ISOMETRIC ACTIONS OF LIE GROUPS AND INVARIANTS

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1. Introduction

Let $S(n)$ denote the space of symmetric $n \times n$ matrices with entries in $\mathbb{R}$ and $O(n)$ the orthogonal group. Consider the action:

$$\ell : O(n) \times S(n) \to S(n)$$
$$(A, B) \mapsto ABA^{-1} = ABA^t$$

If Σ is the space of all real diagonal matrices and $\mathfrak{S}_n$ the symmetric group on n letters, then we have the following

1.1. Theorem.

(1) *Σ meets every $O(n)$-orbit.*

(2) *If $B \in \Sigma$, then $\ell(O(n), B) \cap \Sigma$, the intersection of the $O(n)$-orbit through B with Σ, equals the $\mathfrak{S}_n$-orbit through B, where $\mathfrak{S}_n$ acts on $B \in \Sigma$ by permuting the eigenvalues.*

(3) *Σ intersects each orbit orthogonally in terms of the inner product $\langle A, B\rangle = \operatorname{tr}(AB^t) = \operatorname{tr}(AB)$ on $S(n)$.*

(4) *$\mathbb{R}[S(n)]^{O(n)}$, the space of all $O(n)$-invariant polynomials in $S(n)$ is isomorphic to $\mathbb{R}[\Sigma]^{\mathfrak{S}_n}$, the symmetric polynomials in Σ (by restriction).*

(5) *The space $C^\infty(S(n))^{O(n)}$ of $O(n)$-invariant C^∞-functions is isomorphic to $C^\infty(\Sigma)^{\mathfrak{S}_n}$, the space of all symmetric C^∞-functions in Σ (again by restriction), and these again are isomorphic to the C^∞-functions in the elementary symmetric polynomials.*

(6) *The space of all $O(n)$-invariant horizontal p-forms on $S(n)$, that is the space of all $O(n)$-invariant p-forms ω with the property $i_X\omega = 0$ for all $X \in T_A(O(n).A)$, is isomorphic to the space of $\mathfrak{S}_n$-invariant p-forms on Σ:*

$$\Omega^p_{hor}(S(n))^{O(n)} \cong \Omega^p(\Sigma)^{\mathfrak{S}_n}$$

Proof. (1). Clear from linear algebra.

(2) The transformation of a symmetric matrix into normal form is unique except for the order in which the eigenvalues appear.

(3) Take an A in Σ. For any $X \in \mathfrak{o}(n)$, that is for any skew-symmetric X, let ζ_X denote the corresponding fundamental vector field on $S(n)$. Then we have

$$\begin{aligned}\zeta_X(A) &= \frac{d}{dt}\Big|_{t=0} \exp_e(tX) A \exp_e(tX^t) = \\ &= XA\,id + id\,AX^t = XA - AX.\end{aligned}$$

Now the inner product with $\eta \in T_A\Sigma \cong \Sigma$ computes to

$$\begin{aligned} \langle \zeta_X(A), \eta\rangle &= \operatorname{tr}(\zeta_X(A)\eta) = \operatorname{tr}((XA - AX)\eta) = \\ &= \operatorname{tr}(\underbrace{XA\eta}_{=X\eta A}) - \operatorname{tr}(AX\eta) = \operatorname{tr}(X\eta A) - \operatorname{tr}(X\eta A) = 0. \end{aligned}$$

(4) If $p \in \mathbb{R}[S(n)]^{O(n)}$ then clearly $\tilde{p} := p|_\Sigma \in \mathbb{R}[\Sigma]^{\mathfrak{S}_n}$. To construct p from $\tilde{p}$ we use the result from algebra, that $\mathbb{R}[\mathbb{R}^n]^{\mathfrak{S}_n}$ is just the ring of all polynomials in the elementary symmetric functions. So if we use the isomorphism:

$$A := \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & a_n \end{pmatrix} \mapsto (a_1, a_2, \dots, a_n) =: a$$

to replace $\mathbb{R}^n$ by Σ, we find that each symmetric polynomial $\tilde{p}$ on Σ is of the form

$$\tilde{p}(A) = \bar{p}(\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A)),$$

it can be expressed as a polynimial $\bar{p}$ in the elementary symmetric functions

$$\begin{aligned} \sigma_1 &= -x^1 - x^2 - \dots - x^n \\ \sigma_2 &= x^1x^2 + x^1x^3 + \dots \\ &\dots \\ \sigma_k &= (-1)^k \sum_{j_1 < \dots < j_k} x^{j_1} \dots x^{j_k} \\ &\dots \\ \sigma_n &= (-1)^n X^1 \dots x^n. \end{aligned}$$

Let us consider the characteristic polynomial of the diagonal matrix X with eigenvalues $x^1, \dots, x^n$:

$$\begin{aligned} \prod_{i=1}^n (t - x^i) &= t^n + \sigma_1.t^{n-1} + \dots + \sigma_{n-1}.t + \sigma_n \\ &= \det(t.Id - X) \\ &= \sum_{i=0}^n t^i c_{n-i}(X), \qquad \text{where} \\ c_k(Y) &= \operatorname{tr}(\Lambda^k Y : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n) \end{aligned}$$

is the k-th characteristic coefficient of a matrix A. So the σ_i extend to $O(n)$-invariant polynomials c_i on $S(n)$. So we can now extend $\tilde{p}$ to a polynomial on $S(n)$ by

$$\tilde{p}(H) := \bar{p}(c_1(H), c_2(H), \dots, c_n(H)) \qquad \text{for all } H \in S(n),$$

and $\tilde{p}$ is an $O(n)$-invariant polynomial on $S(n)$, and unique as such due to (1).

(5) Again we have that $f \in C^\infty(S(n))^{O(n)}$ implies $\tilde{f} := f|_\Sigma \in C^\infty(\Sigma)^{\mathfrak{S}_n}$. Finding an inverse map $\tilde{f} \mapsto f$ as above is possible due to the following theorem by Gerald Schwarz (see chapter 3) :

Let G be a compact Lie group with a finite-dimensional representation $G \to GL(V)$, and $\rho_1, \rho_2, \dots, \rho_k$ generators for the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V (this space is finitely generated as an algebra due to Hilbert, see chapter 2). Then, for any smooth function $h \in C^\infty(V)^G$, there is a function $\bar{h} \in C^\infty(\mathbb{R}^k)$ such that $h(v) = \bar{h}(\rho_1(v), \dots, \rho_k(v))$.

Now we can prove the assertion as in (4) above. Again we take the symmetric polynomials $\sigma_1, \dots, \sigma_n$ as generators of $\mathbb{R}[\Sigma]^{\mathfrak{S}_n}$. By Schwarz' theorem $\tilde{f} \in C^\infty(\Sigma)^{\mathfrak{S}_n}$ can be written as a smooth function in $\sigma_1, \dots, \sigma_n$. So we have an $\bar{f} \in C^\infty(\mathbb{R}^n)$ such that

$$\tilde{f}(A) = \bar{f}(\sigma_1(A), \dots \sigma_n(A)) \qquad \text{for all } A \in \Sigma$$

If we extend the σ_i onto $S(n)$ as in (4), we can define

$$f(H) := \bar{f}(c_1(H), c_2(H), \dots, c_n(H)) \qquad \text{for } H \in S(n).$$

f is again a smooth function and the unique $O(n)$-invariant extension of $\tilde{f}$.

(6) Consider $\sigma = (\sigma_1, \dots, \sigma_n) : \Sigma \to \mathbb{R}^n$ and put $J(x) := \det(d\sigma(x))$. For each $\alpha \in \mathfrak{S}_n$ we have

$$\begin{aligned} J.dx^1 \wedge \dots \wedge dx^n &= d\sigma_1 \wedge \dots \wedge d\sigma_n \\ &= \alpha^* d\sigma_1 \wedge \dots \wedge d\sigma_n \\ &= (J \circ \alpha).\alpha^* dx^1 \wedge \dots \wedge dx^n \\ &= (J \circ \alpha).\det(\alpha).dx^1 \wedge \dots \wedge dx^n \end{aligned}$$

$$J \circ \alpha = \det(\alpha^{-1}).J \tag{7}$$

From this we see firstly that J is a homogeneous polynomial of degree

$$0 + 1 + \dots + (n-1) = \tfrac{n(n-1)}{2} = \binom{n}{2}.$$

The mapping σ is a local diffeomorphism on the open set $U = \Sigma \setminus J^{-1}(0)$, thus $d\sigma_1, \dots, d\sigma_n$ is a coframe on U, i.e. a basis of the cotangent bundle everywhere on U. Let (ij) be the transpositions in $\mathfrak{S}_n$, let $H_{(ij)} := \{x \in \Sigma : x^i - x^j = 0\}$ be the reflection hyperplanes of the (ij). If $x \in H_{(ij)}$ then by (7) we have $J(x) = J((ij)x) = -J(x)$, so $J(X) = 0$. Thus $J|H_{(ij)} = 0$, so the polynomial J is divisible by the linear form $x^i - x^j$, for each $i < j$. By comparing degrees we see that

$$J(x) = c.\prod_{i<j}(x^i - x^j), \quad \text{where } 0 \neq c \in \mathbb{R}. \tag{8}$$

By the same argument we see that:

(9) If $g \in C^\infty(\Sigma)$ satisfies $g \circ \alpha = \det(\alpha^{-1}).g$ for all $\alpha \in \mathfrak{S}_n$, then $g = J.h$ for $h \in C^\infty(\Sigma)^{\mathfrak{S}_n}$.

(10) *Claim* (10): Let $\omega \in \Omega^p(\Sigma)^{\mathfrak{S}_n}$. Then we have

$$\omega = \sum_{j_1<j_2<\dots<j_p} \omega_{j_1,\dots,j_p}\, d\sigma_{j_1} \wedge \dots \wedge d\sigma_{j_p}$$

on Σ, for $\omega_{j_1,\dots,j_p} \in C^\infty(\Sigma)^{\mathfrak{S}_n}$.

To prove claim (10) recall that $d\sigma_1, \dots, d\sigma_n$ is an $\mathfrak{S}_n$-invariant coframe on the $\mathfrak{S}_n$-invariant open set U. Thus

$$\begin{aligned} \omega|U &= \sum_{j_1<j_2<\dots<j_p} \underbrace{g_{j_1,\dots,j_p}}_{\in C^\infty(U)}\, d\sigma_{j_1} \wedge \dots \wedge d\sigma_{j_p} \\ &= \sum_{j_1<j_2<\dots<j_p} \underbrace{\left(\frac{1}{n}\sum_{\alpha\in\mathfrak{S}_n} \alpha^* g_{j_1,\dots,j_p}\right)}_{h_{j_1,\dots,j_p}\in C^\infty(U)^{\mathfrak{S}_n}} d\sigma_{j_1} \wedge \dots \wedge d\sigma_{j_p} \end{aligned} \tag{11.}$$

Now choose $I = \{i_1 < \dots < i_p\} \subseteq \{1, \dots, n\}$ and let $\bar I = \{1, \dots, n\} \setminus I = \{i_{p+1} < \dots < i_n\}$. Then we have for a sign $\varepsilon = \pm 1$

$$\begin{aligned} \omega|U \wedge \underbrace{d\sigma_{i_{p+1}} \wedge \dots \wedge d\sigma_{i_n}}_{d\sigma^{\bar I}} &= \varepsilon.h_I.d\sigma_1 \wedge \dots \wedge d\sigma_n \\ &= \varepsilon.h_I.J.dx^1 \wedge \dots \wedge dx^n. \end{aligned}$$

On the whole of Σ we have

$$\omega \wedge d\sigma^{\bar I} = \varepsilon.k_I.dx^1 \wedge \dots \wedge dx^n$$

for suitable $k_I \in C^\infty(\Sigma)$. By comparing the two expression on U we see from (7) that $k_I \circ \alpha = \det(\alpha^{-1}).k_I$ since U is dense in Σ. So from (9) we may conclude that $k_I = J.\omega_I$ for $\omega_I \in C^\infty(\Sigma)^{\mathfrak{S}_n}$, but then $h_I = \omega_I|U$ and $\omega = \sum_I \omega_I\, d\sigma^I$ as asserted in claim (10).

Now we may finish the proof. By the theorem of G. Schwartz there exist $f_I \in C^\infty(\mathbb{R}^n)$ with $\omega_I = f_I(\sigma_1, \dots, \sigma_n)$. Recall now the characteristic coefficients $c_i \in \mathbb{R}[S(n)]$ from the proof of (4) which satisfy $c_i|\Sigma = \sigma_i$. If we put now

$$\tilde\omega := \sum_{i_1<\dots<i_p} f_{i_1,\dots,i_p}(c_1, \dots, c_n)\, dc_{i_1} \wedge \dots \wedge dc_{i_p} \in \Omega^p_{\text{hor}}(S(n))^{O(n)}$$

then the pullback of $\tilde\omega$ to Σ equals ω. $\square$

2. Polynomial Invariant Theory

2.1. Theorem of Hilbert and Nagata. *Let G be a Lie group with a finite-dimensional representation $G \to GL(V)$ and let one of the following conditions be fulfilled:*

(1) *G is semisimple and has only a finite number of connected components*
(2) *V and $\langle G.f\rangle_{\mathbb{R}}$ are completely reducible for all $f \in \mathbb{R}[V]$ (see Nagata's lemma)*

Then $\mathbb{R}[V]^G$ is finitely generated as an algebra, or equivalently, there is a finite set of polynomials $\rho_1, \dots, \rho_k \in \mathbb{R}[V]^G$, such that the map $\rho := (\rho_1, \dots, \rho_k) : V \to \mathbb{R}^k$ induces a surjection

$$\mathbb{R}\left[\mathbb{R}^k\right] \xrightarrow{\rho^*} \mathbb{R}[V]^G .$$

Remark. The first condition is stronger than the second since for a connected, semisimple Lie group, or for one with a finite number of connected components, every finite dimensional representation is completely reducible. To prove the theorem we will only need to know complete reducibility for the finite dimensional representations V and $\langle G.f\rangle_{\mathbb{R}}$ though (as stated in (2)).

2.2. Lemma. *Let $A = \oplus_{i\geq 0} A_i$ be a connected graded $\mathbb{R}$-algebra (that is $A_0 = \mathbb{R}$). If $A_+ := \oplus_{i>0} A_i$ is finitely generated as an A-module, then A is finitely generated as an $\mathbb{R}$-algebra.*

Proof. Let $a_1, \dots, a_n \in A_+$ be generators of A_+ as an A-module. Since they can be chosen homogeneous, we assume $a_i \in A_{d_i}$ for positive integers d_i.
Claim: The a_i generate A as an $\mathbb{R}$-algebra: $A = \mathbb{R}[a_1, \dots, a_n]$

We will show by induction that $A_i \subseteq \mathbb{R}[a_1, \dots, a_n]$ for all i. For $i = 0$ the assertion is clearly true, since $A_0 = \mathbb{R}$. Now suppose $A_i \subseteq \mathbb{R}[a_1, \dots, a_n]$ for all $i < N$. Then we have to show that

$$A_N \subseteq \mathbb{R}[a_1, \dots, a_n]$$

as well. Take any $a \in A_N$. Then a can be expressed as

$$a = \sum_{i,j} c^i_j a_i \qquad c^i_j \in A_j$$

Since a is homogeneous of degree N we can discard all $c^i_j a_i$ with total degree $j + d_i \neq N$ from the righthand side of the equation. If we set $c^i_{N-d_i} =: c^i$ we get

$$a = \sum_i c^i a_i$$

In this equation all terms are homogeneous of degree N. In particular, any occurring a_i have degree $d_i \leq N$. Consider first the a_i of degree $d_i = N$. The corresponding c^i then automatically lie in $A_0 = \mathbb{R}$, so $c^i a_i \in \mathbb{R}[a_1, \dots, a_n]$. To handle the remaining a_i we use the induction hypothesis. Since a_i and c^i are of degree $< N$, they are both contained in $\mathbb{R}[a_1, \dots, a_n]$. Therefore, $c^i a_i$ lies in $\mathbb{R}[a_1, \dots, a_n]$ as well. So $a = \sum c^i a_i \in \mathbb{R}[a_1, \dots, a_n]$, which completes the proof. $\square$

Remark. If we apply this lemma for $A = \mathbb{R}[V]^G$ we see that to prove 2.1 we only have to show that $\mathbb{R}[V]^G_+$, the algebra of all invariant polynomials of strictly positive degree, is finitely generated as a module over $\mathbb{R}[V]^G$. The first step in this direction will be to prove the weaker statement:

$$B := \mathbb{R}[V].\mathbb{R}[V]^G_+ \text{ is finitely generated as an ideal.}$$

It is a consequence of a well known theorem by Hilbert:

2.3. Theorem. *(Hilbert's ideal basis theorem) If A is a commutative Noetherian ring, then the polynomial ring $A[x]$ is Noetherian as well.*

A ring is Noetherian if every strictly ascending sequence of left ideals $I_0 \subset I_1 \subset I_2 \subset \dots$ is finite, or equivalently, if every left ideal is finitely generated. If we choose $A = \mathbb{R}$, the theorem states that $\mathbb{R}[x]$ is again Noetherian. Now consider $A = \mathbb{R}[x]$, then $\mathbb{R}[x][y] = \mathbb{R}[x, y]$ is Noetherian, and so on. By induction, we see that $\mathbb{R}[V]$ is Noetherian. Therefore, any left ideal in $\mathbb{R}[V]$, in particular B, is finitely generated.

Proof of 2.3. Take any ideal $I \subseteq A[x]$ and denote by A_i the set of leading coefficients of all i-th degree polynomials in I. Then A_i is an ideal in A, and we have a sequence of ideals

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A.$$

Since A is Noetherian, this sequence stabilizes after a certain index r, i.e. $A_r = A_{r+1} = \cdots$. Let $\{a_{i1}, \dots, a_{in_i}\}$ be a set of generators for A_i $(i = 1, \dots, r)$, and p_{ij} a polynomial of degree i in I with leading coefficient a_{ij}.
Claim: These polynomials generate I.

Let $\mathcal{P} = \langle p_{ij} \rangle_{A[x]} \subseteq A[x]$ be the ideal generated by the p_{ij}. $\mathcal{P}$ clearly contains all constants in I $(A_0 \subseteq I)$. Let us show by induction that it contains all polynomials in I of degree $d > 0$ as well. Take any polynomial p of degree d. We distinguish between two cases.
(1) Suppose $d \leq r$. Then we can find coefficients $c_1, \dots, c_{n_d} \in A$ such that

$$\tilde{p} := p - c_1 p_{d1} - c_2 p_{d2} - \dots - c_{n_d} p_{dn_d}$$

has degree $< d$.
(2) Suppose $d > r$. Then the leading coefficients of $x^{d-r}p_{r1}, \dots, x^{d-r}p_{rn_r} \in I$ generate A_d. So we can find coefficients $c_1, \dots, c_{n_r} \in A$ such that

$$\tilde{p} := p - c_1 x^{d-r} p_{r1} - c_2 x^{d-r} p_{r2} - \dots - c_{n_r} x^{d-r} p_{rn_r}$$

has degree $< d$.

In both cases we have $p \in \tilde{p} + \mathcal{P}$ and $\deg \tilde{p} < d$. Therefore by the induction hypothesis $\tilde{p}$, and with it p, lies in $\mathcal{P}$. $\square$

To prove theorem 2.1 it remains only to show the following

2.4. Lemma. *Let G be a Lie group acting on V such that the same conditions as in Hilbert and Nagata's theorem are satisfied. Then for $f_1,\dots,f_k \in \mathbb{R}[V]^G$:*

$$\mathbb{R}[V]^G \cap \langle f_1,\dots,f_k\rangle_{\mathbb{R}[V]} = \langle f_1,\dots,f_k\rangle_{\mathbb{R}[V]^G}$$

where the brackets denote the generated ideal (module) in the specified space.

2.5. Remark. In our case, if we take $f_i = \rho_i \in \mathbb{R}[V]^G_+$ to be the finite system of generators of B as an ideal in $\mathbb{R}[V]$, we get:

$$\mathbb{R}[V]^G_+ = \mathbb{R}[V]^G \cap B = \langle \rho_1,\dots,\rho_k\rangle_{\mathbb{R}[V]^G}$$

That is, the ρ_i generate $\mathbb{R}[V]^G_+$ as a $\mathbb{R}[V]^G$-module. With lemma 2.2, Hilbert and Nagata's theorem follows immediately.

2.6. Remark. The inclusion ($\supseteq$) in lemma 2.4 is trivial. If G is compact, then the opposite inclusion

$$\mathbb{R}[V]^G \cap \langle f_1,\dots,f_k\rangle_{\mathbb{R}[V]} \subseteq \langle f_1,\dots,f_k\rangle_{\mathbb{R}[V]^G}$$

is easily seen as well. Take any $f \in \mathbb{R}[V]^G \cap \langle f_1,\dots,f_k\rangle_{\mathbb{R}[V]}$. Then f can be written as

$$f = \sum p_i f_i \qquad p_i \in \mathbb{R}[V].$$

Since G is compact, we can integrate both sides over G using the Haar measure dg to get

$$f(x) = \int_G f(g.x)dg = \sum_i \int_G p_i(g.x) f_i(g.x)dg = \sum_i \underbrace{(\int_G p_i(g.x)dg)}_{=:p_i^*(x)} f_i(x).$$

The p_i^* are G-invariant polynomials, therefore f is in $\langle f_1,\dots,f_k\rangle_{\mathbb{R}[V]^G}$.

To show the lemma in its general form we will need to find a replacement for the integral. This is done in the central

2.7. Lemma [26]. *Under the same conditions as theorem 2.1, to any $f \in \mathbb{R}[V]$ there is an $f^* \in \mathbb{R}[V]^G \cap \langle G.f\rangle_{\mathbb{R}}$ such that*

$$f - f^* \in \langle Gf - Gf\rangle_{\mathbb{R}}.$$

Proof. Take $f \in \mathbb{R}[V]$. Clearly, f is contained in $M_f := \langle G.f\rangle_{\mathbb{R}}$, where f^* is supposed to lie as well. M_f is a finite dimensional subspace of $\mathbb{R}[V]$ since it is contained in

$$M_f \subseteq \bigoplus_{i\le \deg f} \mathbb{R}[V]_i.$$

In addition we have that

$$\langle G.f - G.f\rangle_{\mathbb{R}} =: N_f \subseteq M_f$$

is an invariant subspace. So we can restrict all our considerations to the finite dimensional G-space M_f which is completely reducible by our assumption.

If $f \in N_f$, then we can set $f^* = 0$ and are done. Suppose $f \notin N_f$. Then the f^* we are looking for must also lie in $M_f \setminus N_f$. From the identity

$$g.f = f + \underbrace{(g.f - f)}_{\in N_f} \qquad \text{for all } g \in G$$

it follows that

$$M_f = N_f \oplus \mathbb{R}.f.$$

In particular, N_f has codimension 1 in M_f.

Since we require of f^* to be G-invariant, $\mathbb{R}.f^*$ will be a one dimensional G-invariant subspace of M_f (not contained in N_f). As we just saw, N_f has codimension 1 in M_f, therefore $\mathbb{R}.f^*$ will be a complementary subspace to N_f.

If we now write M_f as the direct sum

$$M_f = N_f \oplus P,$$

where P is the invariant subspace complementary to N_f guaranteed by the complete irreducibility of M_f, then P is a good place to look for f^*.

