conjugated into a special subgroup of $\mathrm{Isom}(P)$? This must be true. How would go about formulating it?

Recall we can almost decide when a certain extension does lift. We have the condition that Q is liftable to P if and only if there exists P' a principal T^k-bundle over W_Q such that, under

$$H^2(W_Q; \mathbb{Z}^k) \xrightarrow{j} H^2(W; \mathbb{Z}^k)^Q$$

$c_1(P) = j(c_1(P))$.

And I am trying to work out exactly from this data, when or how we can decide that there exist $\Phi : \Pi \to \mathrm{TOP}^0_G(P)$.

Even in the product case, we have not gotten a definitive statement. It would be nice to have some stuff worked out. What are your thoughts on this mater?

June 9, 1999, A-I

3.6

We analyze all the $\mathbb{Z}_2$-liftings to principal S^1-bundles over S^2. We could put the first part in the book and state the others as exercises. This also could come earlier when we first discuss the discrete case.

The exact sequence in section **??** is subtle. Let us explain the $E_2^{p,q}$ spectral sequence.

$$E_2^{p,q} = H^p(Q; H^q(W; \mathfrak{Z}^k))$$

where $\mathfrak{Z}^k$ is the constant sheaf $\mathcal{Z}^k \times W$ over W and $\boxed{\mathcal{Z}^k}$ is .... Q acts on this constant sheaf by

$$\alpha \times (\eta, w) = (\alpha \cdot \eta,\ \alpha \cdot w)$$

where $\alpha \in Q$, $\eta \in T^k$, $w \in W$. The action of α on T^k is given by $Q \to \mathrm{Aut}(T^k) = \mathrm{Aut}(\mathbb{Z}^k)$ and that on W by $Q \to \mathrm{TOP}(W)$. So in this way, $H^q(W; \mathfrak{Z}^k) = H^q(W; \mathbb{Z}^k)$ becomes a Q-module. In particular, $H^0(Q; H^2(W; \mathfrak{Z}^k)) = H^2(W; \mathbb{Z}^k)^Q$ and $H^2(Q; H^0(W; \mathfrak{Z}^k)) = H^2(W; \mathbb{Z}^k)$ where $\mathbb{Z}^k$ is a Q-module via $\phi : Q \to \mathrm{Aut}(\mathbb{Z}^k)$. So the sequence in section **??** becomes

$$0 \to H^2(Q; \mathbb{Z}^k) \to H^2(Q; \mathfrak{Z}^k) \to H^2(W; \mathbb{Z}^k) \to H^3_\phi(Q; \mathbb{Z}^k) \to H^3(Q; \mathfrak{Z}^k).$$

Let's examine a couple of cases that we understand pretty well. Let $\mathbb{Z}_2$ act on S^2 and P a principal S^1-bundle over S^2. Let Q be generated by A, the antipodal map on S^2. A is orientation-reversing on S^2.

(i) Suppose A reverses the orientation of S^1, i.e., $\mathbb{Z}_2 \to \mathrm{Aut}(S^1)$ is injective. Then

$$H^2(\mathbb{Z}_2; \mathbb{Z}) = 0,\ H^3(\mathbb{Z}_2; \mathbb{Z}) = \mathbb{Z}_2,\ H^2(S^2; \mathbb{Z})^{\mathbb{Z}_2} = \mathbb{Z}.$$

To calculate $H^2(S^2; \mathbb{Z})^{\mathbb{Z}_2}$, we recall that this comes from $H^0(\mathbb{Z}_2; \mathrm{M}(W, W \times \mathbb{Z}))$, where $W \times \mathbb{Z}$ is a constant sheaf over W. Moreover, $W \times \mathbb{Z} \to W$ is a $\mathbb{Z}_2$-module where A acts on $W = S^2$ and on $\mathrm{Aut}(S^1)$. Since $H^2(S^2; \mathbb{Z})$ is the coefficients group and we want the fixed subgroup under the action of $\mathbb{Z}_2$, we see that

$$H^2(S^2; \mathbb{Z}) \xrightarrow{A_*} H^2(S^2; \mathbb{Z}) \xrightarrow{-\mathrm{id}_{\mathbb{Z}}} H^2(S^2; \mathbb{Z})$$

is the identity automorphism so that $H^2(S^2; \mathbb{Z})^Q \cong \mathbb{Z}$. Our exact sequence then becomes

$$0 \to H^2(\mathbb{Z}_2; \mathfrak{Z}) \xrightarrow{j} H^0(\mathbb{Z}_2; H^2(S^2; \mathbb{Z})) \longrightarrow \mathbb{Z}_2 \longrightarrow H^3(\mathbb{Z}_2; \mathfrak{Z}).$$