Now $P \cong M_f/N_f$ as a G-module, so let us take a look at the action of G on M_f/N_f. Every element of M_f/N_f has a representative in $\mathbb{R}.f$, so we need only consider elements of the form $\lambda f + N_f$ ($\lambda \in \mathbb{R}$). For arbitrary $g \in G$ we have:

$$g.(\lambda f + N_f) = \lambda g.f + N_f = \lambda f + \underbrace{(\lambda g.f - \lambda f)}_{\in N_f} + N_f = \lambda f + N_f.$$

So G acts trivially on M_f/N_f and therefore on P. This is good news, since now every $f' \in P$ is G-invariant and we only have to project f onto P (along N_f) to get the desired $f^* \in \mathbb{R}[V]^G \cap M_f$. $\square$

Proof of lemma 2.4. Recall that for arbitrary $f_1, \dots, f_n$ we have to show

$$\mathbb{R}[V]^G \cap \langle f_1, \dots, f_n \rangle_{\mathbb{R}[V]} \subseteq \langle f_1, \dots, f_n \rangle_{\mathbb{R}[V]^G}.$$

We will do so by induction on n. For $n = 0$ the assertion is trivial.
Suppose the lemma is valid for $n = r - 1$. Consider $f_1, \dots, f_r \in \mathbb{R}[V]^G$ and $f \in \mathbb{R}[V]^G \cap \langle f_1, \dots, f_r \rangle_{\mathbb{R}[V]}$. Then

$$f = \sum_{i=1}^{r} p_i f_i \qquad p_i \in \mathbb{R}[V].$$

By Nagata's lemma 2.7, we can approximate p_i up to $\langle G.p_i - G.p_i \rangle_{\mathbb{R}}$ by a $p_i^* \in \mathbb{R}[V]^G$. So for some finite subset $F \subset G \times G$ we have

$$p_i = p_i^* + \sum_{s,t \in F} \lambda^i_{s,t}(s.p_i - t.p_i) \qquad \lambda^i_{s,t} \in \mathbb{R}.$$

Therefore we have

$$f-\sum_{i=1}^{r}p_i^*f_i=\sum_{i=1}^{r}\sum_{s,t\in F}\lambda_{s,t}^i(s.p_i-t.p_i)f_i\in\mathbb{R}[V]^G\,.$$

It remains to show that the righthand side of this equation lies in $\langle f_1,\dots,f_r\rangle_{\mathbb{R}[V]^G}$. Notice that by the G-invariance of f:

$$\sum_{i=1}^{r}(sp_i-tp_i)f_i=0.$$

For all $s,t\in G$. Therefore

$$\sum_{i=1}^{r-1}(s.p_i-t.p_i)f_i=(t.p_r-s.p_r)f_r.$$

Now we can use the induction hypothesis on

$$\sum_{i=1}^{r}\sum_{s,t\in F}\lambda_{s,t}^i(s.p_i-t.p_i)f_i=$$
$$=\sum_{i=1}^{r-1}\sum_{s,t\in F}(\lambda_{s,t}^i-\lambda_{s,t}^r)(s.p_i-t.p_i)f_i\in\mathbb{R}[V]^G\cap\langle f_1,\dots,f_{r-1}\rangle_{\mathbb{R}[V]}$$

to complete the proof. $\square$

2.8. Remark. With lemma 2.4, Hilbert and Nagata's theorem is proved as well. So in the setting of 2.1 we now have an exact sequence

$$0\to\ker\rho^*\to\mathbb{R}\left[\mathbb{R}^k\right]\xrightarrow{\rho^*}\mathbb{R}[V]^G\to 0$$

where $\ker\rho^*=\{R\in\mathbb{R}\left[\mathbb{R}^k\right]:R(\rho_1,\dots,\rho_k)=0\}$ is just the finitely generated ideal consisting of all relations between the ρ_i.

Since the action of G respects the grading of $\mathbb{R}[V]=\oplus_k\mathbb{R}[V]_k$ it induces an action on the space of all power series, $\mathbb{R}[[V]]=\Pi_{k=1}^{\infty}\mathbb{R}[V]_k$, and we have the following

2.9. Theorem. *Let $G\to GL(V)$ be a representation and $\rho_1,\dots,\rho_k$ a system of generators for the algebra $\mathbb{R}[V]^G$. Then the map $\rho:=(\rho_1,\dots,\rho_k):V\to\mathbb{R}^k$ induces a surjection*

$$\mathbb{R}\left[\left[\mathbb{R}^k\right]\right]\xrightarrow{\rho^*}\mathbb{R}[[V]]^G\,.$$

Proof. Write the formal power series $f\in\mathbb{R}[[V]]^G$ as the sum of its homogeneous parts.

$$f(x)=f_0+f_1(x)+f_2(x)+\dots$$

Then to each $f_i(x) \in \mathbb{R}[V]_i^G$ there is a $g_i(y) \in \mathbb{R}\left[\mathbb{R}^k\right]$ such that

$$f_i(x) = g_i(\rho_1(x), \dots, \rho_k(x)).$$

Before we can set

$$g(y) = g_0 + g_1(y) + g_2(y) + \dots$$

to finish the proof, we have to check whether this expression is finite in each degree. This is the case, since the lowest degree λ_i that can appear in g_i goes to infinity with i:

Write explicitly $g_i = \sum_{|\alpha| \le i} A_{i,\alpha} y^\alpha$ and take an $A_{i,\alpha} \neq 0$. Then $\deg f_i = i = \alpha_1 d_1 + \dots \alpha_k d_k$ where $d_i = \deg \rho_i$ and

$$\lambda_i = \inf\{|\alpha| : i = \sum \alpha_j d_j\} \to \infty \qquad (i \to \infty) \qquad \square$$

The following corollary is an immediate consequence.

2.10. Corollary. *If G is a Lie group with a finite dimensional representation $G \to GL(V)$, then under the same conditions as Hilbert and Nagata's theorem there is a finite set of polynomials $\rho_1, \dots, \rho_k \in \mathbb{R}[V]^G$ such that the map $\rho := (\rho_1, \dots, \rho_k) : V \to \mathbb{R}^k$ induces a surjection*

$$\mathbb{R}[[\mathbb{R}^k]] \xrightarrow{\rho^*} \mathbb{R}[[V]]^G. \qquad \square$$

3. C^∞-Invariant Theory of Compact Lie Groups

If G is a Lie group acting smoothly on a manifold M, then the orbit space M/G is not generally again a smooth manifold. Yet, it still has a functional structure induced by the smooth structure on M simply by calling a function $f : M/G \to \mathbb{R}$ smooth iff $f \circ \pi : M \to \mathbb{R}$ is smooth (where $\pi : M \to M/G$ is the quotient map). That is, the functional structure on M/G is determined completely by the smooth G-invariant functions on M. For compact Lie groups, the space of all G-invariant C^∞-functions on $\mathbb{R}^n$ is characterized in the theorem of Gerald Schwarz (1975), which we already used in 1.1(4). In this chapter we will present the proof as found in [34], Chap. IV. In the following, let G always denote a compact Lie group, $\ell : G \to GL(V)$ a representation on $V = \mathbb{R}^n$. Let $\rho_1, \dots, \rho_k \in \mathbb{R}[V]^G$ denote a finite system of generators for the algebra $\mathbb{R}[V]^G$, and let ρ denote the polynomial mapping:

$$\rho := (\rho_1, \dots, \rho_k) : V \to \mathbb{R}^k.$$

3.1. Definition. *A mapping between two topological spaces $f : X \to Y$ is called* **proper**, *if $K \subseteq Y$ compact implies $f^{-1}(K) \subseteq X$ is compact.*

3.2. Lemma. *Let G be a compact Lie group. Then we have*

(1) *ρ is proper.*
(2) *ρ separates the orbits of G.*
(3) *There is a map $\rho' : V/G \to \mathbb{R}^k$ such that the following diagram commutes,*

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\pi"'] & \mathbb{R}^k \\ V/G \arrow[ur, dashed, "\rho'"'] & \end{tikzcd}$$

and ρ' is a homeomorphism onto its image.

Proof.

(1) Let $r(x) = |\,x\,|^2 = \langle x, x\rangle$, where $\langle\,.\,,\,.\,\rangle$ is an invariant inner product on V. Then $r \in \mathbb{R}[V]^G$. By Hilbert's theorem there is a polynomial $p \in \mathbb{R}\left[\mathbb{R}^k\right]$ such that $r(x) = p(\rho(x))$. If $(x_n) \in V$ is an unbounded sequence, then $r(x_n)$ is unbounded. Therefore $p(\rho(x_n))$ is unbounded, and, since p is a polynomial, $\rho(x_n)$ is also unbounded. With this insight we can conclude that any compact and hence bounded set in $\mathbb{R}^k$ must have a bounded inverse image. By continuity of ρ, it must be closed as well. So the inverse image of a compact set under ρ is again compact, that is, ρ is proper.

(2) Choose two different orbits $G.x \neq G.y$ $(x, y \in V)$ and consider the map:

$$f : G.x \cup G.y \to \mathbb{R} \qquad f(v) := \begin{cases} 0 \text{ for } v \in G.x \\ 1 \text{ for } v \in G.y \end{cases}$$

Both orbits are closed, so f is continuous. Furthermore, both orbits and with them their union are compact, since G is compact. Therefore, by the Weierstrass approximation theorem, there is a polynomial $p \in \mathbb{R}[V]$ such that

$$\| p - f \|_{G.x \cup G.y} = \sup\{ | p(z) - f(z) | : z \in G.x \cup G.y \} < \frac{1}{10}$$

Now we can average p over the group using the Haar measure dg on G to get a G-invariant function.

$$q(v) := \int_G p(g.v) dg$$

Note that since the action of G is linear, q is again a polynomial. Now let us check that q approximates f equally well. For $v \in G.x \cup G.y$, we have

$$\Big| \underbrace{\int_G f(g.v) dg}_{=f(v)} - \int_G p(g.v) dg \Big| \leq \int_G | f(g.v) - p(g.v) | dg \leq \frac{1}{10} \underbrace{\int_G dg}_{=1}$$

Recalling how f was defined, we get

$$\begin{aligned} | q(v) | &\leq \frac{1}{10} \qquad &&\text{for } v \in G.x \\ | 1 - q(v) | &\leq \frac{1}{10} \qquad &&\text{for } v \in G.y. \end{aligned}$$

Therefore $q(G.x) \neq q(G.y)$, and since q can be expressed in the Hilbert generators, we can conclude that $\rho(G.x) \neq \rho(G.y)$.

(3) Clearly, ρ' is well defined. By (2) ρ' is injective and, with the quotient topology on V/G, continuous. So on every compact subset of V/G we know that ρ' is a homeomorphism onto its image. Now take any diverging sequence in V/G. It is the image under π of some equally diverging sequence in V. If this sequence has an unbounded subsequence, then by (1), its image under ρ is unbounded as well, in particular divergent. If the diverging sequence in V (therefore its image under π, our starting sequence) is bounded, then it is contained in a compact subset of V, our starting sequence is contained in a compact subset of V/G, and here ρ' is a homeomorphism. Thereby, its image under ρ' is divergent as well. So we have shown that a sequence in V/G is convergent iff its image under ρ' in $\mathbb{R}^k$ is convergent and, with that, that ρ' is a homeomorphism onto its image. $\square$

3.3. Remark.

(1) If $f : V \to \mathbb{R}$ is in $C^0(V)^G$, then f factors over π to a continuous map $\tilde{f} : V/G \to \mathbb{R}$. By 3.2(3) there is a continuous map $\bar{f} : \rho(V) \to \mathbb{R}$ given by $\bar{f} = \tilde{f} \circ {\rho'}^{-1}$. It has the property $f = \bar{f} \circ \rho$. Since $\rho(V)$ is closed, $\bar{f}$ extends to a continuous function $\bar{f} \in C^0(\mathbb{R}^k)$ (Tietze-Urysohn). So for continuous functions we have the assertion that
$$\rho^* : C^0(\mathbb{R}^k) \to C^0(V)^G \qquad \text{is surjective.}$$

(2) $\rho(V)$ is a real semi algebraic variety, that is it is described by a finite number of polynomial equations and inequalities. In the complex case, the image of an algebraic variety under a polynomial map is again an algebraic variety, meaning it is described by polynomial equations only. In the real case this is already disproved by the simple polynomial map: $x \mapsto x^2$.

3.4. Before we turn to Schwarz' theorem, let us state here the extension theorem of Whitney as found in [46], pp. 68–78. For $K \subseteq \mathbb{R}^n$ compact and $m \in \mathbb{N}$, assign to each multi-index $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $|k| = |k_1| + \dots + |k_n| \le m$ a continuous function F^k on K. Then the family of functions $(F^k)_{|k| \le m}$ is called an m-jet on K. The space of all m-jets on K endowed with the norm
$$|F|_m^K := \sup_{x \in K, |k| \le m} \left| F^k(x) \right|$$
shall be denoted by $J^m(K)$. There is a natural map
$$J^m : C^m(\mathbb{R}^n) \to J^m(K) : f \mapsto \Big(\frac{\partial^{|k|} f}{\partial x^k}\Big|_K\Big)_{|k| \le m}.$$
By Whitney's first extension theorem its image is the subspace of all Whitney jets defined as follows. For each $a \in K$ there is a map $T_a^m : J^m(K) \to \mathbb{R}[\mathbb{R}^n]$ given (in multi-index notation) by
$$T_a^m F(x) = \sum_{|k| \le m} \frac{(x-a)^k}{k!} F^k(a)$$
which assigns to each m-jet its would-be Taylor polynomial of degree m. With it we can define as the remainder term (an m-jet again):
$$R_a^m F := F - J^m(T_a^m F).$$
If F is the set of partial derivatives restricted to K of some C^m-function then in particular

(W) $\quad (R_a^m F)^k(y) = o(|a-y|^{m-|k|}) \qquad$ for $a, y \in K$, $|k| \le m$ and $|a - y| \to 0$

holds by Taylor's theorem. We will call (W) the Whitney condition, and any m-jet on K which satisfies (W) Whitney jet of order m on K. The space of all Whitney jets again forms a vector space and we endow it with the norm:
$$\|F\|_m^K = |F|_m^K + \sup\Big\{\frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, \, x \ne y, \, |k| \le m\Big\}$$
The space of all Whitney jets with the above norm is a Banach space and will be denoted by $\mathcal{E}^m(K)$.

Whitney's Extension Theorem for $\mathcal{E}^m(K)$. *For $K \subset \mathbb{R}^n$ compact, there is a continuous linear map*

$$W : \mathcal{E}^m(K) \to C^m(\mathbb{R}^n)$$

such that for all Whitney jets $F \in \mathcal{E}^m(K)$ and for all $x \in K$

$$D^k W(F)(x) = F^k(x) \qquad |\,k\,| \le m$$

holds and the restriction of $W(F)$ on $\mathbb{R}^n \setminus K$ is smooth.

If we define $J^\infty(K)$ (resp. $\mathcal{E}^\infty(K)$) as the projective limit of the spaces $J^m(K)$ ($\mathcal{E}^m(K)$) we can extend the above theorem to the following

Whitney's Extension Theorem for $\mathcal{E}^\infty(K)$. *For $K \subset \mathbb{R}^n$ compact, there is a linear map*

$$W_\infty : \mathcal{E}^\infty(K) \to C^\infty(\mathbb{R}^n)$$

such that for all Whitney jets $F \in \mathcal{E}^\infty(K)$ and for all $x \in K$

$$D^k W_\infty(F)(x) = F^k(x) \qquad \text{for all } k \in \mathbb{N}_0^n$$

holds.

3.5. Remark. In general, the norm $\|\,.\,\|_m^K$ generates a finer topology on $\mathcal{E}^m(K)$ than $|\,.\,|_m^K$, yet there is a case when we can show that they are equal. If K is connected with respect to rectifiable curves and the Euclidean distance on K is equivalent to the geodesic distance (such a K is called **1-regular**), then the two norms coincide. This is shown roughly as follows.
By definition

$$|\,F\,|_m^K \le \|\,F\,\|_m^K = |\,F\,|_m^K + \sup\{\frac{|\,(R_x^m F)^k(y)\,|}{|\,x-y\,|^{m-|\,k\,|}}\}.$$

So if we approximate $\sup\{\frac{|\,(R_x^m F)^k(y)\,|}{|\,x-y\,|^{m-|\,k\,|}}\}$ by $C.|\,F\,|_m^K$, then we are done. For a fixed $x \in K$ let us denote

$$g := D^k(W(F) - T_x^m F).$$

Then g is in $C^{m-|\,k\,|}(\mathbb{R}^n)$ and flat of order $m - |\,k\,| - 1$ at x. On K, g coincides with $(R_x^m F)^k$. Now, by a somewhat generalized mean value theorem, we have for any rectifiable curve σ connecting x with y:

$$|\,g(y) - g(x)\,| \le \sqrt{n}|\,\sigma\,|\sup\{\big|\,D^j g(\xi)\,\big| : \xi \in \sigma,\ |\,j\,| = 1\}$$

Since $D^k g(x) = 0$ for all $|\,k\,| < m - |\,k\,|$ we can iterate this inequality $m - |\,k\,| - 1$ times, to get

$$|\,g(y)\,| \le n^{\frac{m-|\,k\,|}{2}}|\,\sigma\,|^{m-|\,k\,|}\sup\{\big|\,D^j g(\xi)\,\big| : \xi \in \sigma,\ |\,j\,| = m - |\,k\,|\}$$

Furthermore, we can replace $|\sigma|$ by the geodesic distance $\delta(x,y)$, which is the infimum over all $|\sigma|$, σ as chosen above. Now, if we choose x,y in K and substitute back for g, then the above inequality implies:

$$\begin{aligned}
&\left|(R_x^m F)^k(y)\right| \leq \\
&\qquad \leq n^{\frac{m-|k|}{2}}\delta(x,y)^{m-|k|}\sup\{\left|F^j(\xi)-F^j(x)\right| : \xi\in K, |j|=m\}\leq \\
&\qquad\qquad\qquad\qquad \leq 2n^{\frac{m-|k|}{2}}\delta(x,y)^{m-|k|}|F|_m^K
\end{aligned}$$

Since $\delta(x,y)\leq c|x-y|$ for all $x,y\in K$, this gives us an approximation

$$\sup\{\frac{\left|(R_x^m F)^k(y)\right|}{|x-y|^{m-|k|}}\}\leq C|F|_m^K$$

which completes our proof.

So, for a 1-regular K, we have that for every $m\in\mathbb{N}$, $\mathcal{E}^m(K)$ carries the "usual" topology of uniform convergence in each "derivative". In this case the assertion that the operator W of the first Whitney extension theorem is continuous implies that a sequence of functions in $W(\mathcal{E}^m(K))\subseteq C^m(\mathbb{R}^k)$ which converges uniformly in all derivatives on K does so on every other compact set as well.

If the $|\,.\,|_m^K$-topology coincides with the usual topology on $\mathcal{E}^m(K)$ for all m as in the above case, then the topology on the projective limit

$$\tilde{\mathcal{E}}^\infty(K) := \underset{m\to\infty}{\operatorname{proj}}\,(\mathcal{E}^m(K), |\,.\,|_m^K)$$

coincides with the usual topology on $\mathcal{E}^\infty(K)$ as well. So the topology on $\mathcal{E}^\infty(K)$ is generated by the family of seminorms $\{|\,.\,|_m^K : m\in\mathbb{N}_0\}$. Although there is a natural inclusion $i:\mathcal{E}^\infty(K)\hookrightarrow\mathcal{E}^m(K)$, the restriction i^*W of $W:\mathcal{E}^m(K)\to C^m(\mathbb{R}^n)$ does not coincide with W_∞. If it did, then W_∞ would have to be continuous as well, which is generally not the case.

There is one more result we will need. It is a direct consequence of Whitney's extension theorem if we take $K=\{x\}$ (then $\mathcal{E}^\infty(K)\cong\mathbb{R}^\infty$), but was discovered and proved independently and much earlier (1898) by Emile Borel.

Theorem of E. Borel. *To any formal power series $p\in\mathbb{R}[[\mathbb{R}^n]]$ and $x\in\mathbb{R}^n$ there is a smooth function $f\in C^\infty(\mathbb{R}^n)$ with formal Taylor development p at x.* $\square$

Here we can see directly that the extension operator W_∞ is not continuous, because if it were, it would give an embedding of $\mathbb{R}^\infty$ into $C^\infty(K)$ (where $K\subset\mathbb{R}^n$ is any compact set containing x). But this is impossible, since $\mathbb{R}^\infty$ has no continuous norm.

3.6. Theorem. Multidimensional Faa di Bruno formula. *Let $f\in C^\infty(\mathbb{R}^k)$, let $g=(g_1,\dots,g_k)\in C^\infty(\mathbb{R}^n,\mathbb{R}^k)$. Then for a multiindex $\gamma\in\mathbb{N}^n$ the partial derivative $\partial^\gamma(f\circ g)(x)$ of the composition is given by the following formula, where* *we use multiindex-notation heavily.*

$$\partial^\gamma(f\circ g)(x) =$$

$$= \sum_{\beta\in\mathbb{N}^k} (\partial^\beta f)(g(x)) \sum_{\substack{\lambda=(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_\alpha \lambda_{i\alpha}=\beta_i\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}} \frac{\gamma!}{\lambda!} \prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}} \left(\frac{1}{\alpha!}\right)^{\sum_i \lambda_{i\alpha}} \prod_{i,\alpha>0} (\partial^\alpha g_i(x))^{\lambda_{i\alpha}}$$

$$= \sum_{\substack{\lambda=(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}} \frac{\gamma!}{\lambda!} \prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}} \left(\frac{1}{\alpha!}\right)^{\sum_i \lambda_{i\alpha}} \left(\partial^{\sum_\alpha \lambda_\alpha} f\right)(g(x)) \prod_{i,\alpha>0} (\partial^\alpha g_i(x))^{\lambda_{i\alpha}}$$

This formula is attributed to Reinhold Hoppe (1845) and in the form used above to Faa di Bruno (1855).

Proof. The proof of this goes roughly as follows. The infinite Taylor development of the composition is the composition of the Taylor developments,

$$j^\infty(f\circ g)(x) = j^\infty f(g(x)) \circ j^g(x),$$

$$j^\infty f(y)(z) = \sum_{\beta\in\mathbb{R}^k} \frac{1}{\beta!}\partial^\beta f(y) z^\beta$$

$$= \sum_{\beta\in\mathbb{R}^k} \frac{1}{\beta_1!\dots\beta_k!}\partial^\beta f(y) z_1^{\beta_1}\dots z_k^{\beta_k}$$

So we write down the Taylor series and compose them, using multinomial theorems, and compute then one of the coefficients. The formula above comes up. □

3.7. Theorem of Schwarz. *(*[38]*)*
Let G be a compact Lie group, $\ell : G \to O(V)$ a finite-dimensional representation, and $\rho_1,\rho_2,\dots,\rho_k$ generators for the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V (this space is finitely generated as an algebra due to Hilbert; see chapter 2). If $\rho := (\rho_1,\dots,\rho_k) : V \to \mathbb{R}^k$, then

$$\rho^* : C^\infty\left(\mathbb{R}^k\right) \to C^\infty(V)^G \qquad \textit{is surjective.}$$

The actual proof of Gerald Schwarz' theorem will take us the rest of this section. But let us just begin now with some remarks and make some simplifications.

(1) For the action of $G = \{\pm 1\}$ on $\mathbb{R}^1$ the result is due to Whitney [47].
(2) If $G = \mathfrak{S}_n$ acting on $\mathbb{R}^n$ by the standard representation it was shown by G.Glaeser [15]
(3) It is easy to see that $\rho^* C^\infty\left(\mathbb{R}^k\right)$ is dense in $C^\infty(V)^G$ in the compact C^∞-topology. Therefore, Schwarz' theorem is equivalent to the assertion: $\rho^* C^\infty\left(\mathbb{R}^k\right)$ is closed in $C^\infty(V)^G$. If $\rho_1,\dots,\rho_k$ can be chosen algebraically independent, then this follows from a theorem by Glaeser (see [15]).
(4) To start out with, notice that the Hilbert polynomials can be chosen homogeneous and of positive degree: Since the action of G is linear, the degree of a polynomial $p \in \mathbb{R}[V]$ is invariant under G. Therefore, if we split each Hilbert polynomial up into its homogeneous parts, we get a new set of Hilbert polynomials. Let us denote these by ρ_i and the corresponding degrees by $d_i > 0$.

3.8. Corollary. *Under the same conditions as 3.7:*

$$\rho^* : C_0^\infty(\mathbb{R}^k) \to C_0^\infty(V)^G \qquad \textit{is surjective,}$$

where C_0^∞ *denotes the space of all germs at* 0 *of* C^∞*.*

Proof. $C^\infty(V)^G \xrightarrow{[\,]_0} C_0^\infty(V)^G$ is surjective, since for any $f \in C_0^\infty(V)^G$ there is a representative $f' \in C^\infty(V)$, and with it $f'' := \int_G \ell(g)^* f' dg \in C^\infty(V)^G$ also represents f. By Schwarz' theorem, $f'' = h \circ \rho$ for some $h \in C^\infty(\mathbb{R}^k)$. $\square$

3.9. Corollary. *Under the same conditions as 3.7, also for spaces of smooth functions with compact supports we have:*

$$\rho^* : C_c^\infty(\mathbb{R}^k) \to C_c^\infty(V)^G \qquad \textit{is surjective.}$$

Proof. For $f \in C_C^\infty(V)^G$ by 3.7 there is an $\tilde f \in C^\infty(\mathbb{R}^k)$ such that $f = \rho^* \tilde f = \tilde f \circ \rho$. Since $f = \tilde f \circ \rho$ has compact support it vanishes outside some large compact ball $B \subset V$. Then $\rho(B)$ is contained in some larger ball $B_1 \subset \mathbb{R}^k$. Take $h \in C_c^\infty(\mathbb{R}^k)$ with $h|B_1 = 1$. Then $(h \circ \rho)|B = 1$ and thus $(h.\tilde f) \circ rh = \tilde f \circ \rho = f$. $\square$

3.10. Lemma. *It suffices to prove theorem 3.7 for representations with zero as the only fixed point.*

Proof. Decompose V into the subspace of all fixed points and its orthogonal complement:

$$V = \underbrace{\operatorname{Fix}(G)}_{=:U} \oplus \underbrace{\operatorname{Fix}(G)^\perp}_{=:W}$$

Then W is an invariant subspace with only the one fixed point: 0. Let $\sigma_1, \dots, \sigma_n$ be generators of $\mathbb{R}[W]^G$ such that $\sigma^* : C^\infty(R^n) \to C^\infty(W)^G$ is surjective. Consider the following diagram, where $\hat\otimes$ denotes projective tensor product. Note, that in this case it coincides with the injective tensor product, since $C^\infty(V)$ is a nuclear Fréchet space. From this it follows that the horizontal maps on the bottom and on the top are homeomorphisms.

$$\begin{tikzcd}
C^\infty(U)\hat\otimes C^\infty(\mathbb{R}^n) \arrow[r, "\cong"] \arrow[d, "C^\infty(U)\hat\otimes\sigma^*"'] & C^\infty(U \times \mathbb{R}^n) \arrow[d] \\
C^\infty(U)\hat\otimes C^\infty(W)^G \arrow[r, two heads] & C^\infty(U \times W)^G \\
C^\infty(U)\hat\otimes C^\infty(W) \arrow[r, "\cong"] \arrow[u, "C^\infty(U)\hat\otimes \int_G \ell(g)^* dg"] & C^\infty(U \times W) \arrow[u, "\int_G \check\ell(g)^* dg"']
\end{tikzcd}$$

Starting from the bottom, notice that $C^\infty(U)\hat\otimes \int_G \ell(g)^* dg$ and $\int_G \check\ell(g)^* dg$ are surjective. Therefore, the horizontal map in the center is surjective. By our assumption, $C^\infty(U)\hat\otimes\sigma^*$ is also surjective, so we can conclude that the map on the top right is surjective as well. But this map is just ρ^* for $\rho := (id_U, \sigma)$, and we are done. $\square$

3.11. We shall use the following notation: For a manifold M and a closed submanifold $K \subseteq M$ let

$$C^\infty(M;K) := \{f \in C^\infty(M) : f \text{ is flat along } K\}.$$

Lemma. *For the proof of theorem 3.7 it suffices to show that*

$$C^\infty(V;0)^G \xleftarrow{\rho^*} C^\infty(\mathbb{R}^k;0) \qquad \textit{is surjective.}$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(\mathbb{R}^n;0)^G \arrow[r, "i"] & C^\infty(\mathbb{R}^n)^G \arrow[r, "t"] & \mathbb{R}[[V]]^G \arrow[r] & 0 \\
0 \arrow[r] & C^\infty(\mathbb{R}^k;0) \arrow[r] \arrow[u, "\rho^*"] & C^\infty(\mathbb{R}^k) \arrow[r, "T"] \arrow[u, "\rho^*"] & \mathbb{R}[[\mathbb{R}^k]] \arrow[r] \arrow[u, "\rho^*"] & 0
\end{tikzcd}$$

The right ρ^* is surjective by corollary 2.10. The map T on the lower righthand side assigns to each function its formal Taylor series at zero. It is surjective by the theorem of E. Borel. The same goes for the map t above it. Just take any smooth function $f \in C^\infty(\mathbb{R}^n)$ with a given formal Taylor series in $\mathbb{R}[[V]]^G$ and integrate it over G. The resulting function lies in $C^\infty(\mathbb{R}^n)^G$ and has the same formal Taylor development since this was G-invariant to begin with. Clearly, the space $C^\infty(\mathbb{R}^n;0)^G$ embedded in $C^\infty(\mathbb{R}^n)^G$ is just the kernel of t. So the top sequence is exact. The same goes for the bottom sequence. Now suppose we knew that the left ρ^* is surjective as well, then we could conclude that the ρ^* in the middle is surjective by the following diagram chase. Take any $f \in C^\infty(\mathbb{R}^n)^G$ and consider $t(f)$. Then there is a power series $p \in \mathbb{R}[[\mathbb{R}^k]]$ with $\rho^*(p) = t(f)$ and a smooth function $g \in C^\infty(\mathbb{R}^k)$ with $T(g) = p$. Now $f - \rho^* g \in \operatorname{Ker} t = \operatorname{Im} i$, and by the surjectivity of the ρ^* on the lefthand side of the diagram, we can find an $h \in C^\infty(\mathbb{R}^k)$ such that $\rho^* h = f - \rho^* g$. So $f = \rho^*(g+h)$ and the central ρ^* is surjective as well. □

The proof will involve transforming everything into polar coordinates, so let us start with the following lemma.

3.12. Lemma. *Let $\varphi : [0,\infty) \times S^{n-1} \to \mathbb{R}^n$ be the polar coordinate transformation $\varphi(t,x) = tx$. Then*

$$C^\infty([0,\infty) \times S^{n-1}) \xleftarrow{\varphi^*} C^\infty(\mathbb{R}^n)$$

satisfies

(1) φ^* *is injective.*
(2) $\varphi^*(C^\infty(\mathbb{R}^n;0)) = C^\infty([0,\infty) \times S^{n-1}; 0 \times S^{n-1})$.

Proof. (1) is clear since φ is surjective. Let us go on to (2). Here it is easy to see the inclusion

$$\varphi^*(C^\infty(\mathbb{R}^n;0)) \subseteq C^\infty([0,\infty) \times S^{n-1}; 0 \times S^{n-1}).$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and flat at zero, then $\varphi^*(f) = f \circ \varphi$ is smooth and flat at $\varphi^{-1}(0) = 0 \times S^{n-1}$. Now let us show the converse. On $(0,\infty) \times S^{n-1}$, φ has an inverse $\varphi^{-1} : \mathbb{R}^n \setminus \{0\} \to (0,\infty) \times S^{n-1}$ given by $\varphi^{-1}(x) = (|\,x\,|, \frac{1}{|\,x\,|}x)$. Take a chart (U_i, u_i) of S^{n-1} and define $\varphi_i^{-1} = (id_{\mathbb{R}}, u_i) \circ \varphi^{-1}$. Then we can find $C_\alpha, A_\alpha > 0$ such that

$$\left|\,\partial^\alpha \varphi_i^{-1}(x)\,\right| \le C_\alpha |\,x\,|^{-A_\alpha}.$$

Choose $f \in C^\infty\left([0,\infty) \times S^{n-1}; \{0\} \times S^{n-1}\right)$, then since f is flat along $\{0\} \times S^{n-1}$ we have

$$\partial^\alpha f(t, u_i^{-1}(x)) \le B(\alpha, N) t^N \qquad \forall N, \forall \alpha \in \mathbb{N}^n.$$

All together this gives us via the Faa di Bruno formula 3.6

$$\begin{aligned}
&|\partial^\gamma (f \circ \varphi_i^{-1})(x)| = \\
&= \Bigg| \sum_{\substack{(\lambda_{i\alpha}) \in \mathbb{N}^{k \times (\mathbb{N}^n \setminus 0)} \\ \sum_{i\alpha} \lambda_{i\alpha}\alpha = \gamma}} \frac{\gamma!}{\lambda!} \prod_{\substack{\alpha \in \mathbb{N}^n \\ \alpha > 0}} \left(\frac{1}{\alpha!}\right)^{\sum_i \lambda_{i\alpha}} \left(\partial^{\sum_\alpha \lambda_\alpha} f\right)(\varphi_i^{-1}(x)) \prod_{i,\alpha>0} (\partial^\alpha(\varphi_i^{-1})(x))^{\lambda_{i\alpha}} \Bigg| \\
&\le C(\gamma, N) |\,x\,|^N
\end{aligned}$$

for $|\,x\,| \le 1$. Therefore $f \circ \varphi^{-1}$ can be extended at 0 to $f \in C^\infty(\mathbb{R}^n; \{0\})$. $\square$

3.13. Now let us extend the result of this lemma somewhat. If G is a compact Lie group acting orthogonally on $\mathbb{R}^n$, then G acts on S^{n-1} and trivially on $\mathbb{R}$, so it acts on $\mathbb{R} \times S^{n-1}$. Consider the $\mathbb{Z}_2$-action on $\mathbb{R} \times S^{n-1}$ given by

$$\bar{A} : (t, \theta) \to (-t, -\theta)$$

It clearly commutes with the G-action, so we get a $\mathbb{Z}_2 \times G$-action on $\mathbb{R} \times S^{n-1}$. Now consider

$$\phi : \mathbb{R} \times S^{n-1} \to \mathbb{R}^n \qquad \phi(t,\theta) := t.\theta\,.$$

Then ϕ is $\mathbb{Z}_2 \times G$-equivariant if we let $\mathbb{Z}_2$ act trivially on $\mathbb{R}^n$. Therefore, it induces a homomorphism:

$$\phi^* : C^\infty(\mathbb{R}^n)^{\mathbb{Z}_2 \times G} \to C^\infty\left(\mathbb{R} \times S^{n-1}\right)^{\mathbb{Z}_2 \times G},$$

and we have the following

Lemma.

(1) *ϕ^* is injective.*

(2) $C^\infty\left(\mathbb{R} \times S^{n-1}; \{0\} \times S^{n-1}\right)^{\mathbb{Z}_2} = \phi^* C^\infty(\mathbb{R}^n; \{0\})$
$C^\infty\left(\mathbb{R} \times S^{n-1}; \{0\} \times S^{n-1}\right)^{\mathbb{Z}_2 \times G} = \phi^* C^\infty(\mathbb{R}^n; \{0\})^G$

Remark. By (1) it is sufficient to prove 3.7 in polar coordinates. That is, we only have to show that $\phi^* C^\infty(\mathbb{R}^n; \{0\})^G = \phi^* \rho^* C^\infty\left(\mathbb{R}^k; \{0\}\right)$. The first step in this direction is taken in (2).

Proof.

(1) As in 3.12(1) it is sufficient to note that ϕ is surjective.

(2) If we define $\psi : \mathbb{R} \times S^{n-1} \to [0, \infty) \times S^{n-1} : (t, \theta) \mapsto \operatorname{sgn} t.(t, \theta) = (|t|, \operatorname{sgn}(t).\theta)$, then we have $\phi = \varphi \circ \psi$, where φ is the polar coordinate transformation as in 3.12. Therefore:

$$\begin{aligned} \phi^*(C^\infty(\mathbb{R}^n; 0)) &= \psi^* \circ \varphi^*(C^\infty(\mathbb{R}^n; \{0\})) \\ &= \psi^*(C^\infty\left([0, \infty) \times S^{n-1}; \{0\} \times S^{n-1}\right)) \quad \text{by } 3.12 \end{aligned}$$

Now take any $f \in C^\infty\left([0, \infty) \times S^{n-1}; \{0\} \times S^{n-1}\right)$. Since $\psi |_{[0,\infty)\times S^{n-1}}$ and $\psi |_{(-\infty,0]\times S^{n-1}}$ are diffeomorphisms onto $[0, \infty) \times S^{n-1}$, $\psi^* f$ is smooth on $(-\infty, 0] \times S^{n-1}$ as well as on $[0, \infty) \times S^{n-1}$. Since f is flat at $\{0\} \times S^{n-1}$, $\psi^* f$ is smooth altogether. Furthermore, $\psi^*(f)$ is $\mathbb{Z}_2$-invariant, since ψ is $\mathbb{Z}_2$-invariant. So we have

$$\psi^* C^\infty\left([0, \infty) \times S^{n-1}; \{0\} \times S^{n-1}\right) \subseteq C^\infty\left(\mathbb{R} \times S^{n-1}; \{0\} \times S^{n-1}\right)^{\mathbb{Z}_2}$$

The opposite inclusion is clear, since any $f \in C^\infty\left(\mathbb{R} \times S^{n-1}; \{0\} \times S^{n-1}\right)^{\mathbb{Z}_2}$ is the image under ψ^* of its restriction to $[0, \infty) \times S^{n-1}$.

The assertion with added G-invariance follows easily from this. That $f := \phi^* \tilde f$ is G-invariant with $\tilde f$ is clear, since ϕ is G-equivariant. Now if f is G-invariant, then for all $g \in G$ we have $\tilde f(g.\phi(x)) = \tilde f(\phi(x))$, so by the surjectivity of ϕ we can conclude that $\tilde f$ is G-invariant as well. □

3.14. The next step, roughly, is to translate the $\mathbb{Z}_2$-action $\bar A$ as well as the polar coordinate transformation to the image of $\mathbb{R} \times S^{n-1}$ under $id \times \rho$. This is done in the following two diagrams, where $r : \mathbb{R}^n \to \mathbb{R}$ stands for the polynomial map $x \mapsto |x|^2$.

$$\begin{tikzcd} \mathbb{R} \times S^{n-1} \arrow[r, "\bar A"] \arrow[d, "\mathbb{R} \times \rho"'] & \mathbb{R} \times S^{n-1} \arrow[d, "\mathbb{R} \times \rho"'] \\ \mathbb{R} \times \rho(S^{n-1}) \arrow[r, "A"] & \mathbb{R} \times \rho(S^{n-1}) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{R} \times S^{n-1} \arrow[r, "\phi"] \arrow[d, "\mathbb{R} \times \rho"'] & \mathbb{R}^n \arrow[d, "r \times \rho"'] \\ \mathbb{R} \times \rho(S^{n-1}) \arrow[r, "B"] & \mathbb{R} \times \rho(\mathbb{R}^n) \end{tikzcd}$$

Recall that the ρ_i were chosen homogeneous of degree d_i. With this, A and B are given by:

$$\begin{aligned} A(t, y) &:= (-t, (-1)^{d_1} y_1, \dots, (-1)^{d_k} y_k) \\ B(t, y) &:= (t^2, t^{d_1} y_1, \dots, t^{d_k} y_k) \end{aligned}$$

With this definition, we can let A and B have domain $\mathbb{R}\times\mathbb{R}^k$. The choice of $(t,y)\mapsto t^2$ for the first component of B lets B retain the $\mathbb{Z}_2$-invariance under the $\mathbb{Z}_2$-action given by A. Indeed, $B\circ A=B$:

$$\begin{aligned}
B\circ A(t,y) &= B(-t,(-1)^{d_1}y_1,\dots,(-1)^{d_k}y_k)=\\
&=\big((-t)^2,(-t)^{d_1}(-1)^{d_1}y_1,\dots,(-t)^{d_k}(-1)^{d_k}y_k\big)=\\
&=(t^2,t^{d_1}y_1,\dots,t^{d_k}y_k)=B(t,y)
\end{aligned}$$

Now we can state the following

Lemma. *The map B as defined above induces a mapping B^* on $C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\right)$ into $C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k\right)^{\mathbb{Z}_2}$ such that*

$$\begin{tikzcd}
C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\right) \arrow[r, "B^*"] \arrow[dr] & C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k\right)^{\mathbb{Z}_2} \arrow[d, "restr"'] \\
 & C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k\right)^{\mathbb{Z}_2}\Big|_{\mathbb{R}\times\rho(S^{n-1})}
\end{tikzcd}$$

The map $restr\circ B^*: C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\right)\to C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k\right)^{\mathbb{Z}_2}\Big|_{\mathbb{R}\times\rho(S^{n-1})}$ is surjective.

Proof. The inclusion $B^*C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\right)\subseteq C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k\right)^{\mathbb{Z}_2}$ is clear since, first of all, B maps $0\times\mathbb{R}^k$ to 0, so if f is flat at 0, then B^*f is flat at $0\times\mathbb{R}^k$. Secondly, B^*f is $\mathbb{Z}_2$-invariant, since B is $\mathbb{Z}_2$-invariant.

For the surjectivity, choose any $h\in C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k\right)^{\mathbb{Z}_2}$. Then we need to find an $H\in C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\right)$ such that $B^*H|_{\mathbb{R}\times\rho(S^{n-1})}=h|_{\mathbb{R}\times\rho(S^{n-1})}$. Formally, that would give us

$$H(t,y)=h(t^{\frac12},t^{-\frac{d_1}{2}}y_1,\dots,t^{-\frac{d_1}{2}}y_1).$$

For $t>0$, this is well defined. With the $\mathbb{Z}_2$-symmetry, we know how to define $\bar h$ for $t<0$ as well. To handle the case $t=0$ we will need Whitney's extension theorem.

Let Δ be a k-dimensional cube in $\mathbb{R}^k$ with center 0 which contains $\rho(S^{n-1})$. Consider $K:=[-1,1]\times\Delta\subseteq\mathbb{R}\times\mathbb{R}^k$ and set $L:=B(K)\subset\mathbb{R}\times\mathbb{R}^k$. More precisely, L is a compact subset of $[0,\infty)\times\mathbb{R}^k$. Now define on $[0,\infty)\times\mathbb{R}^k\supset L$ the function

$$H_\varepsilon(t,y_1,\dots,y_k):=h\left((t+\varepsilon)^{\frac12},(t+\varepsilon)^{-\frac{d_1}{2}}y_1,\dots,(t+\varepsilon)^{-\frac{d_k}{2}}y_k\right).$$

H_ε is smooth on $[0,\infty)\times\mathbb{R}^k\supset L$, so $J^\infty H_\varepsilon\in J^\infty(L)$ is a Whitney jet on L. Now we will have to study the behavior of H_ε for $\varepsilon\to 0$. Our strategy will be as follows:

(1) Show that L is 1-regular. Referring back to 3.5, the topology on $\mathcal{E}^\infty(L)$ is then generated by the family of seminorms $\{|\,.\,|_m^L : m\in\mathbb{N}_0\}$.
(2) Show that $J^\infty H_\varepsilon$ is a Cauchy sequence for $\varepsilon\to 0$ in terms of the family of seminorms $\{|\,.\,|_m^L : m\in\mathbb{N}_0\}$.

(3) Since $\mathcal{E}^\infty(L)$ is complete, (1) and (2) together imply that $J^\infty H_\varepsilon$ converges to some Whitney ∞-jet$H = (H^\alpha)_{\alpha\in\mathbb{N}_0^{k+1}}$ on L. In this situation, Whitney's extension theorem implies that H^0 is the restriction onto L of some smooth function we will again call $H \in C^\infty\left(\mathbb{R}\times\mathbb{R}^k\right)$.

(4) Show that H is flat at zero and after some slight modifications satisfies $B^*H = h$ on $\mathbb{R}\times\rho(S^{n-1})$ to finish the proof.

Let us now go ahead and show (1).

Let $\operatorname{dist}_L(l,l')$ denote the shortest length of any rectifiable curve in L joining l with l'. Then we will show that

$$\operatorname{dist}(l,l') \le \operatorname{dist}_L(l,l') \le 2\operatorname{dist}(l,l')$$

The lefthand side of this inequality is clear. To show the righthand side let $l = (t,y)$ and $l' = (t',y')$ and suppose without loss of generality that $t' \le t$. Recall once more how L was defined ($L = B(K)$ where $K = [-1,1]\times\triangle$). Consider the line segments $[(t,y),(t,y')]$ and $[(t,y'),(t',y')]$. Both are contained in L:

To see this, take any $(s,y') \in [(t,y'),(t',y')]$, that is $t' \le s \le t$. Then

$$(s,y') = B(\sqrt{s}, s^{-\frac{d_1}{2}}y'_1,\dots,s^{-\frac{d_k}{2}}y'_k)$$

Since $(t',y') \in L$, we have $(t'^{-\frac{d_1}{2}}y'_1,\dots,t'^{-\frac{d_k}{2}}y'_k) \in \triangle$. With $t' \le s$, that is $t'^{-\frac{d_k}{2}} \ge s^{-\frac{d_k}{2}}$, this implies that $(s^{-\frac{d_1}{2}}y'_1,\dots,s^{-\frac{d_k}{2}}y'_k)$ lies in $\triangle$ as well. That $\sqrt{s}\in[-1,1]$ is clear from $(t,y)\in L$. In particular, we now have that (t,y') lies in L. Therefore, by the linearity of B in the second variable, the first line segment $[(t,y),(t,y')]$ is also contained in L.

Since the line segments $[l,(t,y')]$ and $[(t,y'),l']$ are the sides of a right triangle with hypotenuse $[l,l']$, this immediately implies

$$\operatorname{dist}_L(l,l') \le \operatorname{dist}(l,(t,y')) + \operatorname{dist}((t,y'),l') \le 2\operatorname{dist}(l,l')$$

and (1) is proved.

Now let us turn to (2). Write H_ε as composition $H_\varepsilon = h\circ\beta_\varepsilon$ where the map $\beta_\varepsilon : \mathbb{R}_+\times\mathbb{R}^k \to \mathbb{R}_+\times\mathbb{R}^k$ is given by

$$\beta_\varepsilon : (t,y_1,\dots,y_k) \mapsto ((t+\varepsilon)^{\frac12}, (t+\varepsilon)^{-\frac{d_1}{2}}y_1,\dots,(t+\varepsilon)^{-\frac{d_k}{2}}y_k).$$

By definition, every $(t,y)\in L$ is image under B of some $(\tau,z)\in K = [0,1]\times\triangle$. That is:

$$(t,y) = (\tau^2, \tau^{d_1}z_1,\dots,\tau^{d_k}z_k)$$

which makes

$$\beta_\varepsilon(t,y) = \left((\tau^2+\varepsilon)^{\frac12}, \frac{\tau^{d_1}}{(\tau^2+\varepsilon)^{\frac{d_1}{2}}}z_1,\dots,\frac{\tau^{d_k}}{(\tau^2+\varepsilon)^{\frac{d_k}{2}}}z_k\right).$$

From this formula we see that for $\varepsilon\to0$ there is a compact subset P of $\mathbb{R}\times\mathbb{R}^k$ such that $\beta_\varepsilon(L)$ lies in P for all ε.