♣ Now $H^2(\mathbb{Z}_2; \mathfrak{Z}) \cong \mathbb{Z}$ and either j is an isomorphism or multiplication by 2. We claim that it is multiplication by 2. If not, then there is a $\mathbb{Z}_2$-lifting to S^3 as a group of weak bundle automorphisms. The $\mathbb{Z}_2$ action would have to be acting freely. Then this will be orientation-preserving and would commute with the S^1-action since such an action is known to be equivalent to the linear action $-I_4 \in \mathrm{GL}(4, \mathbb{R})$ on S^3, which contradicts that the involution acts non-trivially on the S^1-fiber. Therefore, the homomorphism j is multiplication by 2. Hence the $\mathbb{Z}_2$-action generated by A is liftable to a group of weak bundle automorphisms on each principal S^1-bundle P over S^2 whose characteristic class is even. Furthermore each such $\mathbb{Z}_2$-lifting is

where is the exact sequence?

unique (up to conjugation by elements of $\mathrm{M}(S^2, S^1)$ because $H^1(\mathbb{Z}_2; \mathrm{M}(S^2, S^1)) \cong H^2(\mathbb{Z}_2; \mathbb{Z}) = 0$.

We can be specific about the lifted $\mathbb{Z}_2$-action which is normalizing the principal S^1-action on P. Since the lifted $\mathbb{Z}_2$-action is free, $\mathbb{Z}_2 \backslash P$ is an orientable 3-manifold which fibers as an S^1-bundle over $\mathbb{R}P_2$. This fibering is *not* a principal S^1-fibering but it is the projection of the induced Seifert fibering $\mathbb{Z}_2 \backslash P \to \mathbb{R}P_2$. Let us summarize by the following

3.6.1. **Observation**. Let $\mathbb{Z}_2$ be generated by the antipodal map on S^2. Then the action is $\mathbb{Z}_2$-liftable to a principal S^1-bundle P over S^2 if and only if the characteristic class, $c_1(P) \in H^2(S^2; \mathbb{Z})$ is even. This lifted action is unique up to equivalence by conjugation by elements of $\mathrm{M}(S^2, S^1)$. Furhtermore, the lifted action is free and P doubly covers a non-principal S^1-bundle over $\mathbb{R}P_2$. This bundle projection is, of course, the resulting Seifert fibering over $\mathbb{R}P_2$. In terms of its Seifert invariants, $\mathbb{Z}_2 \backslash P$ is given by $(O, n, II; 1|\bar{b})$, with $2\bar{b}$ being the Chern class of P. This bundle is isomorphic to $L(4,1)$ if $|\bar{b}| = 1$, and to $\mathbb{R}P_3 \# \mathbb{R}P_3$ if $\bar{b} = 0$, (and hence $P = S^2 \times S^1$). If $|\bar{b}| > 1$, then, it is homeomorphic to some prism manifolds, which is a subclass of the 3-dimensional spherical space forms. Its corresponding description in terms of its Seifert invariants is given by

$$\begin{array}{ll} (O, o; g = 0 \mid -1; (2,1), (2,1), (|\bar{b}|, 1)), & \text{if } \bar{b} > 1 \text{ and} \\ (O, o; g = 0 \mid 1; (2,-1), (2,-1), (|\bar{b}|, -1)), & \text{if } \bar{b} < 0. \end{array}$$

These homeomorphisms can be found in a paper by Threlfall, "Topologische Untersuchung der Diskcontuitäts bereicher Bewegungs-gruppen des dreidimensionalen Spherischen Raumes, II" p. 575, Math Annalen (107) 193–?.♣

Look it up

[Calculations] On page 571, we find that $(O, n; 1|\ \bar{b}; (\alpha_1, \beta_1))$ is homeomorphic to the prism manifold characterized by the pair of integers $[p, q]$. Here $p \geq 1$. If $p = 0$, it is not a prism manifold but corresponds to $\mathbb{R}P_3 \# \mathbb{R}P_3$. Here $q = \alpha_1$ and $p = |\alpha_1 b + \beta_1|$. But we want the action of $\mathbb{Z}_2$ on P_n to be free so therefore α_1 cannot biger than 1, and since normalization means that we can consider only $(1, k)$ otherwise.♣ If we take $(1, k)$, all this does is to rewrite this as $(O, n, 1|\ \bar{b} + k)$ (no α_1). But to follow the computation on page 571, we need to take $\alpha_1 = 1$ and $\beta_1 = 0$. Then $(1, 0)$'s can be ignored in a Seifert presentation.

Badly written.

Then we see that for our case, $[p, q]$ becomes $[|\bar{b}|, 1]$ because $q = \alpha_1 = 1$, $\beta_1 = 0$. Now on p. 575, we have two families $\mathfrak{D}_I$ and $\mathfrak{D}_{II}$. $\mathfrak{D}_{II}$ is ruled out immediately since it only considers $[|\bar{b}|, 1] = [n', \mu]$, where μ is even in their notation. So we need only examine $\mathfrak{D}_I$. In Seifert-Threlfall notation, they require

$$[n', m] \longleftrightarrow (O, o, 0|\ b; (2,1), (2,1), (n', \beta_3)), \quad \text{where } m = \{(b+1)n' + \beta_3\}.$$

Our $m = 1$ so we need to look at $1 = (b+1)n' + \beta_3$. But $n' =$ our $|\bar{b}|$. Now $0 < \beta_3 < \bar{b}$. So we examine $|\bar{b}| = 1$ first. Chose $b = 0$, $\beta_3 = 0$, so we get $(0, (2,1), (2,1), (1,0)) = (0, (2,1), (2,1))$. This is the lens space $L(4,1)$, covered by P_2. This also can be written as $(-1, (2,1), (2,1), (1,1))$.

More generally, fixing $|\bar{b}| = 1$, we get

$$(b+1)(1) + \beta_3 = 1 \text{ or } \beta_3 = 1 - (b+1).$$

Thus we get the prism manifold $[|\overline{b}|, 1] = [1, 1]$ corresponding to $(O, n, II, 1 \mid 1)$ homeomorphic to $(O, o; 0 \mid -1, (2, 1), (2, 1), (1, 1))$, which is the same lens space $L(4, 1)$.

Similarly, fixing $|\overline{b}| > 1$, we get

$$(b + 1)|\overline{b}| + \beta_3 = 1 \text{ or } \beta_3 = 1 - |\overline{b}|(b + 1).$$

Therefore, corresponding to the prism manifold $[|\overline{b}|, 1]$, $|\overline{b}| > 1$, we get the non-principal S^1-bundle $(O, n, II; 1 \mid |\overline{b}|)$ over $\mathbb{R}P_2$ with obstruction class $|\overline{b}|$ and covered by the principal S^1-bundle whose Chern class is $-2|b|$, diffeomorphic to the spherical space form $(O, o, 0| -1; (2, 1), (2, 1), (|\overline{b}|, 1))$. We have not been too careful with orientations having taken always $\overline{b} \geq 0$. If we took the oppsitely oriented bundle $(O, n, II; 1\| -|\overline{b}|)$, then the corresponding spherical space form would also be oppositely oriented and would be given by $(O, o, 0| \ 1; (2, -1), (2, -1), (|\overline{b}|, -1))$ which, in normal form, is $(O, o, 0| -2; (2, 1), (2, 1), (|\overline{b}|, |\overline{b}| - 1))$.

(ii) A preserves the orientation of S^1, i.e., $\mathbb{Z}_2 \to \mathrm{Aut}(S^1)$ is trivial. Here

$$H^2(\mathbb{Z}_2; \mathbb{Z}) = \mathbb{Z}_2, \quad H^3(\mathbb{Z}_2; \mathbb{Z}) = 0, \quad H^2(S^2; \mathbb{Z})^{\mathbb{Z}_2} = 0.$$

Our exact sequence becomes

$$0 \to \mathbb{Z}_2 \to H^2(\mathbb{Z}_2; \mathfrak{Z}) \longrightarrow 0 \longrightarrow 0 \longrightarrow H^3(\mathbb{Z}_2; \mathfrak{Z}).$$

This says that action of A on S^2 cannot be lifted to any principal S^1-bundle over S^2 if $c_1(P) \neq 0$.