Now to h. Since h is flat at $0\times\mathbb{R}^k$ we have that for all compact $P\subseteq\mathbb{R}\times\mathbb{R}^k$, $\alpha\in\mathbb{N}^n$ and $N>0$ there is a constant $C=C(P,\alpha,N)$ such that

$$|\,\partial^\alpha h(t,y)\,|\le C(P,\alpha,N)t^N \qquad \forall(t,y)\in P$$

Now we have all we need to approximate $\sup\limits_{(t,y)\in L}|\partial^\gamma(H_\varepsilon(t,y)-H_\mu(t,y))|$. If we choose P as described above we may apply Faa di Bruno's formula 3.6 and we see that for $(t,y)\in L$

$$\begin{aligned}
&|\partial^\gamma(h\circ\beta_\varepsilon(t,y)-h\circ\beta_\mu(t,y))|\le\\
&\le\sum_{\substack{\lambda=(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}}\frac{\gamma!}{\lambda!}\prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}}\left(\frac{1}{\alpha!}\right)^{\sum_i\lambda_{i\alpha}}\Bigg|\left(\partial^{\sum_\alpha\lambda_\alpha}h\right)(\beta_\varepsilon(t,y))\prod_{\alpha>0}(\partial^\alpha\beta_\varepsilon(t,y))^{\lambda_\alpha}-\\
&\qquad\qquad-\left(\partial^{\sum_\alpha\lambda_\alpha}h\right)(\beta_\mu(t,y))\prod_{\alpha>0}(\partial^\alpha\beta_\mu(t,y))^{\lambda_\alpha}\Bigg|\le\\
&\le\sum_{\substack{(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}}\frac{\gamma!}{\lambda!}\prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}}\left(\frac{1}{\alpha!}\right)^{\sum_i\lambda_{i\alpha}}C(P,{\textstyle\sum_\alpha}\lambda_\alpha,N)\cdot\\
&\qquad\qquad\cdot\left|(t+\varepsilon)^{\frac{N}{2}}\prod_{\alpha>0}(\partial^\alpha\beta_\varepsilon(t,y))^{\lambda_\alpha}-(t+\mu)^{\frac{N}{2}}\prod_{\alpha>0}(\partial^\alpha\beta_\mu(t,y))^{\lambda_\alpha}\right|.
\end{aligned}$$

At this point we must distinguish between two cases.
$(t\ge\delta>0)$ In this case we choose $C_\eta:=C_{\eta,2}$ so that by the above considerations we have

$$\begin{aligned}
&|\partial^\gamma(H_\varepsilon(t,y)-H_\mu(t,y))|\le\\
&\le\sum_{\substack{(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}}\frac{\gamma!}{\lambda!}\prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}}\left(\frac{1}{\alpha!}\right)^{\sum_i\lambda_{i\alpha}}C(P,{\textstyle\sum_\alpha}\lambda_\alpha,2)\delta\cdot\\
&\qquad\qquad\cdot\left|\prod_{\alpha>0}(\partial^\alpha\beta_\varepsilon(t,y))^{\lambda_\alpha}-\prod_{\alpha>0}(\partial^\alpha\beta_\mu(t,y))^{\lambda_\alpha}\right|.
\end{aligned}$$

Since $|\,\partial^\alpha\beta_\varepsilon(t,y)-\partial^\alpha\beta_\mu(t,y)\,|\to 0$ for $\lambda,\mu\to 0$ we may conclude that the expression $|\partial^\gamma(H_\varepsilon(t,y)-H_\mu(t,y))|$ goes to zero with ε and μ uniformly in $(t,y)\in L\cap\{t\ge\delta\}$
$(\delta\ge t\ge 0)$ In this case we have

$$\begin{aligned}
&|\,\partial^\gamma(H_\lambda(t,y)-H_\mu(t,y))\,|\le\\
&\le\sum_{\substack{(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}}\frac{\gamma!}{\lambda!}\prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}}\left(\frac{1}{\alpha!}\right)^{\sum_i\lambda_{i\alpha}}C({\textstyle\sum_\alpha}\lambda_\alpha,N)\cdot\\
&\qquad\qquad\cdot\left|(t+\varepsilon)^{\frac{N}{2}}\prod_{\alpha>0}(\partial^\alpha\beta_\varepsilon(t,y))^{\lambda_\alpha}-(t+\mu)^{\frac{N}{2}}\prod_{\alpha>0}(\partial^\alpha\beta_\mu(t,y))^{\lambda_\alpha}\right|.
\end{aligned}$$

Recalling how β_ε was defined, we see that the sums on the righthand side are basically polynomials in $(t+\varepsilon)^{-1}$ (resp. $(t+\mu)^{-1}$) and y. So we only need to choose N sufficiently large to have the above term converge to zero uniformly in (t,y) for $\varepsilon,\mu \to 0$.

This completes the proof that $J^\infty H_\varepsilon$ is a Cauchy sequence with regard to the seminorms $|\,.\,|_m^L$. By (3) it has a limit in the space of Whitney jets on L which we extend to a smooth function $H \in C^\infty\left(\mathbb{R}\times\mathbb{R}^k\right)$ using Whitney's extension theorem. We now turn to (4).

On L, $\partial^\gamma H$ is the limit of $\partial^\gamma H_\varepsilon$ for $\varepsilon\to 0$. Since $0\in L$, it suffices to show

$$\partial^\gamma H_\varepsilon(0) \to 0 \qquad \text{for all } \gamma \in \mathbb{N}^{k+1}$$

to imply that H is flat at 0. This is seen as in (2) above: By setting $(t,y)=0$ in $|\,\partial^\gamma(h\circ\beta_\varepsilon)(t,y)\,| \le$

$$(t+\varepsilon)^{\frac{N}{2}} \sum_{\substack{(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}} \frac{\gamma!}{\lambda!}\prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}}\left(\frac{1}{\alpha!}\right)^{\sum_i\lambda_{i\alpha}} C(P,\textstyle\sum_\alpha\lambda_\alpha,N)\cdot\left|\prod_{\alpha>0}(\partial^\alpha\beta_\varepsilon(t,y))^{\lambda_\alpha}\right|.$$

we get

$$|\,\partial^\gamma H_\varepsilon(0)\,| \le$$

$$\le \varepsilon^{\frac{N}{2}} \sum_{\substack{(\lambda_{i\alpha})\in\mathbb{N}^{k\times(\mathbb{N}^n\setminus 0)}\\ \sum_{i\alpha}\lambda_{i\alpha}\alpha=\gamma}} \frac{\gamma!}{\lambda!}\prod_{\substack{\alpha\in\mathbb{N}^n\\ \alpha>0}}\left(\frac{1}{\alpha!}\right)^{\sum_i\lambda_{i\alpha}} C(P,\textstyle\sum_\alpha\lambda_\alpha,N)\left|\prod_{\alpha>0}(\partial^\alpha\beta_\varepsilon(0))^{\lambda_\alpha}\right|.$$

Again, the righthand sum is a polynomial in ε^{-1}, and if N is chosen large enough, we see that the whole expression converges to zero with $\varepsilon\to 0$.

Next and final point of the proof is to check inhowfar B^* maps H to h. On $L\setminus\{0\}$, β_ε converges to $\beta_0 : L\setminus\{0\}\to(0,1]\times\triangle$. So restricted to $L\setminus\{0\}$, we have $H = h\circ\beta_0$. By definition of β_0,

$$B^*H = B^*(h\circ\beta_0) = h \quad\text{on}\quad (0,1]\times\triangle.$$

Therefore, by continuity, $B^*H = h$ on $[0,1]\times\triangle$; in particular

$$B^*H\,|_{[0,1]\times\rho(S^{n-1})} = h\,|_{[0,1]\times\rho(S^{n-1})}\,.$$

Since h as well as B^*H are A-invariant, their values on $A([0,1]\times\rho(S^{n-1})) = [-1,0]\times\rho(S^{n-1})$ are uniquely determined by their restriction to $[0,1]\times\rho(S^{n-1})$. So we even have

$$B^*H\,|_{[-1,1]\times\rho(S^{n-1})} = h\,|_{[-1,1]\times\rho(S^{n-1})}\,.$$

Since B is a diffeomorphism on $(1,\infty)\times\rho(S^{n-1})$ as well as on $(-\infty,-1)\times\rho(S^{n-1})$ we can change H outside of $B([-1,1]\times\rho(S^{n-1}))$ to

$$H = \begin{cases} h\circ B|^{-1}_{(1,\infty)\times\rho(S^{n-1})} & \text{on } B((1,\infty)\times\rho(S^{n-1}))\\ H & \text{on } B([-1,1]\times\rho(S^{n-1}))\\ h\circ B|^{-1}_{(-\infty,-1)\times\rho(S^{n-1})} & \text{on } B((-\infty,-1)\times\rho(S^{n-1}))\end{cases}$$

This H is in $C^\infty\left(\mathbb{R}\times\mathbb{R}^k;0\right)$, and it has the desired property:

$$B^*H\,|_{\mathbb{R}\times\rho(S^{n-1})} = h\,|_{\mathbb{R}\times\rho(S^{n-1})}\,. \qquad \square$$

3.15. The main part of the proof of Schwarz' theorem will be carried out by induction. To be able to state the induction hypothesis, we make the following definition:

For two compact Lie groups G and G' we will call $G < G'$ if

(a) $\dim G < \dim G'$ or

(b) if $\dim G = \dim G'$, then G has less connected components than G'.

We will continue the proof of 3.7 under the following two hypotheses:

I (Induction hypothesis) The compact Lie group G is such that theorem 3.7 is valid for all compact Lie groups $G' < G$ (and each orthogonal representation of G').

II The orthogonal representation has 0 as only fixed point (see 3.10).

The next step will be to prove the

Key lemma. *Under the hypotheses* **I** *and* **II***:*

$$\rho^* C^\infty\left(\mathbb{R}^k \setminus \{0\}\right) = C^\infty\left(V \setminus \{0\}\right)^G.$$

In particular: $\left(\rho|_{S^{n-1}}\right)^* C^\infty\left(\mathbb{R}^k\right) = C^\infty\left(S^{n-1}\right)^G$.

Before we get involved in a complicated proof, let us draw some conclusions from this.

3.16. Corollary. *Under the hypotheses* **I** *and* **II** *we have*

(a)

$$(id \times \rho\,|_{S^{n-1}})^* \, C^\infty(\mathbb{R}\times\mathbb{R}^k; \{0\}\times\mathbb{R}^k) = C^\infty(\mathbb{R}\times S^{n-1}; \{0\}\times S^{n-1})^G$$

(b)

$$(id \times \rho\,|_{S^{n-1}})^* \, C^\infty(\mathbb{R}\times\mathbb{R}^k; \{0\}\times\mathbb{R}^k)^{\mathbb{Z}_2} = C^\infty(\mathbb{R}\times S^{n-1}; \{0\}\times S^{n-1})^{\mathbb{Z}_2\times G}$$

where the $\mathbb{Z}_2$-action on $\mathbb{R}\times\mathbb{R}^k$ is given by A and on $\mathbb{R}\times S^{n-1}$ by $\bar{A}$.

Remark. 3.16(b) is the missing link between 3.13(2) and 3.14. Together the three lemmas give the equation

$$\begin{aligned}
\phi^* C^\infty(\mathbb{R}^n; 0)^G &= C^\infty(\mathbb{R}\times S^{n-1}; \{0\}\times S^{n-1})^{\mathbb{Z}_2\times G} && \text{by } 3.13(2)\\
&= (id \times \rho\,|_{S^{n-1}})^* \, C^\infty(\mathbb{R}\times\mathbb{R}^k; \{0\}\times\mathbb{R}^k)^{\mathbb{Z}_2} && \text{by (b)}\\
&= (id \times \rho\,|_{S^{n-1}})^* \, B^* C^\infty(\mathbb{R}\times\mathbb{R}^k; \{0\}) && \text{by } 3.14.
\end{aligned}$$

This is already a big step forward in the proof of Schwarz' theorem.

Proof of the Corollary. In (a) as well as in (b) the inclusion "$\subseteq$" is clear. So let us just concern ourselves with the surjectivity of $(id \times \rho\,|_{S^{n-1}})^*$ in each case.

(a) is a consequence of the identity

$$C^\infty(\mathbb{R}\times\mathbb{R}^k; \{0\}\times\mathbb{R}^k) \cong C^\infty(\mathbb{R}^k, C^\infty(\mathbb{R}; \{0\})) \cong C^\infty(\mathbb{R}^k)\hat{\otimes}\, C^\infty(\mathbb{R}; \{0\})$$

and the resulting commutative diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R};0)\hat{\otimes}\, C^\infty(\mathbb{R}^k) \arrow[r, "\cong"] \arrow[d, two heads, "id\hat{\otimes}(\rho|S^{n-1})^*"'] & C^\infty(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k) \arrow[d, "(id\times\rho|S^{n-1})^*"] \\
C^\infty(\mathbb{R};0)\hat{\otimes}\, C^\infty(S^{n-1})^G \arrow[r, two heads] & C^\infty(\mathbb{R}\times S^{n-1};0\times S^{n-1})^G \\
C^\infty(\mathbb{R};0)\hat{\otimes}\, C^\infty(S^{n-1}) \arrow[r, "\cong"] \arrow[u, two heads, "id\hat{\otimes}\int_G"] & C^\infty(\mathbb{R}\times S^{n-1};0\times S^{n-1}) \arrow[u, two heads, "\int_G"']
\end{tikzcd}$$

Here, the map on the upper lefthand side, $id\,\hat{\otimes}(\rho|S^{n-1})^*$, is surjective by 3.15. The surjectivity of the maps on the bottom is clear and implies that the horizontal map in the middle is also surjective. From this we can deduce that $(id\times\rho|S^{n-1})^*$ on the upper righthand side is surjective as well. This proves (a).

(b) is now a consequence of (a). To any $\varphi\in C^\infty(\mathbb{R}\times S^{n-1};\{0\}\times S^{n-1})^{\mathbb{Z}_2\times G}$ assertion (a) supplies a $\psi\in C^\infty(\mathbb{R}\times\mathbb{R}^k;0\times\mathbb{R}^k)$ which is mapped to φ under $(id\times\rho\,|_{S^{n-1}})^*$. It remains to make ψ $\mathbb{Z}_2$-invariant. On $\mathbb{R}\times\rho(S^{n-1})$ ψ is automatically $\mathbb{Z}_2$-invariant:

$$\begin{aligned}(id\times\rho\,|_{S^{n-1}})^*\,(\psi\circ A)&=\psi\circ A\circ(id\times\rho\,|_{S^{n-1}})=\\ =\psi\circ(id\times\rho\,|_{S^{n-1}})\circ\bar{A}&=\varphi\circ\bar{A}=\varphi=(id\times\rho\,|_{S^{n-1}})^*\,\psi\end{aligned}$$

Since A maps $\mathbb{R}^+\times\mathbb{R}^k$ onto $\mathbb{R}^-\times\mathbb{R}^k$ and ψ is flat at $\{0\}\times\mathbb{R}^k$, we can change ψ on $\mathbb{R}^-\times\mathbb{R}^k$ to make it $\mathbb{Z}_2$-invariant everywhere. This way we retain its smoothness, its flatness at $\{0\}\times\mathbb{R}^k$ and since ψ isn't changed on $\mathbb{R}\times\rho(S^{n-1})$ we also retain $\varphi=(id\times\rho\,|_{S^{n-1}})^*\,\psi$. $\square$

Notation. In the following we will sometimes write $\mathbb{R}[x]$ for $\mathbb{R}[\mathbb{R}^n]$ where ($x=(x_1,\dots,x_n)$) stands for the variable. The linear subspace of homogeneous polynomials of degree i will be denoted by $\mathbb{R}[x]_i$, so that we have

$$\mathbb{R}[V]=\bigoplus_{i\geq 1}\mathbb{R}[x]_i$$
$$\mathbb{R}[[V]]=\prod_{i\geq 1}\mathbb{R}[x]_i$$

Furthermore, we will abbreviate the ideal of all polynomials with no constant term by

$$\bigoplus_{i>1}\mathbb{R}[x]_i=:\mathbb{R}[x]_+$$

3.17. Definition. *We will call a system of generators $\{\sigma_1,\dots,\sigma_m\}$ of an algebra* **minimal**, *if there is no nontrivial polynomial relation of the type*

$$\sigma_j=P(\sigma_1,\dots,\sigma_{j-1},\sigma_{j+1},\dots,\sigma_k).$$

Remark. If an algebra is finitely generated, then it automatically possesses a minimal system of generators. We only have to take an arbitrary finite set of generators and recursively drop any elements which can be expressed as polynomials in the others.

Proof of 3.15. Let us get an idea of how this proof will work before we go into the technical lemmas it requires.

Choose an arbitrary $\psi \in C^\infty(\mathbb{R}^n \setminus \{0\})^G$ and take $p \in \mathbb{R}^n \setminus \{0\}$. By hypothesis II, p is not fixed under G. Therefore $G_p < G$ and Schwarz' theorem is satisfied for any representation of G_p by the induction hypothesis. In particular, take a slice S at p small enough not to meet 0 (this also implies $0 \notin G.S$). S is contained in an affine subspace $p + L(\mathbb{R}^q) \subseteq \mathbb{R}^n$, where L is a linear embedding $L : \mathbb{R}^q \hookrightarrow \mathbb{R}^n$. The slice action gives a representation of G_p on $\mathbb{R}^q$. Restrict $p+L$ to $L^{-1}(S-p) =: \tilde{S} \subseteq \mathbb{R}^q$ (open) to get the map $\tilde{\lambda} : \tilde{S} \xrightarrow{\cong} S$. We then have $\tilde{\lambda}^*(\psi\mid_S) \in C^\infty(\tilde{S})^{G_p}$. Consider a minimal system of generators $\sigma_1, \dots, \sigma_s$ of $\mathbb{R}[\mathbb{R}^q]^{G_p}$, then by Schwarz' theorem there is an $\alpha \in C^\infty(\mathbb{R}^s)$ such that

$$\tilde{\lambda}^*\psi(t) = \alpha(\sigma_1(t), \dots, \sigma_s(t)) \qquad \text{for all } t \in \tilde{S}$$

(since $\tilde{\lambda}^*\psi$ can be extended to a G_p-invariant function on $\mathbb{R}^k$). Now we require the following

Lemma 3.20. In the above situation (where here it is important that $\{\sigma_i\}$ be a minimal system of generators), denote by $\bar{\sigma}_i$ (resp. $\bar{\mu}_i$) the germ of σ_i (resp. $\mu_i := \rho_i \circ (p+L)$) at 0. Then there are germs of smooth functions $\bar{B}_i \in C_0^\infty(\mathbb{R}^k)$ such that

$$\bar{\sigma}_j = \bar{B}_j(\bar{\mu}_1, \dots, \bar{\mu}_k).$$

Let us first finish the proof of 3.15 assuming the lemma and then return to it. Recall that on $\tilde{S}$ we were able to express $\psi \circ \tilde{\lambda}$ in the Hilbert generators $\sigma_1, \dots, \sigma_s$.

$$\psi \circ \tilde{\lambda} = \alpha(\sigma_1, \dots, \sigma_s)$$

In a sufficiently small neighborhood U_0 of 0 we can now replace σ_i by $B_i \circ \mu$, where B_i is a suitable representative of the germ $\bar{B}_i$ and has domain $V_p = \mu(U_0)$ (notice that $\mu(U_0) = \rho(p + L(U_0)) = \rho(G.(p + L(U_0)))$ is open since ρ is open by 3.2(3)).

$$\psi \circ \tilde{\lambda}\mid_{U_0} = \alpha \circ (B_1 \circ \mu\mid_{U_0}, \dots, B_k \circ \mu\mid_{U_0}).$$

Since $\tilde{\lambda}$ is a diffeomorphism and $\mu\mid_{\tilde{S}} = \rho\mid_S \circ \tilde{\lambda}$, we can drop the $\tilde{\lambda}$ on each side. With $\tilde{U}_p := \tilde{\lambda}(U_0)$ this gives us:

$$\psi\mid_{\tilde{U}_p} = \alpha \circ (B_1 \circ \rho\mid_{\tilde{U}_p}, \dots, B_k \circ \rho\mid_{\tilde{U}_p})$$

Since both sides are G-invariant, we can extend the above equation to the tubular neighborhood $U_p := G.\tilde{U}_p$ of p. To simplify the formula, we set

$$C^\infty(V_p) \ni \varphi_p : x \mapsto \alpha(B_1(x), \dots, B_k(x)).$$

So we get:

$$(*)\qquad \psi\,|_{U_p} = \rho^*\varphi_p\,|_{U_p}$$

In this way we can assign to each $p \in \mathbb{R}^n \setminus \{0\}$ neighborhoods $U_p \ni p$ and $V_p \ni \rho(p)$ as well as a map $\varphi_p \in C^\infty(V_p)$ with the above property. Let us consider a partition of unity (h_p) of $\rho(\mathbb{R}^n) \setminus \{0\}$ which corresponds to the covering V_p. Then we can define

$$\varphi := \sum h_p \varphi_p \in C^\infty(\mathbb{R}^k \setminus \{0\}).$$

Now $\rho^* h_p$ is a G-invariant partition of unity on $\mathbb{R}^n \setminus \{0\}$. It corresponds to the covering (U_p) since $\rho(U_p) = V_p$ and ρ separates the orbits by 3.2(2). So with (*) we get

$$\rho^*\varphi = \rho^*(\sum h_p\varphi_p) = \sum(\rho^* h_p)(\rho^*\varphi_p) = \sum(\rho^* h_p)\psi\,|_{U_p} = \psi \qquad \square$$

Before we can prove the key lemma's key lemma (3.20) we need two supporting lemmas:

3.18. Lemma. *Let $\sigma_1, \dots, \sigma_k$ be a system of homogeneous generators of $\mathbb{R}[x]^G$. Then the following two conditions are equivalent:*

(1) *$\{\sigma_1, \dots, \sigma_k\}$ is a minimal system; that is there is no nontrivial polynomial relation of the type*

$$\rho_j = P(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_k)$$

(2) *$\rho_1, \dots, \rho_k$ are an $\mathbb{R}$-basis of $\mathbb{R}[x]^G_+/(\mathbb{R}[x]^G_+)^2$.*

Proof.

($\Uparrow$) Suppose there is a nontrivial relation. It can be written as

$$\rho_j = \sum_{i\neq j}\lambda_i\rho_i + \sum \mu_\alpha \rho^\alpha$$

where the second summation is taken over all multi-indices $\alpha \in \mathbb{N}^k$ with $|\,\alpha\,| \geq 2$ and $\alpha_j = 0$. This immediately implies

$$\rho_j \equiv \sum_{i\neq j}\lambda_i\rho_i \mod (\mathbb{R}[x]^G_+)^2.$$

So the ρ_j are linearly dependent $\mod (\mathbb{R}[x]^G_+)^2$.