Actually we know more, in fact. Since all principal S^1-bundles over S^2 with $c_1(P) \neq 0$, has the same rational homology as the 3-sphere then using the Lefschetz number as in 4...♣ We know there eixsts no $f \in \mathrm{TOP}_{S^1}(P)$ such that f maps to $\mathrm{id}_{S^1} \times A \in \mathrm{Aut}(S^1) \times \mathrm{TOP}(S^2)$. There are 2 distinct actions on $S^1 \times S^2$ covering the $\mathbb{Z}_2$ action on S^2 generated by A. They are given by 4?

$$\alpha \times (z, w) \mapsto (z, Aw) \quad \text{and} \quad \alpha \times (z, w) \mapsto (-z, Aw).$$

Let R be the reflection across the equator in S^2. Once again consider the $\mathbb{Z}_2$-liftings to principal S^1-bundles P over S^2. The groups are identical to the cases above, but $H^2(\mathbb{Z}_2; \mathfrak{Z}) \cong \mathbb{Z} \xrightarrow{j} H^2(S^2; \mathbb{Z})^{\mathbb{Z}_2} \cong \mathbb{Z}$ is different because the Borel spaces are different.

(iii) Assume R maps into a non-trivial element of $\mathrm{Aut}(S^1)$. We claim the j is an onto isomorphism. Let

$$(z_1, z_2) \xrightarrow{\alpha} (\overline{z}_1, \overline{z}_2), \quad z_1\overline{z}_1 + z_2\overline{z}_2 = 1.$$

This defines involution on the 3-sphere which is orientation-preserving. Furthermore, $\alpha(zz_1, zz_2) = \overline{z}(\overline{z}_1, \overline{z}_2)$ and so α is a weak bundle automorphism with respect to the Hopf bundle over S^2. The homeomorphism projects to R on the base of the Hopf bundle and is given by $\alpha \mapsto R : \frac{z_1}{z_2} \mapsto \frac{\overline{z}_1}{\overline{z}_2}$, the reflection across the equator of S^2. Therefore j is an onto isomorphism.

(iv) Now suppose R maps into the trivial element of $\mathrm{Aut}(S^1)$. If there is a lifted action to any P with $c_1(P) \neq 0$, then it is free over $(S^2 - \text{equator})$. Over the equator, each fiber is mapped onto itself, either by rotation through π in each fiber, or it is fixed. If it rotates in the fiber, the lifted action is free which can only happen if the involution if orientation preserving, (which it is not). If the involution is fixed on each fiber, then the fixed set for this involution is a 2-torus, which is impossible for these principal bundles. On $S^2 \times S^1$, there are exactly 2 actions given by

$$\alpha \times (z, w) \mapsto (-z, Rw) \quad \text{and} \quad \alpha \times (z, w) \mapsto (z, Rw).$$

(v) Let T be the rotation of π about the polar axis on S^2. Assume a lift α of T maps into $\operatorname{Aut}(S^1)$ non-trivially. Therefore, α reverses the orientation on P. Now $H^2(\mathbb{Z}_2; \mathbb{Z}) = 0$ and $H^0(\mathbb{Z}_2; H^2(S^2; \mathbb{Z})) = 0$, and so the only lifting is the unique one to $S^2 \times S^1$ given by $\alpha \times (z, w) \mapsto (\overline{z}, Tw)$.

(vi) Assume a lift α of T maps into $\operatorname{Aut}(S^1)$ trivially. Therefore, α preserves the orientation on P. From our exact sequence

$$0 \to H^2(\mathbb{Z}_2; \mathbb{Z}) = \mathbb{Z}_2 \to H^2(\mathbb{Z}_2; \mathfrak{Z}) \to H^2(S^2; \mathbb{Z})^{\mathbb{Z}_2} = \mathbb{Z}_2 \to 0,$$

we have $H^2(\mathbb{Z}_2; \mathfrak{Z}) = \mathbb{Z} \times \mathbb{Z}_2$.

We may describe the liftings to $P_1 = S^3$ by

$$\alpha \times (z_1, z_2) \mapsto (z_1, -z_2) \quad \text{and} \quad \alpha \times (z_1, z_2) \mapsto (-z_1, z_2), \quad \text{where } z_1\overline{z}_2 + z_2\overline{z}_2 = 1.$$

Because α commutes with the diagonal (Hopf) S^1-action on S^3, we can first divide out S^3 by $\mathbb{Z}_n = \langle e^{\frac{2\pi i}{n}} \rangle$ given by

$$e^{\frac{2\pi i}{n}} \times (z_1, z_2) \mapsto (z_1 e^{\frac{2\pi i}{n}}, z_2 e^{\frac{2\pi i}{n}}).$$

The resulting quotient space is the lens space $L(n, 1) = P_n$. Then α projects to P_n and is a lift of T on S^2. The actions on $S^1 \times S^2$ are given by

$$(z, w) \mapsto (-z, Tw) \quad \text{and} \quad (z, w) \mapsto (z, Tw).$$

3.7. Injective Holomorphic Seifert Fiberings

We assume that $G = \mathbb{C}^k$, W is a complex manifold, $\rho : Q \to \operatorname{TOP}(W)$ has image in the holomorphic homeomorphisms of W, and $\mathcal{H}(W, \mathbb{C}^k) \subset \mathrm{M}(W, \mathbb{C}^k)$ are the holomorphic maps.

We also assume that the map $\mathbb{C}^k \times W \to W$ is holomorphically trivial and so $\mathcal{U} = \operatorname{Hol}_{\mathbb{C}^k}(\mathbb{C}^k \times W)$ then becomes

$$\mathcal{H}(W, \mathbb{C}^k) \rtimes (\operatorname{GL}(k, \mathbb{C}^k) \times \operatorname{Hol}(W)),$$

where $\operatorname{Hol}(W)$ is the group of holomorphic automorphisms of W. Existence, uniqueness and rigidity do not necessarily hold because we do not have holomorphic partitions of unity and the groups $H^i(Q; \mathcal{H}(W, \mathbb{C}^k))$ does not vanish in general.

The reader is referred to [**?**] where a general and comprehensive theory of holomorphic Seifert fiberings whose universal space is a holomorphic fiber bundle over W with fiber a complex torus or $\mathbb{C}^k$ is given. We shall restrict ourselves here to a special case closely related to the classical Seifert 3-manifolds.

Let $(\mathbb{C}^*, M)$ be an injective, proper, holomorphic $\mathbb{C}^*$ action on a complex 2-manifold M so that the quotient space is compact. As in the case of an injective S^1

action, an injective proper $\mathbb{C}^*$ action lifts to the covering space of M corresponding to the image of the evaluation homomorphism, and yields a splitting $(\mathbb{C}^*, \mathbb{C}^* \times W)$ where W is a simply connected complex 1-manifold, (cf. section 1.14). Therefore, W is $\mathbb{C}$, D the open unit disk, or $\mathbb{C}P_1$. We shall restrict ourselves to W being the unit disk D. The orbit space $Q\backslash W$ is a closed Riemann surface. The action of $Q = \pi_1(M)/\mathbb{Z}$ on D is holomorphic, properly discontinuous, but not necessarily free. Therefore, $M \longrightarrow \mathbb{C}^*\backslash M = Q\backslash W$ is a generalization of a principal holomorphic $\mathbb{C}^*$-bundle over a Riemann surface.

From the exact sequence $1 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 0$, we obtain the exact sequence

$$0 \longrightarrow \mathbb{Z} = M(W, \mathbb{Z}) \longrightarrow \mathcal{H}(W, \mathbb{C}) \longrightarrow \mathcal{H}(W, \mathbb{C}^*) \longrightarrow 0$$

which gives rise to a long exact sequence of cohomology groups

$$\cdots \xrightarrow{\delta^{i-1}} H^i(Q; \mathbb{Z}) \longrightarrow H^i(Q; \mathcal{H}(W, \mathbb{C})) \longrightarrow H^i(Q; \mathcal{H}(W, \mathbb{C}^*)) \xrightarrow{\delta^i} \cdots$$