($\Downarrow$) Since the ρ_i generate $\mathbb{R}[x]^G$, they automatically generate $\mathbb{R}[x]^G_+/(\mathbb{R}[x]^G_+)^2$ as a vector space. So if we suppose (2) false, then there is a nontrivial relation

$$\sum \lambda_i\rho_i \equiv 0 \mod (\mathbb{R}[x]^G_+)^2.$$

Order the ρ_i by degree: $i < j \Rightarrow d_i \le d_j$. Now let i_0 be the smallest i for which $\lambda_i \ne 0$. Then we can express ρ_{i_0} as follows

$$\rho_{i_0} = \sum_{i_0<j} \mu_j \rho_j + \sum_{|\,\alpha\,|\ge 2} \nu_\alpha \rho^\alpha .$$

This equality still holds if we drop all terms of degree $\ne d_{i_0}$, and both sides remain the same. After doing so, we see that ρ_{i_0} does not appear on the righthand side of the equation. Because if it did, then it would be in a term $\nu_\alpha \rho^\alpha$ with $\alpha_{i_0} \ne 0$ in the sum on the far right and this term would have degree $> d_{i_0}$. So we have a nontrivial polynomial relation between the ρ_i and a contradiction to (1). □

3.19. Lemma [15]**.** *Consider $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ open, $f : U \to V$ smooth and $f^* : C^\infty(V) \to C^\infty(U)$ with the compact C^∞-topology on both spaces. Then for each $\varphi \in \overline{f^* C^\infty(V)}$ and for all $a \in U$ there is a $\psi \in C^\infty(V)$ such that*

$$T_a^\infty \varphi = T_{f(a)}^\infty \psi \circ T_a^\infty f,$$

where $T_a^\infty \varphi \in \mathbb{R}[[x-a]]$ denotes the formal Taylor series of φ at a and by the composition on the right we mean the insertion of $T_a^\infty f \in \mathbb{R}[[x-a]]$ for y in $T_{f(a)}^\infty \psi \in \mathbb{R}[[y-f(a)]]$.

Proof. The assertion of the lemma is equivalent to the statement

$$T_a^\infty(f^* C^\infty(V)) = T_a^\infty(\overline{f^* C^\infty(V)}),$$

since $T_a^\infty(f^* C^\infty(V))$ is simply the set of all jets which can be written as a composition like in the lemma. Due to the fact that T_a^∞ is continuous, we have the inclusions:

$$T_a^\infty(f^* C^\infty(V)) \subseteq T_a^\infty(\overline{f^* C^\infty(V)}) \subseteq \overline{T_a^\infty(f^* C^\infty(V))}.$$

Therefore, it is sufficient to show that $T_a^\infty \circ f^*$ has a closed image. Since $C^\infty(V)$ is a reflexive Fréchet space, we can show instead that the dual map $(T_a^\infty \circ f^*)'$ has a closed image.

$$(T_a^\infty)' : \mathbb{R}[[x-a]]' \to C^\infty(V)'$$

$\mathbb{R}[[x-a]]'$ is the space of all distributions with support a. Let $\sum \lambda_\beta \delta_a^{(\beta)}$ be such a distribution, and take any $\alpha \in C^\infty(V)$. Then

$$\begin{aligned}\Big\langle \alpha, (T_a^\infty \circ f^*)' \sum \lambda_\beta \delta_a^{(\beta)} \Big\rangle &= \Big\langle (T_a^\infty \circ f^*)(\alpha), \sum \lambda_\beta \delta_a^{(\beta)} \Big\rangle = \\ = \sum_\beta \lambda_\beta (\alpha \circ f)^{(\beta)}(a) &= \sum_\gamma \mu_\gamma \partial^\gamma \alpha(f(a)) = \Big\langle \alpha, \sum \mu_\gamma \delta_{f(a)}^{(\gamma)} \Big\rangle .\end{aligned}$$

So the image of $\mathbb{R}[[x-a]]'$ under $(T_a^\infty \circ f^*)'$ is contained in the space of all distributions concentrated at $f(a)$ which is isomorphic to a countable sum of $\mathbb{R}$ with the finest locally convex topology. But in this topology, every linear subspace is closed (since every quotient mapping is continuous), so $(T_a^\infty \circ f^*)'(\mathbb{R}[[x-a]]')$ is closed as well. □

Now let us state again

3.20. Lemma. *Consider $\lambda : \tilde{S} \xrightarrow{\tilde{\lambda}} S \hookrightarrow G.S$ as in the proof of 3.15, and define $\mu := \rho\,|_{G.S} \circ \lambda : \tilde{S} \to \mathbb{R}^k$. The σ_i form a minimal system of generators for $\mathbb{R}\left[\mathbb{R}^k\right]$ and we denote the germ of σ_i (resp. μ_i) by $\bar{\sigma}_i$ (resp. $\bar{\mu}_i$). Then there are germs of smooth functions $\bar{B}_i \in C_0^\infty(\mathbb{R}^k)$ such that*

$$\bar{\sigma}_j = \bar{B}_j(\bar{\mu}_1, \dots, \bar{\mu}_k).$$

Proof of lemma 3.20. Since μ is a G_p-invariant polynomial (or the restriction of one), we can express μ_i in the Hilbert generators as follows:

$$(*) \qquad \mu_i = \mu_i(0) + A_i(\sigma_1, \dots, \sigma_s) \qquad A_i \in \mathbb{R}[\mathbb{R}^s]$$

So our goal is to find a local inverse for A. With the help of Glaeser's lemma 3.19 let us now try to construct a formal power series inverse. λ induces an isomorphism by which

$$C^\infty(\tilde{S})^{G_p} = \lambda^* C^\infty(G.S)^G.$$

Without loss of generality let us now assume S was chosen compact. Then $G.S$ is compact as well and we can apply the Weierstrass approximation theorem to get

$$C^\infty(G.S)^G = \overline{\mathbb{R}[x]\,|_{G.S}}^{\,G} = \overline{\rho\,|_{G.S}^*\mathbb{R}[t]} = \overline{\rho\,|_{G.S}^* C^\infty(\mathbb{R}^k)}.$$

If we use the fact that λ^* is a homeomorphism, the two equations taken together yield

$$C^\infty(\tilde{S})^{G_p} = \lambda^*\overline{(\rho|G.S)^* C^\infty(\mathbb{R}^k)} = \overline{\lambda^*(\rho|G.S)^* C^\infty(\mathbb{R}^k)} = \overline{\mu^* C^\infty(\mathbb{R}^k)}.$$

So we have that $\sigma_i \in C^\infty(\tilde{S})^{G_p}$ is "almost" some smooth function of μ. Now we can use Glaeser's lemma. Take σ_i and $0 \in \tilde{S}$. Then there is a smooth function $\psi_i \in C^\infty(\mathbb{R}^k)$ such that

$$T_0^\infty \sigma_i = T_{\mu(0)}^\infty \psi_i \circ T_0^\infty \mu.$$

Since both σ_i and μ are polynomials, we can disregard the T_0^∞. $T_{\mu(0)}^\infty \psi_i$ is a power series in $(t - \mu(0))$. If we take $\varphi_i \in \mathbb{R}[t]$ to be the power series in t with the same coefficients, then the above formula turns into

$$(**) \qquad \sigma_i = \varphi_i(\mu - \mu(0)).$$

Since σ_i is homogeneous of degree > 0, φ_i has no constant term. So we can write it as

$$\varphi_i = L_i + \text{higher order terms} \qquad L_i \in \mathbb{R}[t]_1$$

In particular, if we insert (*) into (**) this implies

$$(***) \qquad \sigma_i - L_i(A_1(\sigma), \dots, A_k(\sigma)) \in (\mathbb{R}[t]_+^{G_p})^2.$$

Since the σ_i were chosen to be a minimal system of generators, lemma 3.18 implies that the $\sigma_i + (\mathbb{R}[t]_+^{G_p})^2$ form a basis of $\mathbb{R}[t]_+^{G_p}/(\mathbb{R}[t]_+^{G_p})^2$. Therefore we have a well defined algebra isomorphism:

$$\mathbb{R}[t]_+^{G_p}/(\mathbb{R}[t]_+^{G_p})^2 \xrightarrow{\cong} \mathcal{A} := \mathbb{R}[z_1,\dots,z_s]_+/\langle z^2\rangle$$
$$\sigma_i + (\mathbb{R}[t]_+^{G_p})^2 \mapsto [z_i]$$

Now (***) translated to $\mathcal{A}$ gives

$$L_i(A_1(z),\dots,A_k(z)) = z_i + O(z^2) \qquad \text{in } \mathbb{R}[z]$$

Therefore

$$DL(0)\circ DA(0) = Id_{\mathbb{R}^k},$$

and by the inverse function theorem A has a local inverse. So, locally, we can solve the equation (*) in terms of σ_i, which proves the lemma. □

This completes the proof of the key lemma. So far, we have shown (see remark 3.16) that under the hypotheses I and II

$$\phi^* C^\infty(\mathbb{R}^n;\{0\})^G = (id\times\rho\,|_{S^{n-1}})^*\, B^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\})$$

holds. We have been able to pull out ρ, but the polar coordinate transformation is now encoded in B. We must now pull the B^* out in front of the $(id\times\rho\,|_{S^{n-1}})^*$ where it will appear again as ϕ^* and then get rid of the excess dimension.

Recall that B was defined to satisfy the diagram:

$$\begin{CD}
\mathbb{R}\times S^{n-1} @>{\phi}>> \mathbb{R}^n\\
@V{id\times(\rho\,|_{S^{n-1}})}VV @VV{(r,\rho)}V\\
\mathbb{R}\times\mathbb{R}^k @>{B}>> \mathbb{R}\times\mathbb{R}^k
\end{CD}$$

where r denoted the polynomial map $r(x) = |\,x\,|^2$ on $\mathbb{R}^n$. Thus $B\circ(id\times\rho\,|_{S^{n-1}}) = (r,\rho)\circ\phi$. And therefore

$$\begin{aligned}\phi^* C^\infty(\mathbb{R}^n;\{0\})^G &= (id\times\rho\,|_{S^{n-1}})^*\, B^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}) =\\ &= \phi^*\circ(r,\rho)^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}).\end{aligned}$$

Since ϕ^* was injective, we can now discard it to get

$$C^\infty(\mathbb{R}^n;\{0\})^G = (r,\rho)^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}).$$

That takes care of B as well as ϕ, so let us now tackle r.

r is an $O(n)$-invariant polynomial, in particular it is G-invariant. Therefore by Hilbert:

$$r = \psi\circ\rho \qquad \text{for some } \psi \text{ in } C^\infty(\mathbb{R}^k).$$

So $(r,\rho) = (\psi, id)\circ\rho$ and we get

$$C^\infty(\mathbb{R}^n;\{0\})^G = \rho^*\circ(\psi,id)^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}).$$

Now we are just one easy lemma away from the desired result

$$C^\infty(\mathbb{R}^n;\{0\})^G = \rho^* C^\infty(\mathbb{R}^k;\{0\})$$

under hypotheses I and II. That is.

3.21. Lemma.
$$(\psi, id)^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}) = C^\infty(\mathbb{R}^k;\{0\})$$

Proof. Taking a closer look at (ψ, id), we see that it is a composition of maps

$$(\psi, id) : \mathbb{R}^k \underset{g}{\xrightarrow{\cong}} \operatorname{Graph}\psi \overset{i}{\hookrightarrow} \mathbb{R}\times\mathbb{R}^k$$

where i is the embedding of the closed submanifold $\operatorname{Graph}\psi$ into $\mathbb{R}\times\mathbb{R}^k$. Therefore

$$(\psi, id)^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}) = g^* i^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}).$$

Since $0 = r(0) = \psi\circ\rho(0) = \psi(0)$, we see that $g(0) = 0$. So we also have

$$C^\infty(\mathbb{R}^k;\{0\}) = g^* C^\infty(\operatorname{Graph}\psi;\{0\}).$$

Therefore it remains to prove that

$$i^* C^\infty(\mathbb{R}\times\mathbb{R}^k;\{0\}) = C^\infty(\operatorname{Graph}\psi;\{0\}).$$

Now take an arbitrary $f \in C^\infty(\operatorname{Graph}\psi;\{0\})$. There is a smooth extension $\tilde f$ of f on $\mathbb{R}\times\mathbb{R}^k$ but it need not be flat at zero. So consider a submanifold chart (ξ, U) of $\operatorname{Graph}\psi$ around 0 and define

$$f_U : U \xrightarrow{\xi} \mathbb{R}\times\mathbb{R}^k \xrightarrow{pr_2} \mathbb{R}^k \xrightarrow{(\psi,id)} \operatorname{Graph}\psi \xrightarrow{f} \mathbb{R}.$$

Then f_U is a smooth extension of f on U and is flat at zero. Now $\tilde f$ and f_U patched together with a suitable partition of unity give a function $\bar f \in C^\infty(\mathbb{R}\times\mathbb{R}^k;0)$ such that $i^*\bar f = f$. □

End of the Proof of 3.7. Recall from lemma 3.10 that it is sufficient to prove the theorem of Schwarz, assuming hypothesis II. We will now carry out induction over G. For $G = \{\}id$, 3.7 holds trivially. Now for any compact Lie group G satisfying hypothesis II we showed above that under the induction hypothesis (I)

$$\rho^* C^\infty(\mathbb{R}^k;\{0\}) = C^\infty(\mathbb{R}^n;\{0\})^G.$$

From this, together with our considerations from the beginning of the proof (3.11), we see that Schwarz' theorem is valid for G. □

There is one more Corollary to be gained from all of this. Notice that up to now we have not shown

$$(*)\qquad \rho^* C^\infty(\mathbb{R}^k;\{0\}) = C^\infty(\mathbb{R}^n;\{0\})^G$$

in general. Although we worked on this throughout the proof of 3.7, we were only able to show it under the hypotheses I and II. Now that Schwarz' theorem is proved, the hypothesis I is automatically satisfied so we can disregard it. But we have to look more deeply into the proof to be able to see whether (*) is satisfied for representations of compact Lie groups with more than one fixed point. It turns out that it is.

3.22. Corollary. *Let G be a compact Lie group with an orthogonal representation on $\mathbb{R}^n$ and $\rho = (\rho_1, \dots, \rho_k)$ the corresponding Hilbert generators, homogeneous and of positive degree. Then*

$$\rho^* C^\infty(\mathbb{R}^k; \{0\}) = C^\infty(\mathbb{R}^n; \{0\})^G.$$

Proof. Schwarz' theorem implies that

$$(\rho\,|_{S^{n-1}})^* \, C^\infty(\mathbb{R}^k) = C^\infty(S^{n-1})^G.$$

By backtracing we see that before we knew theorem 3.7 this was a consequence of the key lemma 3.15 which was based on the two hypotheses. In fact, it was the only assertion of 3.15 that was needed to prove the corollary 3.16. So we now know that 3.16 does not require the hypotheses after all. But the remainder of the proof for $\rho^* C^\infty(\mathbb{R}^k; \{0\}) = C^\infty(\mathbb{R}^n; \{0\})^G$ did not use 3.15 at all, it only used 3.16. Therefore, it is independent of the hypotheses as well. $\square$

Further results in this direction were obtained by Luna who, among other things, generalized the theorem of Schwarz to reductive Lie groups losing only the property of the Hilbert generators separating the orbits (see [20]).

Luna's Theorem (1976). *Consider a representation of a reductive Lie group G on $\mathbb{K}^m$ (where $\mathbb{K} = \mathbb{C}, \mathbb{R}$), and let $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{K}^m \to \mathbb{K}^n$, where $\sigma_1, \dots, \sigma_n$ generate the algebra $\mathbb{K}[\mathbb{K}^m]^G$. Then the following assertions hold:*

(1) $\mathbb{K} = \mathbb{C} \Rightarrow \sigma^* : \mathbb{H}(\mathbb{C}^n) \to \mathbb{H}(\mathbb{C}^m)^G$ *is surjective.*
(2) $\mathbb{K} = \mathbb{R} \Rightarrow \sigma^* : \mathbb{C}^\omega(\mathbb{R}^n) \to \mathbb{C}^\omega(\mathbb{R}^m)^G$ *is surjective.*
(3) $\mathbb{K} = \mathbb{R}$ *implies that*

$$\sigma^* : C^\infty(\mathbb{R}^n) \to \left\{ f \in C^\infty(\mathbb{R}^m)^G : f \text{ is constant on } \sigma^{-1}(y) \text{ for all } y \in \mathbb{R}^n \right\}$$

is surjective.

4. Transformation Groups

4.1. Definition. *Let G be a Lie group, M a C^∞-manifold. A smooth map $\ell : G\times M \to M$ (we will write $\ell_g(x), \ell^x(g)$ as well as $g.x$ for $\ell(g,x)$), defines a* **smooth action** *of G on M if it satisfies*

(1) $e.x = x$, *for all $x \in M$ where $e \in G$ is the unit element.*
(2) $(g_1 \cdot g_2).x = g_1.(g_2.x)$, *for all $g_1, g_2 \in G$, $x \in M$.*

We will also say G acts on M, M is a G-manifold or M is a smooth G-space.

4.2. Definition.

(1) *For $x \in M$ the set $G.x = \{g.x : g \in G\}$ is called the G-orbit through x.*
(2) *A G-action on M is called* **transitive** *if the whole of M is one G-orbit.*
(3) *A G-action on M is called* **effective** *if the homorphism $G \to \operatorname{Diff}(M)$ into the diffeomorphism group is injective: If $g.x = x$ for all $x \in M$ then $g = e$.*
(4) *A G-action on M is called* **free** *if $\ell^x : G \to M$ is injective for each $x \in M$: $g.x = x$ for one $x \in M$ already implies $g = e$.*
(5) *A G-action on M is called* **infinitesimally free** *if $T_e(\ell^x) : \mathfrak{g} \to T_xM$ is injective for each $x \in M$.*
(6) *A G-action on M is called* **infinitesimally transitive** *if $T_e(\ell^x) : \mathfrak{g} \to T_xM$ is surjective for each $x \in M$.*
(7) *A G-action on M is called* **linear** *if M is a vector space and the action defines a representation.*
(8) *A G-action on M is called* **affine** *if M is an affine space, and every $\ell_g : M \to M$ is an affine map.*
(9) *A G-action on M is called* **orthogonal** *if (M, γ) is a Euclidean vector space and $\ell_g \in O(M, \gamma)$ for all $g \in G$. (Then $\{\ell_g : g \in G\} \subseteq O(M, \gamma)$ is automatically a subgroup).*
(10) *A G-action on M is called* **isometric** *if (M, γ) is a Riemannian manifold and ℓ_g is an isometry for all $g \in G$.*
(11) *A G-action on M is called* **symplectic** *if (M, ω) is a symplectic manifold and ℓ_g is a symplectomorphism for all $g \in G$ (i.e. ℓ_g^* preserves ω).*
(12) *A G-action on M is called a* **principal fiber bundle action** *if it is free and if the projection onto the orbit space $\pi : M \to M/G$ is a principal fiber bundle. This means that that M/G is a smooth manifold, and π is a submersion. By ther implicit function theorem there exit then local sections, and the inverse function theorem the mapping $\tau : M \times_{M/G} M \to G$ which satisfies $x = \tau(x, y).y$ for x and y in the same orbit, is smooth. This is a central notion of differential geometry.*

4.3. Definition. *If M is a G-manifold, then M/G, the space of all G-orbits endowed with the quotient topology, is called the* **orbit space**.

4.4. Examples.

(1) The standard action of $O(n)$ on $\mathbb{R}^n$ is orthogonal. The orbits are the concentric spheres around the fixed point 0 and 0 itself. The orbit space is $\mathbb{R}^n/O(n) \cong [0, \infty)$.

(2) Every Lie group acts on itself by conjugation: $\operatorname{conj} : G \times G \to G$ is defined by $(g, h) \mapsto \operatorname{conj}_g(h) := g.h.g^{-1}$ and is a smooth action of the Lie group on itself.

(3) The adjoint action $Ad : G \to GL(\mathfrak{g})$ of the Lie group G on its Lie algebra $\mathfrak{g}$ is defined as the derivative of conj (interpreted as a map $G \to \operatorname{Aut}(G)$)

$$Ad(g) : X \mapsto \left.\frac{d}{dt}\right|_{t=0} g.\exp^G(tX).g^{-1} = T_e(\operatorname{conj}_g) : \mathfrak{g} \to \mathfrak{g}$$

It is clearly linear. If G is compact, then it is orthogonal with respect to the negative Cartan-Killing form,

$$-B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} : (X, Y) \mapsto -\operatorname{tr}(ad(X) \circ ad(Y)),$$

which in this case defines an inner product on $\mathfrak{g}$.

(4) In particular, the orthogonal group acts orthogonally on $\mathfrak{o}(n)$, the Lie algebra of all antisymmetric $n \times n$-matrices. Not a special case of (3) is the $O(n)$-action on $S(n)$ defined in chapter 1. Yet it is also orthogonal: Let $A \in O(n)$ act on $G, H \in S(n)$ then

$$\begin{aligned} \operatorname{tr}\left(AHA^{-1}(AGA^{-1})^t\right) &= \operatorname{tr}(AHA^{-1}(A^{-1})^t G^t A^t) = \\ \operatorname{tr}(AHA^{-1}AG^tA^{-1}) &= \operatorname{tr}(AHG^tA^{-1}) = \operatorname{tr}(HG^t) \end{aligned}$$

(5) $SU(n)$ acts unitarily on the hermitian $n \times n$ matrices by conjugation (analogous to (4)).

4.5. Definition.
Let M be a G-manifold, then the closed subgroup $G_x = \{g \in G : g.x = x\}$ of G is called the **isotropy subgroup** *of x.*

Remark. The map $i : G/G_x \to M$ defined by $i : g.G_x \mapsto g.x \in M$ is a G-equivariant initial immersion with image $G.x$. [19], Theorem 5.14

$$\begin{tikzcd} G \arrow[r, "\ell^x"] \arrow[d, two heads, "p"'] & M \\ G/G_x \arrow[ur, "i"'] & \end{tikzcd}$$

If G is compact, then clearly i is an embedding.

4.6. Lemma. *Let M be a G-manifold and $x, y \in M$, then*

(1) $G_{gx} = g.G_x.g^{-1}$
(2) $G.x \cap G.y \neq \emptyset \Rightarrow G.x = G.y$
(3) $T_x(G.x) = T_e(\ell^x).\mathfrak{g}$

Proof.

(1) $a \in G_{gx} \Leftrightarrow ag.x = g.x \Leftrightarrow g^{-1}ag.x = x \iff g^{-1}ag \in G_x \Leftrightarrow a \in g\,G_x g^{-1}$.
(2) $z \in G.x \cap G.y \Rightarrow z = g_1.x = g_2.y \Rightarrow x = g_1^{-1}g_2 y =: g.y$, therefore $G.x = G.(g.y) = G.y$.
(3) $X \in T_x(G.x) \Leftrightarrow X = \frac{d}{dt}\big|_{t=0} c(t)$ for some smooth curve $c(t) = g_t.x \in G.x$ with $g_0 = e$. So we have $X = \frac{d}{dt}\big|_{t=0} \ell^x(g_t) \in T_e(\ell^x).\mathfrak{g}$. $\square$

4.7. Conjugacy Classes. The closed subgroups of G can be partitioned into equivalence classes by the following relation:

$$H \sim H' \quad :\iff \quad \exists\, g \in G \text{ for which } H = gH'g^{-1}$$

The equivalence class of H is denoted by (H).
First consequence: (with lemma 4.6(1)) The conjugacy class of an isotropy subgroup is invariant under the action of G : $(G_x) = (G_{gx})$. Therefore we can assign to each orbit $G.x$ the conjugacy class (G_x). We will call (G_x) the **isotropy type** of the orbit through x, and two orbits are said to be of the same type, if they have the same isotropy type.

If G is compact, we can define a partial ordering on the conjugacy classes simply by transferring the usual partial ordering "$\subseteq$" on the subgroups to the classes:

$$(H) \leq (H') \quad :\iff \quad \exists\, K \in (H), K' \in (H') : K \subseteq K'$$

This is equivalent to a shorter definition:

$$(H) \leq (H') \quad :\iff \quad \exists\, g \in G : H \subset gH'g^{-1}$$

If G is not compact this relation may not be antisymmetric. For compact G the antisymmetry of this relation is a consequence of the following

4.8. Lemma [5], **1.9.** *Let G be a compact Lie group, H a closed subgroup of G, then*

$$gHg^{-1} \subseteq H \quad \implies \quad gHg^{-1} = H$$

Proof. By iteration, $gHg^{-1} \subseteq H$ implies $g^nHg^{-n} \subseteq H$ for all $n \in \mathbb{N}$. Now let us study the set $A := \{g^n : n \in \mathbb{N}_0\}$. We will show that g^{-1} is contained in its closure.

Suppose first that e is an accumulation point of $\bar{A}$. Then for any neighborhood U of e there is a $g^n \in U$ where $n > 0$. This implies $g^{n-1} \in g^{-1}U \cap A$. Since the sets $g^{-1}U$ form a neighborhood basis of g^{-1}, we see that g^{-1} is an accumulation point of A as well. That is, $g^{-1} \in \bar{A}$.

Now suppose that e is discrete in $\bar{A}$. Then since G is compact, A is finite. Therefore $g^n = e$ for some $n > 0$, and $g^{n-1} = g^{-1} \in A$.

Since $\operatorname{conj} : G \times G \to G$ is continuous and H is closed, we have

$$\operatorname{conj}(\bar{A}, H) \subseteq H.$$

In particular, $g^{-1}Hg \subseteq H$ which together with our premise implies that $gHg^{-1} = H$. $\square$

4.9. Definition. *Let M and N be G-manifolds. A smooth map $f : M \to N$ is called* **equivariant**, *if it satisfies $f(g.x) = g.f(x)$ for all x in M and g in G.*

4.10. Definition. *Let M be a G-manifold. The orbit $G.x$ is called* **principal orbit**, *if there is an invariant open neighborhood U of x in M and for all $y \in U$ an equivariant map $f : G.x \to G.y$.*

Remark.

(1) The equivariant map $f : G.x \to G.y$ of the definition is automatically surjective :
Let $f(x) =: a.y$. For an arbitrary $z = g.y \in G.y$ this gives us
$z = g.y = ga^{-1}a.y = ga^{-1}f(x) = f(ga^{-1}.x)$.

(2) The existence of f in the above definition is equivalent to the condition : $G_x \subseteq aG_ya^{-1}$ for some $a \in G$:
$(\Rightarrow)$ $g \in G_x \Rightarrow g.x = x \Rightarrow g.f(x) = f(g.x) = f(x)$ and for $f(x) =: a.y$ this implies $ga.y = a.y \Rightarrow g \in G_{ay} = aG_ya^{-1}$(by 4.6(1)).
$(\Leftarrow)$ Define $f : G.x \to G.y$ explicitly by $f(g.x) := ga.y$. Then we have to check that, if $g_1.x = g_2.x$ i.e. $g := g_2^{-1}g_1 \in G_x$, then $g_1a.y = g_2a.y$ or $g \in G_{ay} = aG_ya^{-1}$. This is guaranteed by our assumption.

(3) We call $x \in M$ a **regular point** if $G.x$ is a principal orbit. Otherwise, x is called **singular**. The subset of all regular (singular) points in M is denoted by M_{reg} (M_{sing}).

4.11. Definition. *Let M be a G-manifold and $x \in M$ then a subset $S \subseteq M$ is called a* **slice** *at x, if there is a G-invariant open neighborhood U of $G.x$ and a smooth equivariant retraction $r : U \to G.x$ such that $S = r^{-1}(x)$.*

4.12. Proposition. *If M is a G-manifold and $S = r^{-1}(x)$ a slice at $x \in M$, where $r : U \to G.x$ is the corresponding retraction, then*

(1) *$x \in S$ and $G_x.S \subseteq S$*
(2) *$g.S \cap S \neq \emptyset \Rightarrow g \in G_x$*
(3) *$G.S = \{g.s : g \in G, s \in S\} = U$*

Proof.

(1) $x \in S$ is clear, since $S = r^{-1}(x)$ and $r(x) = x$. To show that $G_x.S \subseteq S$, take an $s \in S$ and $g \in G_x$. Then $r(g.s) = g.r(s) = g.x = x$, and therefore $g.s \in r^{-1}(x) = S$.

(2) $g.S \cap S \neq \emptyset \Rightarrow g.s \in S$ for some $s \in S \Rightarrow x = r(g.s) = g.r(s) = g.x \Rightarrow g \in G_x$.

(3) $(\subseteq)$ Since r is defined on U only, and U is G-invariant, $G.S = G.r^{-1}(x) \subseteq G.U = U$.
$(\supseteq)$ Consider $y \in U$ with $r(y) = g.x$, then $y = g.(g^{-1}.y)$ and $g^{-1}.y \in S$, since $r(g^{-1}.y) = g^{-1}.r(y) = g^{-1}g.x = x$ so $y \in G.S$. □

4.13. Corollary. *If M is a G-manifold and S a slice at $x \in M$, then*

(1) *S is a G_x-manifold.*
(2) *$G_s \subseteq G_x$ for all $s \in S$.*

(3) *If $G.x$ is a principal orbit and G_x compact, then $G_y = G_x$ for all $y \in S$ if the slice S at x is chosen small enough. In other words, all orbits near $G.x$ are principal as well.*
(4) *If two G_x-orbits $G_x.s_1, G_x.s_2$ in S have the same orbit type as G_x-orbits in S, then $G.s_1$ and $G.s_2$ have the same orbit type as G-orbits in M.*
(5) *$S/G_x \cong G.S/G$ is an open neighborhood of $G.x$ in the orbit space M/G.*

Proof.
(1) This is is clear from 4.12(1).
(2) $g \in G_y \Rightarrow g.y = y \in S \Rightarrow g \in G_x$ by 4.12(2).
(3) By (2) we have $G_y \subseteq G_x$, so G_y is compact as well. Because $G.x$ is principal it follows that for $y \in S$ close to x, G_x is conjugate to a subgroup of G_y, $G_y \subseteq G_x \subseteq g.G_y g^{-1}$. Since G_y is compact, $G_y \subseteq g.G_y g^{-1}$ implies $G_y = g.G_y g^{-1}$ by 4.8. Therefore $G_y = G_x$, and $G.y$ is also a principal orbit.
(4) For any $s \in S$ it holds that $(G_x)_s = G_s$, since $(G_x)_s \subseteq G_s$, and, conversely, by (2), $G_s \subseteq G_x$, therefore $G_s \subseteq (G_x)_s$. So $(G_x)_{s_1} = g(G_x)_{s_2} g^{-1}$ implies $G_{s_1} = gG_{s_2}g^{-1}$ and the G-orbits have the same orbit type.
(5) The isomorphism $S/G_x \cong G.S/G$ is given by the map $G_x.s \mapsto G.s$ (it is an injection by 4.12(2)). Since $G.S = U$ is an open G-invariant neighborhood of $G.x$ in M (4.12(3)), we have $G.S/G$ is an open neighborhood of $G.x$ in M/G. $\square$

4.14. Remark. The converse to 4.13(4) is generally false. If the two G-orbits $G.s_1$, $G.s_2$ are of the same type, then the isotropy groups G_{s_1} and G_{s_2} are conjugate in G. They need not be conjugate in G_x. For example, consider the compact Lie group $G := (S^1 \times S^1) \circledS \mathbb{Z}_2$ with multiplication "$\circ$" defined as follows. Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in S^1$ and $\alpha, \beta \in \mathbb{Z}_2$. Take on $S^1 \times S^1$ the usual multiplication by components, and as $\mathbb{Z}_2$-action:

$$\begin{aligned} i :& \bar{0} \mapsto i_0 := id_{S^1 \times S^1} \\ & \bar{1} \mapsto (i_1 : (\varphi_1, \varphi_2) \mapsto (\varphi_2, \varphi_1)). \end{aligned}$$

Then

$$(\varphi_1, \varphi_2, \alpha) \circ (\psi_1, \psi_2, \beta) := ((\varphi_1, \varphi_2).i_\alpha(\psi_1, \psi_2), \alpha + \beta)$$

shall give the multiplication on $(S^1 \times S^1) \circledS \mathbb{Z}_2$.
Now we let G act on $M := V \sqcup W$ where $V = W = \mathbb{R}^2 \times \mathbb{R}^2$. For any element in M we will indicate its connected component by the index $(x, y)_V$ or $(x, y)_W$. The action shall be the following

$$\begin{aligned} (\varphi_1, \varphi_2, \bar{0}).(x, y)_V &:= (\varphi_1.x, \varphi_2.y)_V \\ (\varphi_1, \varphi_2, \bar{1}).(x, y)_V &:= (\varphi_1.y, \varphi_2.x)_W \end{aligned}$$

The action on W is simply given by interchanging the V's and W's in the above formulae. This really defines an action as can be verified directly, for example,

$$\begin{aligned} (\varphi_1, \varphi_2, \bar{1}).((\psi_1, \psi_2, \bar{1}).(x, y)_V) &= (\varphi_1, \varphi_2, \bar{1}).(\psi_1.y, \psi_2.x)_W \\ &= (\varphi_1\psi_2.x, \varphi_2\psi_1.y)_V = (\varphi_1\psi_2, \varphi_2\psi_1, \bar{0})(x, y)_V \\ &= ((\varphi_1, \varphi_2, \bar{1}) \circ (\psi_1, \psi_2, \bar{1})).(x, y)_V. \end{aligned}$$

Denote by H the abelian subgroup $S^1 \times S^1 \times \{\bar{0}\}$. H is the isotropy subgroup of $(0,0)_V$, and V is a slice at $(0,0)_V$. Now consider $s_1 := (0, v^1)_V$ and $s_2 := (v^2, 0)_V$, both not equal to zero. Then let

$$\begin{aligned} H_1 &:= G_{s_1} = S^1 \times \{id\} \times \{\bar{0}\} \\ H_2 &:= G_{s_2} = \{id\} \times S^1 \times \{\bar{0}\} \end{aligned}$$

H_1 and H_2 are conjugate in G by $c = (id, id, \bar{1})$:

$$H_1 \circ c \ni (\varphi, id, \bar{0}) \circ c = (\varphi, id, \bar{1}) = c \circ (id, \varphi, \bar{0}) \in c \circ H_2$$

Yet they are clearly not conjugate in H since H is abelian. So $H.s_1$ and $H.s_2$ have different orbit types in H while $G.s_1$ and $G.s_2$ are of the same G-orbit type.

4.15. Proposition. *Let M be a G-manifold and S a slice at x, then there is a G-equivariant diffeomorphism of the associated bundle $G[S]$ onto $G.S$,*

$$f : G[S] = G \times_{G_x} S \to G.S$$

which maps the zero section $G \times_{G_x} \{x\}$ onto $G.x$.

Proof. Since $\ell(gh, h^{-1}.s) = g.s = \ell(g,s)$ for all $h \in G_x$, there is an $f : G[S] \to G.S$ such that the diagram below commutes.

$$\begin{tikzcd} G \times S \arrow[r, "\ell"] \arrow[d, "q"'] & G.S \\ G \times_{G_x} S \arrow[ur, "f"] & \end{tikzcd}$$

f is smooth because $f \circ q = \ell$ is smooth and q is a submersion. It is equivariant since ℓ and q are equivariant. Also, f maps the zero section $G \times_{G_x} \{x\}$ onto $G.x$. It remains to show that f is a diffeomorphism. f is bijective, since with 4.12(2)

$$\begin{aligned} &g_1.s_1 = g_2.s_2 \quad \iff \quad s_1 = g_1^{-1} g_2.s_2 \quad \iff \\ &g_1 = g_2 h^{-1} \text{ and } s_1 = h.s_2 \text{ for } h = g_1^{-1} g_2 \in G_x \end{aligned}$$

and this is equivalent to

$$q(g_1, s_1) = q(g_2, s_2).$$

To see that f is a diffeomorphism let us prove that the rank of f equals the dimension of M. First of all, note that

$$\begin{aligned} &\operatorname{Rank}(\ell_g) = \dim(g.S) = \dim S \\ &\text{and} \quad \operatorname{Rank}(\ell^x) = \dim(G.x) \end{aligned}$$

Since $S = r^{-1}(x)$ and $r : G.S \to G.x$ is a submersion ($r\,|_{G.x} = id$) it follows that $\dim(G.x) = \operatorname{codim} S$. Therefore,

$$\begin{aligned} &\operatorname{Rank}(f) = \operatorname{Rank}(\ell) = \operatorname{Rank}(\ell_g) + \operatorname{Rank}(\ell^x) = \\ &\dim S + \dim(G.x) = \dim S + \operatorname{codim} S = \dim M. \end{aligned}$$

□

4.16. Remark. The converse also holds. If $\bar f : G \times_{G_x} S \to G.S$ is a G-equivariant diffeomorphism, then for some $\bar g \in G, \bar s \in S$, $\bar f[\bar g, \bar s] = x$. So $f[g,s] := \bar f[g\bar g, s]$ defines a G-equivariant diffeomorphism with the additional property that $x = f[e, \bar s]$.

$$\begin{array}{ccc}
G \times_{G_x} S & \xrightarrow{f} & G.S \\
{\scriptstyle pr_1}\downarrow & & {\scriptstyle r}\downarrow \\
G/G_x & \xrightarrow{i} & G.x
\end{array}$$

If we define $r := i \circ pr_1 \circ f^{-1} : G.S \to G.x$, then r is again a smooth G-equivariant map, and it is a retraction onto $G.x$ since

$$x \xrightarrow{f^{-1}} [e, \bar s] \xrightarrow{pr_1} e.G_x \xrightarrow{i} e.x.$$

Furthermore, $r^{-1}(x) = S$ making S a slice.

5. Proper Actions

In this section we describe and characterize "proper" actions of Lie groups. We will see that the following definition is tailored to generalize compact Lie group actions while retaining many of their nice properties.

5.1. Definition. *A smooth action $\ell : G \times M \to M$ is called* **proper** *if it satisfies one of the following three equivalent conditions:*

(1) $(\ell, id) : G \times M \to M \times M$, $(g,x) \mapsto (g.x, x)$, *is a proper mapping*
(2) $g_n.x_n \to y$ *and* $x_n \to x$ *in* M, *for some* $g_n \in G$ *and* $x_n, x, y \in M$, *implies that these* g_n *have a convergent subsequence in* G.
(3) K *and* L *compact in* M *implies that* $\{g \in G : g.K \cap L \neq \emptyset\}$ *is compact as well.*

Proof.
(1) $\Rightarrow$ (2) is a direct consequence of the definitions.
(2) $\Rightarrow$ (3): Let g_n be a sequence in $\{g \in G : g.K \cap L \neq \emptyset\}$ and $x_n \in K$ such that $g_n.x_n \in L$. Since K is compact, we can choose a convergent subsequence $x_{n_k} \to x \in K$ of x_n. Since L is compact we can do the same for $g_{n_k}.x_{n_k}$ there. Now (2) tells us that in such a case g_n must have a convergent subsequence, therefore $\{g \in G : g.K \cap L \neq \emptyset\}$ is compact.
(3) $\Rightarrow$ (1): Let R be a compact subset of $M \times M$. Then $L := \mathrm{pr}_1(R)$ and $K := \mathrm{pr}_2(R)$ are compact, and $(\ell, id)^{-1}(R) \subseteq \{g \in G : g.K \cap L \neq \emptyset\} \times K$. By (3), $\{g \in G : g.K \cap L \neq \emptyset\}$ is compact. Therefore $(\ell, id)^{-1}(R)$ is compact, and (ℓ, id) is proper. $\square$

5.2. Remark. If G is compact, then every G-action is proper. If $\ell : G \times M \to M$ is a proper action and G is not compact, then for any unbounded $H \subseteq G$ and $x \in M$ the set $H.x$ is unbounded in M. Furthermore, all isotropy groups are compact (most easily seen from 5.1(3) by setting $K = L = \{x\}$).

5.3. Lemma. *A continuous, proper map $f : X \to Y$ between two topological spaces is closed.*

Proof. Consider a closed subset $A \subseteq X$, and take a point y in the closure of $f(A)$. Let $f(a_n) \in f(A)$ converge to y ($a_n \in A$). Then the $f(a_n)$ are contained in a bounded subset $B \subseteq f(A)$. Therefore $a_n \subseteq f^{-1}(B) \cap A$ which is now, since f is proper, a bounded subset of A. Consequently, (a_n) has a convergent subsequence with limit $a \in A$, and by continuity of f, it gives a convergent subsequence of $f(a_n)$ with limit $f(a) \in f(A)$. Since $f(a_n)$ converges to y, we have $y = f(a) \in f(A)$. $\square$

5.4. Proposition. *The orbits of a proper action $\ell : G \times M \to M$ are closed submanifolds.*

Proof. By the preceding lemma, (ℓ, id) is closed. Therefore $(\ell, id)(G, x) = G.x \times \{x\}$, and with it $G.x$ is closed. Next let us show that $\ell^x : G \to G.x$ is an open mapping.

Since ℓ^x is G-equivariant, we only have to show for a neighborhood U of e that $\ell^x(U) = U.x$ is a neighborhood of x. Let us assume the contrary: there is a sequence $g_n.x \in G.x - U.x$ which converges to x. Then by 5.1(2), g_n has a convergent subsequence with limit $g \in G_x$. On the other hand, since $g_n.x \notin U.x = U.G_x.x$ we have $g_n \notin U.G_x$, and, since $U.G_x$ is open, we have $g \notin U.G_x$ as well. This contradicts $g \in G_x$.

Now we see that the orbits of a proper action are closed submanifolds.

$$\begin{tikzcd} G \arrow[rr, "\ell^x"] \arrow[dr, "p"'] & & G.x \\ & G/G_x \arrow[ur, "i"'] & \end{tikzcd}$$

As the integral manifold of fundamental vector fields, $G.x$ is an initial submanifold, and i is an injective immersion [19], Theorem 5.14. Since $i \circ p = \ell^x$ is open, i is open as well. Therefore it is a homeomorphism, and $G.x$ is an embedded submanifold of M. $\square$

5.5. Lemma. *Let (M, γ) be a Riemannian manifold and $\ell : G \times M \to M$ an effective isometric action (i.e. $g.x = x$ for all $x \in M \Rightarrow g = e$), such that $\ell(G) \subseteq \mathrm{Isom}(M, \gamma)$ is closed in the compact open topology. Then ℓ is proper.*

Proof. Let $g_n \in G$ and $x_n, x, y \in M$ such that $g_n.x_n \to y$ and $x_n \to x$ then we have to show that g_n has a convergent subsequence which is the same as proving that $\{g_n : n \in \mathbb{N}\}$ is relatively compact, since $\ell(G) \subseteq \mathrm{Isom}(M, \gamma)$ is closed.

Let us choose a compact neighborhood K of x in M. Then, since the g_n act isometrically, we can find a compact neighborhood $L \subseteq M$ of y such that $\bigcup_{n=1}^{\infty} g_n.K$ is contained in L. So $\{g_n\}$ is bounded. Furthermore, the set of all g_n is equicontinuous as subset of $\mathrm{Isom}(M)$. Therefore, by the theorem of Ascoli-Arzela, $\{g_n : n \in \mathbb{N}\}$ is relatively compact. $\square$

5.6. Theorem (Existence of Slices). [31], *1961*
Let M be a G-space, and $x \in M$ a point with compact isotropy group G_x. If for all open neighborhoods W of G_x in G there is a neighborhood V of x in M such that $\{g \in G : g.V \cap V \neq \emptyset\} \subseteq W$, then there exists a slice at x.

Proof. Let $\tilde{\gamma}$ be any Riemann metric on M. Since G_x is compact, we can get a G_x-invariant metric by integrating over the Haar-measure for the action of G_x.

$$\gamma_x(X, Y) := \int_{G_x} (\ell_a^* \tilde{\gamma})(X, Y) da = \int_{G_x} \tilde{\gamma}(T\ell_a X, T\ell_a Y) da$$

Now if we choose $\varepsilon > 0$ small enough for $\exp_x^\gamma : T_xM \supseteq B_{0_x}(\varepsilon) \to M$ to be a diffeomorphism onto its image, we can define:

$$\tilde{S} := \exp_x^\gamma \left(T_x(G.x)^\perp \cap B_{0_x}(\varepsilon) \right) \subseteq M.$$

$\tilde{S}$ is a submanifold of M and the first step towards obtaining a real slice. Let us show that $\tilde{S}$ is G_x-invariant. Since G_x leaves γ unchanged and $T_x(G.x)$ is invariant under $T_x\ell_g$ (for $g \in G_x$), $T_x\ell_g$ is an isometry and leaves $T_x(G.x)^\perp \cap B_{0_x}(\varepsilon)$ invariant. Therefore:

$$\begin{array}{ccc} T_x(G.x)^\perp \cap B_{0_x}(\varepsilon) & \xrightarrow{T_x\ell_g} & T_x(G.x)^\perp \cap B_{0_x}(\varepsilon) \\ \Big\downarrow{\scriptstyle \exp_x^\gamma} & & \Big\downarrow{\scriptstyle \exp_x^\gamma} \\ \tilde{S} & \xrightarrow{\ell_g} & \tilde{S} \end{array}$$

What is not necessarily true for $\tilde{S}$ is that any $g \in G$ which maps some $s \in \tilde{S}$ back into $\tilde{S}$ is automatically in G_x. This property is necessary for a slice, and we will now try to attain it for a G_x-invariant subset $S \subseteq \tilde{S}$. At this point, the condition that for every open neighborhood W of G_x in G, there is a neighborhood V of x in M such that $\{g \in G : g.V \cap V \neq \emptyset\} \subseteq W$ comes in. The idea is to find a suitable W and corresponding V such that $V \cap \tilde{S}$ has the desired property.

First we must construct a W fitting our purposes. Choose an open neighborhood $U \subseteq G/G_x$ of $e.G_x$ such that there is a smooth section $\chi : U \to G$ of $\pi : G \to G/G_x$ with $\chi(e.G_x) = e$. And let U and possibly $\tilde{S}$ be small enough for us to get an embedding

$$f : U \times \tilde{S} \to M : (u, s) \mapsto \chi(u).s.$$

Our neighborhood of G_x will be $W := \pi^{-1}(U)$. Now by our assumption, there is a neighborhood V of x in M such that $\{g \in G : g.V \cap V \neq \emptyset\} \subseteq W$.

Next we will prove that V can be chosen G_x-invariant. Suppose we can choose an open neighborhood $\tilde{W}$ of G_x in G such that $G_x.\tilde{W} \subseteq W$ (we will prove this below). Then let V' be the neighborhood of x in M satisfying $\{g \in G : g.V' \cap V' \neq \emptyset\} \subseteq \tilde{W}$. Now $V := G_x.V'$ has the desired property, since:

$$\begin{aligned} \{g \in G : g.G_x.V' \cap G_x.V' \neq \emptyset\} &= \bigcup_{g_1, g_2 \in G_x} \{g \in G : g.g_1.V' \cap g_2.V' \neq \emptyset\} = \\ \bigcup_{g_1, g_2 \in G_x} \{g \in G : g_2^{-1} g g_1.V' \cap V' \neq \emptyset\} &= \bigcup_{g_1, g_2 \in G_x} g_2 \{g \in G : g.V' \cap V' \neq \emptyset\} g_1^{-1} = \\ &G_x.\{g \in G : g.V' \cap V' \neq \emptyset\}.G_x \subseteq G_x.\tilde{W}.G_x \subseteq W.G_x \subseteq W \end{aligned}$$

To complete the above argumentation, we have left to prove the

Claim: To any open neighborhood W of G_x in G there is an open neighborhood $\tilde{W}$ of G_x such that $G_x.\tilde{W} \subseteq W$.