The group Q acts on the unit disk D as a cocompact Fuchsian group. That is, $\rho : Q \to \mathrm{Hol}(D)$, the complex automorphisms of the unit disk. The action of Q on $\lambda \in \mathcal{H}(W, \mathbb{C})$ is given by

$${}^{\alpha}\lambda = \lambda \circ (\rho(\alpha))^{-1}.$$

Let us compare the smooth situation with the holomorphic one. We have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^1(Q, \mathbb{Z}) & \longrightarrow & H^1(Q, \mathcal{H}(W, \mathbb{C})) & \longrightarrow & H^1(Q, \mathcal{H}(W, \mathbb{C}^*)) & \xrightarrow{\delta} & H^2(Q, \mathbb{Z}) & \longrightarrow & H^2(Q; \mathcal{H}(W, \mathbb{C})) \\
 & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = & & \downarrow \\
0 & \longrightarrow & H^1(Q, \mathbb{Z}) & \longrightarrow & H^1(Q, \mathcal{C}(W, \mathbb{C})) & \longrightarrow & H^1(Q, \mathcal{C}(W, \mathbb{C}^*)) & \xrightarrow{\delta} & H^2(Q, \mathbb{Z}) & \longrightarrow & H^2(Q; \mathcal{C}(W, \mathbb{C}))
\end{array} \tag{3.7–1}$$

For the smooth case, $H^i(Q; \mathcal{C}(W, \mathbb{C})) = 0$, $i > 0$, and as we shall see, $H^2(Q; \mathcal{H}(W, \mathbb{C})) = 0$.

For each central extension $0 \to \mathbb{Z} \to \Pi \to Q \to 0$ represented by $[f] \in H^2(Q; \mathbb{Z})$, we have smooth Seifert Constructions $\theta : \Pi \to \mathrm{Diff}_{\mathbb{C}}(\mathbb{C} \times D) = \mathrm{Diff}_{\mathbb{C}}(\mathbb{C} \times \mathbb{R}^2)$. If we fix $i : \mathbb{Z} \to \mathbb{C}$ and $\rho : Q \to \mathrm{Diff}(D)$, the construction is unique up to strict equivalences (subsection **??**). We have the smooth Seifert orbifold over $Q\backslash W = Q\backslash D$ with an induced injective $\mathbb{C}^*$ action and therefore an injective S^1 action on $\theta(\Pi)\backslash(\mathbb{C} \times D)$ $(\cong \mathbb{R}^1 \times \theta(\Pi)\backslash(\mathbb{R}^1 \times D) = \mathbb{R}^1 \times N^3$ since $\mathbb{C}^*$ splits smoothly as $\mathbb{R}^1 \times S^1)$. The uniqueness says that for any other embedding $\theta' : \Pi \to \mathrm{Diff}_{\mathbb{C}}(\mathbb{C} \times D)$, keeping i and ρ fixed, the $\mathbb{C}^*$ action on $\theta'(\Pi)\backslash(\mathbb{C} \times D)$ is strictly smoothly equivalent to that on $\theta(\Pi)\backslash(\mathbb{C} \times D)$.

For the same Π, we have the homomorphism $H^2(Q; \mathbb{Z}) \longrightarrow H^2(Q; \mathcal{H}(D, \mathbb{C}))$. The second group fortunately can be identified with the second cohomology of the sheaf of germs of holomorphic functions over $Q\backslash D$. This vanishes since $Q\backslash D$ is (complex) 1-dimensional and the sheaf is coherent (i.e., locally free). This means that $[f] \in H^2(Q; \mathbb{Z})$ maps to $0 \in H^2(Q; \mathcal{H}(W, \mathbb{C}))$. But as the groups are abelian, this becomes exactly the identity $\boxed{(\textbf{??})}$, and we have $\theta : \Pi \to \mathrm{Hol}_{\mathbb{C}}(\mathbb{C} \times W)$. Therefore, each $[f]$ has holomorphic realizations for each fixed i and ρ. Recall from Theorem **??**, the set of all $\theta : \Pi \to \mathrm{Hol}_{\mathbb{C}}(\mathbb{C} \times D)$ with fixed $i : \mathbb{Z} \to \mathbb{C}$ and $\rho : Q \to \mathcal{H}(D)$, up to conjugation by elements of $\mathcal{H}(D, \mathbb{C})$, is in one–one correspondence with $H^1(Q; \mathcal{H}(D, \mathbb{C}))$. (This complex vector space is the same as $H^1(V; \mathcal{H}(D, \mathbb{C}))$, the first cohomology of the sheaf of germs of holomorphic functions where V is treated as the analytic space $V = Q\backslash D$. That is, for each open U in

V, we consider $p^{-1}(U)$ and holomorphic functions $\lambda : p^{-1}(U) \to \mathbb{C}$ such that $\lambda(\rho(\alpha)(w)) = \lambda(w)$, $w \in p^{-1}(U)$ and $p : D \to Q\backslash D$ is the projection. This defines a sheaf over V). This group is isomorphic to $\mathbb{C}^g$, where g is the genus of V.