Proof: The proof relies on the compactness of G_x. Choose for all $(a, b) \in G_x \times G_x$ neighborhoods $A_{a,b}$ of a and $B_{a,b}$ of b, such that $A_{a,b}.B_{a,b} \subseteq W$. This is possible by continuity, since $G_x.G_x = G_x$. $\{B_{a,b} : b \in G_x\}$ is an open covering of G_x. Then since G_x is compact, there is a finite subcovering $\bigcup_{j=1}^N B_{a,b_j} := B_a \supseteq G_x$. Since $A_{a,b_j}.B_{a,b_j} \subseteq W$ we must choose $A_a := \bigcap_{j=1}^N A_{a,b_j}$, to get $A_a.B_a \subseteq W$. Now since A_a is a neighborhood of a in G_x, the A_a cover G_x again. Consider a finite subcovering $A := \bigcup_{j=1}^n A_{a_j} \supseteq G_x$, and as before define $B := \bigcap_{j=1}^n B_{a_j}$, so that $A.B \subseteq W$. In particular, this gives us $G_x.B \subseteq W$, so $\tilde{W} := B$ is an open neighborhood of G_x with the desired property.

We have found a G_x-invariant neighborhood V of x, with $\{g \in G : gV \cap V \neq \emptyset\}$ contained in W. Now we define $S := \tilde{S} \cap V$ and hope for the best. S is an open subset of $\tilde{S}$, and it is again invariant under G_x. Let us check whether we have the converse: $\{g \in G : g.S \cap S \neq \emptyset\} \subseteq G_x$. If $g.s_1 = s_2$ for some $s_1, s_2 \in S$, then $g \in W = \pi^{-1}(U)$ by the above effort. Therefore $\pi(g) \in U$. Choose $h = g^{-1}\chi(\pi(g)) \in G_x$. Then

$$f(\pi(g), h^{-1}s_1) = \chi(\pi(g))h^{-1}s_1 = g.s_1 = s_2 = f(\pi(e), s_2).$$

Since f is a diffeomorphism onto its image, we have shown that $\pi(g) = \pi(e)$, that is $g \in G_x$.

Now, it is easy to see that $F : G \times_{G_x} S \to G.S : [g, s] \mapsto g.s$ is well defined, G-equivariant and smooth. We have the diagram

$$\begin{tikzcd} G \times S \arrow[rr, "\ell"] \arrow[dr, "q"'] & & G.S \\ & G \times_{G_x} S \arrow[ur, "F"'] & \end{tikzcd}$$

To finish the proof, we have to show that F is a diffeomorphism (4.16). F is injective because:

$$\begin{aligned} &F[g, s] = F[g', s'] \Rightarrow g.s = g'.s' \Rightarrow g^{-1}g'.s' = s \\ &\Rightarrow g^{-1}g' \in G_x \Rightarrow [g, s] = [g, g^{-1}g'.s'] = [g', s'] \end{aligned}$$

Next, we notice that $\ell(W, S) = W.S = f(U, S)$ is open in M since $f : U \times \tilde{S} \to M$ is an embedding with an open image. Consequently, $G.S = \ell(G, W.S)$ is open, since ℓ is open, and F is a diffeomorphism. $\square$

5.7. Theorem. *If M is a proper G-manifold, then for all $x \in M$ the conditions of the previous theorem are satisfied, so each x has slices.*

Proof. We have already shown that G_x is compact (5.2(2)). Now for every neighborhood U of G_x in G, for every $x \in M$, it remains to find a neighborhood V of x in M such that

$$\{g \in G : g.V \cap V \neq \emptyset\} \subseteq U.$$

Claim: U contains an open neighborhood $\tilde{U}$ with $G_x\tilde{U} = \tilde{U}$ (so we will be able to assume $G_xU = U$ without loss of generality).

In the proof of theorem 5.6 we showed the existence of a neighborhood B of G_x such that $G_x.B \subseteq U$, using only the compactness of G_x. So $\tilde{U} := G_x.B = \bigcup_{g \in G_x} g.B$ is again an open neighborhood of G_x, and it has the desired properties.

Now we can suppose $U = G_x.U$. Next, we have to construct an open neighborhood $V \subseteq M$ of x, such that $\{g \in G : g.V \cap V \neq \emptyset\} \subseteq U$. This is the same as saying $(G - U).V \cap V$ should be empty. So we have to look for V in the complement of $(G - U).x$.

First we have to check that $M - ((G - U).x)$ really contains an open neighborhood of x. Upon closer inspection, we see that $M - ((G - U).x)$ is open altogether, or rather that $(G-U).x$ is closed. This is because $(G-U).x \times \{x\} = (\ell, id)((G-U) \times \{x\})$ is the image of a closed set under (ℓ, id) which is a closed mapping by lemma 5.3.

Now let us choose a compact neighborhood W of x in $M - ((G-U).x)$. Then since G acts properly, it follows that $\{g \in G : g.W \cap W \neq \emptyset\}$ is compact, in particular $K := \{g \in G - U : g.W \cap W \neq \emptyset\}$ is compact. But what we need is for $\{g \in G - U : g.V \cap V \neq \emptyset\}$ to be empty. An x-neighborhood V contained in W fulfills this, if $K.V \subseteq M - W$. Let us find such a neighborhood.

Our choice of W guarantees $K.x \subseteq M - W$. But $M - W$ is open, therefore for each $k \in K$ we can choose a neighborhood Q_k of k in G and V_k of x in W, such that $Q_k.V_k \subseteq M - W$. The neighborhoods Q_k cover K, and we can choose a finite subcovering $\bigcup_{j=1}^m Q_j$. Then $V := \bigcap_{j=1}^m V_j$ has the desired property : $K.V \subseteq M - W$. $\square$

5.8. Lemma. *Let M be a proper G-manifold , V a linear G-space and $f : M \to V$ smooth with compact support, then*

$$\tilde{f} : x \mapsto \int_G g^{-1} f(g.x) d\mu_r(g)$$

is a G-equivariant C^∞-map with $\tilde{f}(x) = 0$ for $x \notin G.\operatorname{supp} f$ (where $d\mu_r$ stands for the right Haar measure on G).

Proof. Since G acts properly, $\{g \in G : g.x \in \operatorname{supp} f\}$ is compact. Therefore the map $g \mapsto g^{-1}f(g.x)$ has compact support, and $\tilde{f}$ is well defined. To see that $\tilde{f}$ is smooth, let x_0 be in M, and U a compact neighborhood of x_0. Then the set $\{g \in G : g.U \cap \operatorname{supp} f \neq \emptyset\}$ is compact. Therefore, $\tilde{f}$ restricted to U is smooth, in particular $\tilde{f}$ is smooth in x_0. $\tilde{f}$ is G-equivariant, since

$$\tilde{f}(h.x) = \int_G g^{-1} f(gh.x) d\mu_r(g) =$$
$$= \int_G h(gh)^{-1} f(gh.x) d\mu_r(g) = h.\int_G g^{-1} f(g.x) d\mu_r(g) = h\tilde{f}(x).$$

Furthermore, $x \notin G.\operatorname{supp} f \Rightarrow f(g.x) = 0$ for all $g \in G \Rightarrow \tilde{f}(x) = 0$. $\square$

5.9. Corollary. *If M is a proper G-manifold, then M/G is completely regular.*

Proof. Choose $F \subseteq M/G$ closed and $\bar{x}_0 = \pi(x_0) \notin F$. Now let U be a compact neighborhood of x_0 in M fulfilling $U \cap \pi^{-1}(F) = \emptyset$, and $f \in C^\infty(M, [0,\infty))$ with support in U such that $f(x_0) > 0$. If we take the trivial representation of G on $\mathbb{R}$, then from lemma 5.8 it follows that $\tilde{f} : x \mapsto \int_G f(g.x) d\mu_r(g)$ defines a smooth G-invariant function. Furthermore, $\tilde{f}(x_0) > 0$. Since $\operatorname{supp} \tilde{f} \subseteq G.\operatorname{supp} f \subseteq G.U$, we have $\operatorname{supp} \tilde{f} \cap \pi^{-1}(F) = \emptyset$. Because $\tilde{f} \in C^\infty(M, [0,\infty))^G$, f factors over π to a map $\bar{f} \in C^0(M/G, [0,\infty))$, with $\bar{f}(\bar{x}_0) > 0$ and $\bar{f}\big|_F = 0$. $\square$

5.10. Theorem. *If M is a proper G-manifold, then there is a G-invariant Riemann metric on M.*

Proof. By 5.7 there is a slice S_x at x for all $x \in M$. If $\pi : M \to M/G$ is the quotient map, then we will show the existence of a sequence $x_n \in M$ such that $\pi(S_{x_n})$ is a locally finite covering of M/G. To do so, notice first that M/G is locally compact (in particular Hausdorff), σ-compact and therefore normal.

Since M/G is σ-compact and Hausdorff, there is a countable locally finite covering by compact sets C_i. Each C_i, in turn, is covered by $\{\pi(S_x) : x \in \pi^{-1}(C_i)\}$. Since C_i is compact, there is a finite subcovering, and these taken all together give the desired covering of M/G.

Let us now construct a neighborhood K_n of x_n in $S_{x_n}(=: S_n)$ such that K_n has compact closure in S_n and $\{\pi(K_n)\}$ is still a covering.

Take a C_i from above. If $\{\pi(S_j) : j \in F \subset \mathbb{N},\ \text{finite}\}$ covers C_i, then consider the complement of $\bigcup_{j\in F\setminus\{l\}} \pi(S_j)$ in C_i. This is a compact set contained in C_i with open neighborhood $\pi(S_l)$, so it has a relatively compact neighborhood R_l with $\bar{R}_l \subset \pi(S_l)$, since M/G is normal. $K_l := \pi^{-1}(R_l) \cap S_l$ is relatively compact due to the compactness of G_{x_l}: K_i is a subset of S_i, so 4.13(5) states that $R_i \cong K_i/G_{x_i}$, so $\bar{R}_i \cong \bar{K}_i/G_{x_i}$ and with $\bar{R}_i$, $\bar{K}_i$ must be compact, since G_{x_i} is compact.

If we choose $f_n \in C^\infty(M, [0,\infty))$ with $f_n|_{K_n} > 0$ and $\operatorname{supp}(f_n) \subseteq G.S_n$ compact, then

$$\bar{f}_n(x) := \int_G f_n(g.x)\,d\mu_r(g) \in C^\infty(M, [0,\infty))^G$$

is positive on $G.K_n$ and has $\operatorname{supp}(\bar{f}_n) \subseteq G.S_n$. The action of the compact group G_{x_n} on $TM|_{S_n}$ is fiber linear, so there is a G_x-invariant Riemann metric $\gamma^{(n)}$ on the vector bundle $TM|_{S_n}$ by integration. To get a Riemann metric on $TM|_{G.S_n}$ invariant under the whole group G, consider the following diagram.

$$\begin{tikzcd}
G \times TM|_{S_{x_n}} \arrow[r, "T_2\ell"] \arrow[d, "q"'] & TM|_{G.S_{x_n}} \arrow[dd] \\
G \times_{G_{x_n}} TM|_{S_{x_n}} \arrow[ur, "\widetilde{T_2\ell}"] \arrow[d] & \\
G \times_{G_{x_n}} S_{x_n} \arrow[r, "\cong"] & G.S_{x_n}
\end{tikzcd}$$

$T_2\ell : (g, X_s) \mapsto T_s\ell_g.X_s$ factors over q to a map $\widetilde{T_2\ell}$. This map is injective, since if $T_2\ell(g, X_s) = T_2\ell(g', X_{s'})$, then on the one side $\ell(g.s) = \ell(g'.s')$ so $g^{-1}g'.s' = s$ and $g^{-1}g' \in G_x$. On the other side, $T_s\ell_g.X_s = T_{s'}\ell_{g'}.X_{s'}$. So $(g', X_{s'}) = \big(g(g^{-1}g'), T_{s'}\ell_{g'^{-1}}T_s\ell_g.X_s\big)$. And, therefore, $q(g', X_{s'}) = q(g, X_s)$.

The Riemann metric $\gamma^{(n)}$ defines a G-invariant vector bundle metric on $G \times TM|_{S_n} \to G \times S_n$ by

$$\gamma_n(g, X_s, Y_s) := \gamma^{(n)}(X_s, Y_s).$$

It is also invariant under the G_x-action $h.(g, X_s) = (gh^{-1}, T\ell_h.X_s)$ and, therefore, induces a Riemann metric $\tilde{\gamma}_n$ on $G \times_{G_x} TM|_{S_n}$. This metric is again G-invariant, since the actions of G and G_x commute. Now $(\widetilde{T_2\ell})_*\tilde{\gamma}_n =: \bar{\gamma}_n$ is a G-invariant Riemann metric on $TM|_{G.S_n}$, and

$$\gamma := \sum_{n=1}^{\infty} \bar{f}_n(x)\bar{\gamma}_n$$

is a G-invariant Riemann metric on M. $\square$

Remark. By a theorem of Mostow (1957), if G is a compact Lie group, then any G-manifold M with a finite number of orbit types can be embedded into some (higher dimensional) vector space V in such a way that the action of G on M can be extended to a linear action on V (see [5], pp.110–112). A more recent result is the following theorem found in [31].

5.11. Theorem. [31]
Let G be a matrix group, that is a Lie group with a faithful finite dimensional representation, and let M be a G-space with only a finite number of orbit types. Then there is a G-equivariant embedding $f : M \to V$ into a linear G-space V.

6. Riemannian G-manifolds

6.1. Preliminaries. Let (M,γ) be a Riemannian G-manifold. If $\varphi : M \to M$ is an isometric diffeomorphism, then

(1) $\varphi(\exp^M_x(tX)) = \exp^M_{\varphi(x)}(tT_x\varphi.X)$. This is due to the fact that isometries map geodesics to geodesics, and the starting vector of the geodesic $t \mapsto \varphi(\exp^M_x(t.X))$ is $T_x\varphi.X$.

(2) If $\varphi(x) = x$, then, in the chart $(U_x, (\exp^M_x)^{-1})$, φ is a linear isometry (where U_x is neighborhood of x so small, that $(\exp^M_x)^{-1} : U_x \to T_xM$ is a diffeomorphism onto a neighborhood of 0 in T_xM) :

$$\bar\varphi(X) := (\exp^M_x)^{-1} \circ \varphi \circ \exp^M_x(X) = (\exp^M_x)^{-1}\exp^M_x(T_x\varphi.X) = T_x\varphi.X$$

(3) $\operatorname{Fix}(\varphi) = \{x \in M : \varphi(x) = x\}$ is a totally geodesic submanifold of M:
If we choose $X \in T_x\operatorname{Fix}(\varphi)$, then, since $T_x\varphi.X = X$ and by (1), we have

$$\varphi(\exp^M_x(tX)) = \exp^M_x(T_x\varphi.tX) = \exp^M_x(tX).$$

So the geodesic through x with starting vector X stays in $\operatorname{Fix}(\varphi)$.

(4) If H is a set of isometries, then $\operatorname{Fix}(H) = \{x \in M : \varphi(x) = x \text{ for all } \varphi \in H\}$ is also a totally geodesic submanifold in M.

6.2. Definition. *Let M be a proper Riemannian G-manifold, $x \in M$. The normal bundle to the orbit $G.x$ is defined as*

$$\operatorname{Nor}(G.x) := T(G.x)^\perp$$

Let $\operatorname{Nor}_\varepsilon(G.x) = \{X \in \operatorname{Nor}(G.x) : |X| < \varepsilon\}$, *and choose $r > 0$ small enough for* $\exp_x : T_xM \supseteq B_r(0_x) \to M$ *to be a diffeomorphism onto its image and for* $\exp_x(B_r(0_x)) \cap G.x$ *to have only one component. Then, since the action of G is isometric,* $\exp$ *defines a diffeomorphism from* $\operatorname{Nor}_{r/2}(G.x)$ *onto an open neighborhood of $G.x$, so* $\exp\left(\operatorname{Nor}_{r/2}(G.x)\right) =: U_{r/2}(G.x)$ *is a tubular neighborhood of $G.x$. We define the normal slice at x by*

$$S_x := \exp_x\left(\operatorname{Nor}_{r/2}(G.x)\right)_x.$$

6.3. Lemma. *Under these conditions we have*

(1) $S_{g.x} = g.S_x$.
(2) S_x *is a slice at* x.

Proof.

(1) Since G acts isometrically and by 6.1(1) :

$$S_{g.x} = \exp_{g.x}\left(T_x\ell_g\left(\operatorname{Nor}_{r/2}(G.x)\right)_x\right) = \ell_g \exp_x\left(\operatorname{Nor}_{r/2}(G.x)\right)_x = g.S_x$$

(2) $r : G.S_x \to G.x : \exp_{g.x} X \mapsto g.x$ defines a smooth equivariant retraction (note that S_x and S_y are disjoint if $x \neq y$). $\square$

6.4. Definition. *Let M be a G-manifold and $x \in M$, then there is a representation of the isotropy group G_x*

$$G_x \to GL(T_xM) : g \mapsto T_x\ell_g$$

called **isotropy representation**. *If M is a Riemannian G-manifold, then the isotropy representation is orthogonal, and $T_x(G.x)$ is an invariant subspace under G_x. So $T_x(G.x)^\perp$ is also invariant, and*

$$G_x \to O(\operatorname{Nor}_x(G.x)) : g \mapsto T_x\ell_g$$

is called the **slice representation**.

6.5. Example. Let $M = G$ be a compact Lie group with a biinvariant metric. Then $G \times G$ acts on G by $(g_1, g_2).g := g_1 g g_2^{-1}$, making G a Riemannian $(G \times G)$-space. The isotropy group of e is $(G \times G)_e = \{(g,g) : g \in G\}$, and the isotropy representation coincides with the adjoint representation of $G \cong (G \times G)_e$ on $\mathfrak{g} = T_e(G)$.

6.6. Example. Let G/K be a semisimple symmetric space (G compact) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding orthogonal decomposition of the Lie algebra $\mathfrak{g}$ with regard to the negative Cartan-Killing form $-B$. Then $T_e(G/K) \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$, and the isotropy subgroup of G at e is K. The isotropy representation is $Ad^\perp_{K,G} : K \to O(\mathfrak{p})$. The slices are points since the action is transitive.

6.7. Lemma. *Let M be a proper Riemannian G-manifold, $x \in M$. Then the following three statements are equivalent:*

(1) *x is a regular point.*
(2) *The slice representation at x is trivial.*
(3) *$G_y = G_x$ for all $y \in S_x$ for a sufficiently small slice S_x.*

Proof. Clearly, (2) $\iff$ (3). To see (3) $\implies$ (1), let S_x be a small slice at x. Then $U := G.S$ is an open neighborhood of $G.x$ in M, and for all $g.s \in U$ we have $G_{g.s} = gG_sg^{-1} = gG_xg^{-1}$. Therefore $G.x$ is a principal orbit. The converse is true by 4.13(3), since G_x is compact. $\square$

6.8. Definition. *Let M be a Riemannian G-manifold and $G.x$ some orbit, then a smooth section u of the normal bundle* $\operatorname{Nor}(G.x)$ *is called* **equivariant normal field**, *if*

$$T_y(\ell_g).u(y) = u(g.y) \qquad \text{for all } y \in G.x, g \in G.$$

6.9. Corollary. *Let M be a proper Riemannian G-manifold and x a regular point. If $X \in \operatorname{Nor}_x(G.x)$, then $\hat{X}(g.x) := T_x(\ell_g).X$ is a well defined equivariant normal field along $G.x$ in M.*

Proof. If $g.x = h.x$ then $h^{-1}g \in G_x \Rightarrow T_x\left(\ell_{h^{-1}g}\right).X = X$, since the slice representation is trivial by (2) above. Now by the chain rule: $T_x(\ell_g).X = T_x(\ell_h).X$. Therefore $\hat{X}$ is a well defined, smooth section of $\operatorname{Nor}(G.x)$. It is equivariant by definition. □

6.10. Corollary. *Let M be a Riemannian G-manifold, $G.x$ a principal orbit, and $(u_1, \dots, u_n)$ an orthonormal basis of* $\operatorname{Nor}_x(G.x)$. *By corollary 6.9, each u_i defines an equivariant normal field $\hat{u}_i$. So $(\hat{u}_1, \dots, \hat{u}_n)$ is a global equivariant orthonormal frame field for* $\operatorname{Nor}(G.x)$, *and* $\operatorname{Nor}(G.x)$ *is a trivial bundle.* □

This follows also from the tubular neighborhood description $G.S_x \cong G \times_{G_x} S_x$, where S_x is a normal slice at x with trivial G_x-action, see 6.7.

6.11. Definition. *Let (M, γ) be a Riemannian manifold and ∇^M its Levi-Civita covariant derivative. If P is a submanifold of M and ∇^P the induced covariant derivative on P, then the* **second fundamental form** $S \in C^\infty\left(S^2T^*P \otimes \operatorname{Nor}(P)\right)$ *is given by the so called Gauss equation:*

$$\nabla^M_X Y = \nabla^P_X Y + S(X,Y) \qquad \text{for} \quad X, Y \in \mathfrak{X}(P)$$

In other words, $S(X,Y)$ is the part of the covariant derivative in M orthogonal to P.

6.12. Definition. *Let (M, γ) be a Riemannian G-manifold and u an equivariant normal field along an orbit $P := G.x_0$. Then $X_x \in T_xP$ defines a linear form on T_xP by*

$$Y_x \mapsto \gamma(S(X_x, Y_x), u(x)).$$

Therefore, there is a vector $S_{u(x)}(X_x) \in T_xP$ such that

$$\gamma\,|_{TP}\,(S_{u(x)}(X_x), Y_x) = \gamma(S(X_x, Y_x), u(x))$$

This assignment defines a linear map $S_{u(x)} : T_xP \to T_xP$ called the **shape operator** *of P in the normal direction $u(x)$. For hypersurfaces it is also known as the Weingarten endomorphism. Its eigenvalues are called the main curvatures of P along u.*

6.13. Lemma. *Let u be an equivariant normal field along an orbit $P := G.x_0$, then*

(1) $S_{u(g.x)} = T_x(\ell_g).S_{u(x)}.T_{g.x}(\ell_{g^{-1}})$
(2) *The main curvatures of P along u are all constant.*
(3) $\{\exp^M(u(x)) : x \in P = G.x_0\}$ *is another G-orbit.*

Proof.

(1) Since γ is G-invariant and S is G-equivariant:

$$\begin{aligned}
&\gamma\left(S_{u(g.x)}\left(X_{g.x}\right), Y_{g.x}\right) = \gamma\left(S\left(X_{g.x}, Y_{g.x}\right), u(g.x)\right) = \\
&= \gamma\left(T\ell_g S\left(T\ell_{g^{-1}} X_{g.x}, T\ell_{g^{-1}} Y_{g.x}\right), T\ell_g(u(x))\right) = \\
&= \gamma\left(S\left(T\ell_{g^{-1}} X_{g.x}, T\ell_{g^{-1}} Y_{g.x}\right), u(x)\right) = \\
&= \gamma\left(S_{u(x)} \circ T\ell_{g^{-1}}(X_{g.x}), T\ell_{g^{-1}} Y_{g.x}\right) = \gamma\left(T\ell_g \circ S_{u(x)} \circ T\ell_{g^{-1}}(X_{g.x}), Y_{g.x}\right)
\end{aligned}$$

(2) By (1) $S_{u(g.x)}$ results from $S_{u(x)}$ by a linear coordinate transformation, which does not affect the eigenvalues.