3.7.1 THEOREM ([**?**] §13). *For each smooth action* $(\mathbb{C}^*, M)$ *corresponding to the unique strict conjugacy class* $\theta(\Pi)$*, there exists a complex* g*-dimensional family of strictly holomorphically inequivalent* $\mathbb{C}^*$ *actions each strictly smoothly equivalent to the smooth* $(\mathbb{C}^*, M)$.

PROOF. We may interpret $[f'] \in H^1(Q; \mathcal{H}(W, \mathbb{C}^*))$ to represent the holomorphic $\mathbb{C}^*$ action on $\theta(\Pi)\backslash(\mathbb{C} \times D)$ up to strict $\mathbb{C}^*$ equivalence. If Q were torsion free, then Q acts freely and $Q\backslash D = V$ is a closed oriented surface without branch points. The $\mathbb{C}^*$ action is then free and proper yielding a principal holomorphic $\mathbb{C}^*$ bundle, corresponding to a complex line bundle over V. Since Q is not assumed to be torsion free, f' determines the holomorphic $\mathbb{C}^*$ action (cf, [**?**] §5). Given two injective $\mathbb{C}^*$ actions $[f']$ and $[f'']$ with the same "Chern class" $\delta[f'] = \delta[f'']$, there exists a $[\lambda] \in H^2(Q; \mathcal{H}(D, \mathbb{C}))$ such that $[f''] = [f'] + [\lambda]$. Since $H^1(Q; \mathcal{H}(D, \mathbb{C}))$ is a vector space of complex dimension g, there exists a whole g-dimensional family of inequivalent holomorphic $\mathbb{C}^*$ actions starting from $[f']$ and ending with $[f'']$. □

Returning to diagram **??**, define

$$\begin{aligned} \mathrm{Pic}(Q\backslash D) &= H^1(Q; \mathcal{H}(D, \mathbb{C}))/\mathrm{image}(H^1(Q; \mathbb{Z})) \\ &= \text{a complex } g\text{−torus, or real } 2g\text{−torus}, T^{2g}. \end{aligned}$$

The connected component of the group $H^1(Q; \mathcal{H}(D, \mathbb{C}^*))$, in the Q torsion free case, is called the *Picard group* for the line bundles over $Q\backslash D$. In our case, $H^1(Q; \mathcal{H}(D, \mathbb{C}^*))$ is isomorphic to $T^{2g} \oplus \mathbb{Z} \oplus$ finite torsion.

We obtain the exact sequence

$$0 \longrightarrow \mathrm{Pic}(Q\backslash D) \longrightarrow H^1(Q; \mathcal{H}(D, \mathbb{C}^*)) \xrightarrow{\delta} H^2(Q; \mathbb{Z}) \longrightarrow 0,$$

where the middle group is the isomorphism classes of injective holomorphic $\mathbb{C}^*$ actions over $Q\backslash D$, δ sends such an isomorphism class to its "Chern class" and $\mathrm{Pic}(Q\backslash D)$ represents the deformations. As before, $H^2(Q; \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Torsion}$.

In the discussion above, we have fixed $i : \mathbb{Z} \to \mathbb{C}$ and $\rho : Q \to \mathrm{Hol}(D)$. If we vary these choices, we don't get anything new in the smooth case because of smooth rigidity. That is, $\theta(\Pi)$ is conjugate to $\theta'(\Pi)$ in $\mathrm{Diff}_{\mathbb{C}}(\mathbb{C} \times D)$ where conjugation is taken in the whole group and not just in $\mathcal{C}(D, \mathbb{C})$ as for strict equivalence. However, in the holomorphic case, a change in $\rho : Q \to \mathrm{Hol}(D)$ induces a much larger deformation space than treated above. We can see this in our next example of reduction of the universal group where instead of considering complex structures and complex actions, we replace them by essentially equivalent Riemannian metric structures and metric preserving S^1-actions on $N^3, (M = N \times \mathbb{R}^1)$.

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