(3) $\{\exp^M(u(x)) : x \in P = G.x_0\} = G.\exp^M(u(x_0))$, since

$$g.\exp^M(u(x_0)) = \exp^M(T\ell_g.u(x_0)) = \exp^M(u(g.x_0)).$$

□

6.14. Example. Let $N^n(c)$ be the simply connected space form with constant sectional curvature c, that is

$$\begin{aligned}
N^n(c) &= S^n, \text{ sphere with radius } \frac{1}{c} \text{ if } c > 0 \\
&= \mathbb{R}^n \text{ if } c = 0 \\
&= H^n, \text{ hyperbolic sphere with radius } \frac{1}{|c|} \text{ if } c < 0.
\end{aligned}$$

Let G be a closed subgroup of $\operatorname{Isom}(N^n(c))$. If P is a G-orbit, then so is the subset $\{\exp(u(x)) : x \in P\}$ for any equivariant normal field u along P. For instance

(1) If $G = SO(n) \subset \operatorname{Isom}(\mathbb{R}^n)$, then the G-orbits are the spheres with center 0. A radial vector field with constant length on each sphere, $u(x) := f(|x|).x$, defines an equivariant normal field on each orbit. Clearly its flow carries orbits back into orbits.

(2) Another example is the subgroup

$$G = \{f : x \mapsto x + \lambda v : \lambda \in \mathbb{R}, v \in \langle v_1, v_2, \dots, v_m\rangle\}$$

of $\operatorname{Isom}(\mathbb{R}^n)$ consisting only of affine translations in certain fixed directions. Here the orbits of G are then parallel planes of dimension m. An equivariant normal field on an orbit is a constant vector field orthogonal to $v_1, v_2, \dots, v_m$.

6.15. Theorem. *Let M be a proper G-manifold, then the set of all regular points M_{reg} is open and dense in M. In particular, there is always a principal orbit type.*

Proof. Suppose $x \in M_{\text{reg}}$. By 5.7 there is a slice S at x, and by 4.13(3) S can be chosen small enough for all orbits through S to be principal as well. Therefore $G.S$ is an open neighborhood of x in M_{reg} (open by 4.12(3)).

To see that M_{reg} is dense, let $U \subseteq M$ be open, $x \in U$, and S a slice at x. Now choose a $y \in G.S \cap U$ for which G_y has the minimal dimension and the smallest number of connected components for this dimension in all of $G.S \cap U$. Let S_y be a slice at y, then $G.S_y \cap G.S \cap U$ is open, and for any $z \in G.S_y \cap G.S \cap U$ we have $z \in g.S_y = S_{g.y}$, so $G_z \subseteq G_{g.y} = gG_yg^{-1}$. By choice of y, this implies $G_z = gG_yg^{-1}$ for all $z \in G.S_y \cap G.S \cap U$, and $G.y$ is a principal orbit. □

6.16. Theorem. *Let M be a proper G-manifold and $x \in M$. Then there is a G-invariant neighborhood U of x in which only finitely many orbit types occur.*

Proof. By theorem 5.10 there is a G-invariant Riemann metric on M. Let S be the normal slice at x. Then S is again a Riemannian manifold, and the compact group G_x acts isometrically on S. In 4.13(4) we saw that, if $G_x.s_1$ and $G_x.s_2$ have the same orbit type in S, then $G.s_1$ and $G.s_2$ have the same orbit type in M. So the number of G-orbit types in $G.S$ can be no more, than the number of G_x-orbit types in S. Therefore it is sufficient to consider the case where G is a compact Lie group. Let us now prove the assertion under this added assumption. We carry out induction on the dimension of M.

For $n = 0$ there is nothing to prove. Suppose the assertion is proved for $\dim M < n$. Again, it will do to find a slice S at x with only a finite number of G_x-orbit types. If $\dim S < \dim M$, this follows from the induction hypothesis. Now suppose $\dim S = n$. S is equivariantly diffeomorphic to an open ball in T_xM under the slice representation (by exp). Since the slice representation is orthogonal, it restricts to a G_x-action on the sphere S^{n-1}. By the induction hypothesis, locally, S^{n-1} has only finitely many G_x-orbit types. Since S^{n-1} is compact, it has only finitely many orbit types globally. The orbit types are the same on all spheres $r.S^{n-1}$ $(r > 0)$, since $x \mapsto \frac{1}{r}x$ is G-equivariant. Therefore, S has only finitely many orbit types: those of S^{n-1} and the 0-orbit. $\square$

6.17. Theorem. *If M is a proper G-manifold then the set M_{sing}/G of all singular G-orbits does not locally disconnect the orbit space M/G (that is to every point in M/G the connected neighborhoods remain connected even after removal of all singular orbits).*

Proof. As in the previous theorem, we will reduce the statement to an assertion about the slice representation. By theorem 5.10, there is a G-invariant Riemann metric on M. Let S be the normal slice at x. Then S is again a Riemannian manifold, and the compact group G_x acts isometrically on S. A principal G_x-orbit is the restriction of a principal G-orbit, since $G_x.s$ is principal means that all orbits in a sufficiently small neighborhood of $G_x.s$ have the same orbit type as $G_x.s$ (6.7). Therefore, by 4.13(4), the corresponding orbits in $G.U$ are also of the same type, and $G.s$ is principal as well. So there are "fewer" singular G-orbits in $G.S$ than there are singular G_x-orbits in S. Now cover M with tubular neighborhoods like $G.S_x$, and recall that $G.S_x/G \cong S_x/G_x$ by 4.13(5). This together with the above argument shows us that it will suffice to prove the statement for the slice action. Furthermore, like in the proof of theorem 6.18, we can restrict our considerations to the slice representation. So we have reduced the statement to the following:

If V is a real, n-dimensional vector space and G a compact Lie group acting on V, then the set V_{sing}/G of all singular G-orbits does not locally disconnect the orbit space V/G (that is to every point in V/G the connected neighborhoods remain connected even after removal of all singular orbits).

We will prove this by induction on the dimension n of V. For $n = 1$, that is $V = \mathbb{R}$, the only nontrivial choice for G is $O(1) \cong \mathbb{Z}_2$. In this case, $\mathbb{R}/G = [0, \infty)$ and $\mathbb{R}_{\text{sing}}/G = \{0\}$. Clearly, $\{0\}$ does not locally disconnect $[0, \infty)$, and we can proceed to the general case.

Suppose the assertion is proved for all dimensions smaller than n. Now for $G \subseteq O(n)$ we consider the induced action on the invariant submanifold S^{n-1}. For any $x \in S^{n-1}$ we can apply the induction hypothesis to the slice representation $G_x \to O(\operatorname{Nor}_x G.x)$. This implies for the G_x-action on S_x, that $S_x/G_x \cong G.S_x/G$ is not locally disconnected by its singular points. As above, we can again cover S^{n-1} with tubular neighborhoods like $G.S_x$, and we see that all of S^{n-1}/G is not locally disconnected by its singular orbits. Now we need to verify that the orbit space of th unit ball D^n is not locally disconnected by its singular orbits. Since scalar multiplication is a G-equivariant diffeomorphism, the singular orbits in V (not including $\{0\}$) project radially onto singular orbits in S^{n-1}. So if we view the ball D^n as cone over S^{n-1} and denote the cone construction by $\operatorname{cone}S^{n-1}$, then $D^n_{\text{sing}} = \operatorname{cone}S^{n-1}_{\text{sing}}$. Furthermore, we have a homeomorphism

$$\operatorname{cone}S^{n-1}/G \to \operatorname{cone}(S^{n-1}/G) : G.[x,t] \mapsto [G.x,t]$$

since G preserves the "radius" t. Therefore

$$\begin{aligned} D^n/G &= (\operatorname{cone}S^{n-1})/G \cong \operatorname{cone}(S^{n-1}/G) \\ \text{and}\quad D^n_{\text{sing}}/G &= \operatorname{cone}S^{n-1}_{\text{sing}}/G \cong \operatorname{cone}\left(S^{n-1}_{\text{sing}}/G\right). \end{aligned}$$

Since S^{n-1}_{sing}/G does not locally disconnect S^{n-1}/G, we also see that

$$\operatorname{cone}\left(S^{n-1}_{\text{sing}}/G\right) \cong D^n_{\text{sing}}/G$$

does not locally disconnect $\operatorname{cone}(S^{n-1}/G) \cong D^n/G$. $\square$

6.18. Corollary. *Let M be a connected proper G-manifold, then*

(1) *M/G is connected.*
(2) *M has precisely one principal orbit type.*

Proof.

(1) Since M is connected and the quotient map $\pi : M \to M/G$ is continuous, its image M/G is connected as well.
(2) By the last theorem we have that $M/G - M_{\text{sing}}/G = M_{\text{reg}}/G$ is connected. On the other hand by 6.7, the orbits of a certain principal orbit type form an open subset of M/G, in particular of M_{reg}/G. Therefore if there were more than one principal orbit type, these orbit types would partition M_{reg}/G into disjoint nonempty open subsets contradicting the fact that M_{reg}/G is connected. $\square$

6.19. Corollary. *Let M be a connected, proper G-manifold of dimension n and let k be the least number of connected components of all isotropy groups of dimension $m := \inf\{\dim G_x | x \in M\}$. Then the following two assertions are equivalent:*

(1) *$G.x_0$ is a principal orbit.*
(2) *The isotropy group G_{x_0} has dimension m and k connected components.*

If furthermore G is connected and simply connected, these conditions are again equivalent to

(3) *The orbit $G.x_0$ has dimension $n-m$ and for the order of the fundamental group we have: $|\pi_1(G.x_0)| = k$.*

Proof. Recall that we proved the existence of a principal orbit in 6.15 just by taking a G_{x_0} as described above. The other direction of the proof follows from the above corollary. Since there is only one principal orbit type, this must be it.

If moreover G is connected and simply connection we look at the fibration $G_{x_0} \to G \to G/G_{x_0} = G.x_0$ and at the following portion of its long exact homotopy sequence

$$0 = \pi_1(G) \to \pi_1(G.x_0) \to \pi_0(G_{x_0}) \to \pi_0(G) = 0$$

from which we see that $|\pi_1(G.x_0)| = k$ if and only if the isotropy group G_{x_0} has k connected components. $\square$

6.20. Theorem. [37] *Let $\pi : G \to O(V)$ be an orthogonal, real, finite-dimensional representation of a compact Lie group G. Let $\rho_1, \dots, \rho_k \in \mathbb{R}[V]^G$ be homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. For $v \in V$, let $\operatorname{Nor}_v(G.v) := T_v(G.v)^\perp$ be the normal space to the orbit at v, and let $\operatorname{Nor}_v(G.v)^{G_v}$ be the subspace of those vectors which are invariant under the isotropy group G_v.*

Then $\operatorname{grad}\rho_1(v), \dots, \operatorname{grad}\rho_k(v)$ span $\operatorname{Nor}_v(G.v)^{G_v}$ as a real vector space.

Proof. Clearly each $\operatorname{grad}\rho_i(v) \in \operatorname{Nor}_v(G.v)^{G_v}$. In the following we will identify G with its image $\pi(G) \subseteq O(V)$. Its Lie algebra is then a subalgebra of $\mathfrak{o}(V)$ and can be realized as a Lie algebra consisting of skew-symmetric matrices. Let $v \in V$, and let S_v be the normal slice at v which is chosen so small that the projection of the tubular neighborhood (see 4.15) $p_{G.v} : G.S_v \to G.v$ from the diagram

$$\begin{tikzcd}
G \times S_v \arrow[r, "q"] & G \times_{G_v} S_v \arrow[r, "\pi"] \arrow[d, "p"'] & G.S_v \arrow[d, "p_{G.v}"'] \\
 & G/G_v \arrow[r, "\pi", "\cong"'] & G.v
\end{tikzcd}$$

has the property, that for any $w \in G.S_v$ the point $p_{G.v}(w) \in G.v$ is the unique point in the orbit $G.v$ which minimizes the distance between w and the orbit $G.v$.

Choose $n \in \operatorname{Nor}_v(G.v)^{G_v}$ so small that $x := v + n \in S_v$. So $p_{G.v}(x) = v$. For the isotropy groups we have $G_x \subseteq G_v$ by 4.13.(2). But we have also $G_v \subseteq G_v \cap G_n \subseteq G_x$, so that $G_v = G_x$. Let S_x be the normal slice at x which we choose also so small that $p_{G.x} : G.S_x \to G.x$ has the same minimizing property as $p_{G.v}$ above, but so large that $v \in G.S_x$ (choose n smaller if necessary). We also have $p_{G.x}(v) = x$ since for the Euclidean distance in V we have

$$\begin{aligned}
|v - x| &= \min_{g \in G} |g.v - x| && \text{since } v = p_{G.v}(x) \\
&= \min_{g \in G} |h.g.v - h.x| && \text{for all } h \in G \\
&= \min_{g \in G} |v - g^{-1}.x| && \text{by choosing } h = g^{-1}.
\end{aligned}$$

For $w \in G.S_x$ we consider the local, smooth, G-invariant function

$$\begin{aligned}\operatorname{dist}(w, G.x)^2 &= \operatorname{dist}(w, p_{G.x}(w))^2 = \langle w - p_{G.x}(w), w - p_{G.x}(w)\rangle\\ &= \langle w, w\rangle + \langle p_{G.x}(w), p_{G.x}(w)\rangle - 2\langle w, p_{G.x}(w)\rangle\\ &= \langle w, w\rangle + \langle x, x\rangle - 2\langle w, p_{G.x}(w)\rangle.\end{aligned}$$

Its derivative with respect to w is

$$(1)\qquad d(\operatorname{dist}(\quad, G.x)^2)(w)y = 2\langle w, y\rangle - 2\langle y, p_{G.x}(w)\rangle - 2\langle w, dp_{G.x}(w)y\rangle.$$

We shall show below that

$$(2)\qquad \langle v, dp_{G.x}(v)y\rangle = 0 \quad \text{for all } y \in V,$$

so that the derivative at v is given by

$$(3)\qquad d(\operatorname{dist}(\quad, G.x)^2)(v)y = 2\langle v, y\rangle - 2\langle y, p_{G.x}(v)\rangle = 2\langle v - x, y\rangle = -2\langle n, y\rangle.$$

Now choose a smooth G_x-invariant function $f : S_x \to \mathbb{R}$ with compact support which equals 1 in an open ball around x and extend it smoothly (see the diagram above, but for S_x) to $G.S_x$ and then to the whole of V. We assume that f is still equal to 1 in a neighborhood of v. Then $g = f.\operatorname{dist}(\quad, G.x)^2$ is a smooth G-invariant function on V which coincides with $\operatorname{dist}(\quad, G.x)^2$ near v. By the theorem of Schwarz (3.7) there is a smooth function $h \in C^\infty(\mathbb{R}^k, \mathbb{R})$ such that $g = h \circ \rho$, where $\rho = (\rho_1, \dots, \rho_k) : V \to \mathbb{R}^k$. Then we have finally by (3)

$$\begin{aligned}-2n &= \operatorname{grad}(\operatorname{dist}(\quad, G.x)^2)(v) = \operatorname{grad} g(v) =\\ &= \operatorname{grad}(h \circ \rho)(v) = \sum_{i=1}^k \frac{\partial h}{\partial y_i}(\rho(v)) \operatorname{grad} \rho_i(v),\end{aligned}$$

which proves the result.

It remains to check equation (2). Since $T_vV = T_v(G.v) \oplus \operatorname{Nor}_v(G.v)$ the normal space $\operatorname{Nor}_x(G.x) = \ker dp_{G.x}(v)$ is still transversal to $T_v(G.v)$ if n is small enough; so it remains to show that $\langle v, dp_{G.x}(v).X.v\rangle = 0$ for each $X \in \mathfrak{g}$. Since $x = p_{G.x}(v)$ we have $|v - x|^2 = \min_{g\in G} |v - g.x|^2$, and thus the derivative of $g \mapsto \langle v - g.x, v - g.x\rangle$ at e vanishes: for all $X \in \mathfrak{g}$ we have

$$(4)\qquad 0 = 2\langle -X.x, v - x\rangle = 2\langle X.x, x\rangle - 2\langle X.x, v\rangle = 0 - 2\langle X.x, v\rangle,$$

since the action of X on V is skew symmetric. Now we consider the equation $p_{G.x}(g.v) = g.p_{G.x}(v)$ and differentiate it with respect to g at $e \in G$ in the direction $X \in \mathfrak{g}$ to obtain in turn

$$\begin{aligned}dp_{G.x}(v).X.v &= X.p_{G.x}(v) = X.x,\\ \langle v, dp_{G.x}(v).X.v\rangle &= \langle v, X.x\rangle = 0, \quad \text{by (4)}. \quad \square\end{aligned}$$

6.21. Lemma. *Let $\pi : G \to O(V)$ be an orthogonal representation. Let $\omega \in \Omega^p_{\text{hor}}(V)^G$ be an invariant differential form on V which is horizontal in the sense that $i_w\omega_x = 0$ if w is tangent to the orbit $G.x$. Let $v \in V$ and let $w \in T_vV$ be orthogonal to the space $\operatorname{Nor}_v(G.v)^{G^0_v}$ of those orthogonal vectors which are invariant under the connected component G^0_v of the isotropy group G_v.*

Then $i_w\omega_v = 0$.

Proof. We consider the orthogonal decomposition

$$T_vV = T_v(G.v) \oplus W \oplus \operatorname{Nor}_v(G.v)^{G^0_v}.$$

We may assume that $w \in W$ since $i_u\omega_v = 0$ for $u \in T_v(G.v)$.

We claim that each $w \in W$ is a linear combination of elements of the form $X.u$ for $u \in W$ and $X \in \mathfrak{g}_v := \ker(d\pi(\quad)v)$. Since G^0_v is compact, the representation space W has no fixed point other than zero and is completely reducible under G^0_v and thus also under its Lie algebra $\mathfrak{g}_v$, and we may treat each irreducible component separately, or assume that W is irreducible. Then $\mathfrak{g}_v(W)$ is an invariant subspace which is not 0. So it agrees with W, and the claim follows.

So we may assume that $w = X.u$ for $u \in W$. But then

$$(v + \tfrac{1}{n}u, X.u = nX.(v + \tfrac{1}{n}u)) \in T_{v+\frac{1}{n}u}(G.(v + \tfrac{1}{n}u))$$

satisfies $i_{X.u}\omega_{v+u/n} = 0$ by horizontality and thus we have

$$i_w\omega_v = i_{X.u}\omega_v = \lim_n i_{X.u}\omega_{v+u/n} = 0. \quad \square$$

7. Riemannian Submersions

7.1. Definitions. *Let $p : E \to B$ be a submersion of smooth manifolds, that is $Tp : TE \to TB$ surjective. Then*

$$V = V(p) = V(E) := \mathrm{Ker}(Tp)$$

is called the **vertical subbundle** *of E. If E is a Riemannian manifold with metric γ, then we can go on to define the* **horizontal subbundle** *of E.*

$$\mathrm{Hor} = \mathrm{Hor}(p) = \mathrm{Hor}(E) = \mathrm{Hor}(E, \gamma) := V(p)^{\perp}$$

If both (E, γ_E) and (B, γ_B) are Riemannian manifolds, then we will call p a **Riemannian submersion***, if*

$$T_x p : \mathrm{Hor}(p) \to T_{p(x)} B$$

is an isometric isomorphism for all $x \in E$.

Some Simple Examples. For any two Riemannian manifolds M, N, the projection $pr_1 : M \times N \to M$ is a Riemannian submersion. Here Riemann metric on the product $M \times N$ is given by: $\gamma_{M\times N}(X, Y) := \gamma_M(X_M, Y_M) + \gamma_N(X_N, Y_N)$ (where we use $T(M \times N) \cong TM \oplus TN$ to decompose $X, Y \in T(M \times N)$). In particular, $\mathbb{R}^{m+n} \to \mathbb{R}^m$ with the usual metric, or $pr_2 : S^n \times \mathbb{R}^+ \to \mathbb{R}^+$ are Riemannian submersions.

7.2. G-manifold with single orbit type as fiber bundle. Let (M, γ) be a proper Riemannian G-manifold and suppose that M has only one orbit type, (H). We then want to study the quotient map $\pi : M \to M/G$. Let us first consider the orbit space M/G. Choose $x \in M$ and let S_x denote the normal slice at x. Then by 4.13(2) we have $G_y \subseteq G_x$ for all $y \in S_x$. Since G_y must additionally be conjugate to G_x and both are compact, they must be the same (by 4.8). So $G_x = G_y$ and therefore G_x acts trivially on S_x (this can also be seen as a special case of 6.7). From 4.13(5) it follows that $\pi(S_x) \cong S_x/G_x = S_x$, and with 4.15 we have that $G.S_x$ is isomorphic to $G/G_x \times S_x$. Therefore, for any $x \in M$, $\big(\pi(S_x), \exp_x^{-1}|_{S_x}\big)$ can serve as a chart for M/G.

$$\begin{tikzcd}
M \arrow[d, "\pi"'] & S_x \arrow[l, hook'] \arrow[d, "\pi"] \\
M/G & \pi(S_x) = S_x/G_x \arrow[l, hook']
\end{tikzcd}$$

To make an atlas out of these charts, we have to check whether they are compatible — which is obvious. By 5.9 M/G is Hausdorff, and therefore it is a smooth manifold.

Now let us study the smooth submersion $\pi : M \to M/G$. We want to find a Riemannian metric on M/G which will make π a Riemannian submersion.

Claim. For $X_x, Y_x \in \operatorname{Hor}_x(\pi) = \operatorname{Nor}_x(G.x)$, the following inner product is well defined.

$$\bar{\gamma}_{\pi(x)}(T\pi X_x, T\pi Y_x) := \gamma_x(X_x, Y_x)$$

Proof. Choose $X'_{gx}, Y'_{gx} \in \operatorname{Hor}_{gx}(\pi)$ such that $T\pi.X'_{gx} = T\pi.X_x$ and $T\pi.Y'_{gx} = T\pi.Y_x$. Then we see that $X'_{gx} = T(\ell_g)X_x$ by the following argumentation: Clearly $T\pi\left(X'_{gx} - T(\ell_g).X_x\right) = 0$, so the difference $X'_{gx} - T(\ell_g).X_x$ is vertical. On the other hand, X'_{gx} is horizontal, and so is $T(\ell_g).X_x$:
ℓ_g leaves $G.x$ invariant, consequently, $T\ell_g$ maps vertical vectors to vertical vectors and since it is an isometry, it also maps horizontal vectors to horizontal vectors. Therefore $X'_{gx} - T(\ell_g).X_x$ is horizontal as well as vertical and must be zero.
Now we can conclude, that

$$\gamma_{gx}\left(X'_{gx}, Y'_{gx}\right) = \gamma_{gx}(T(\ell_g)X_x, T(\ell_g)Y_x) = \gamma_x(X_x, Y_x). \qquad \square$$

So we have found a Riemannian metric $\bar{\gamma}$ on M/G which makes π a Riemannian submersion.

Let us finally try to understand in which sense $\pi : M \to M/G$ is an associated bundle. Let $x \in M$ be such that $G_x = H$. By 6.1.(4) the set $\operatorname{Fix}(H) = \{x \in M : g.x = x \text{ for all } g \in H\}$ is a geodesically complete submanifold of M. It is $N_G(H)$-invariant, and the restriction $\pi : \operatorname{Fix}(H) \to M/G$ is a smooth submersion since for each $y \in \operatorname{Fix}(H)$ the slice S_y is also contained in $\operatorname{Fix}(H)$. The fiber of $\pi : \operatorname{Fix}(H) \to M/G$ is a free $N_G(H)/H$-orbit: if $\pi(x) = \pi(y)$ and $G_x = H = G_y$ then $g \in N_G(H)$. So $\pi : \operatorname{Fix}(H) \to M/G$ is a principal $N_G(H)/H$-bundle, and M is the associated bundle with fiber G/H as follows:

$$\begin{tikzcd}
\operatorname{Fix}(H) \times G/H \arrow[d] \arrow[dr, "{(x,[g]) \mapsto g.x}"] & \\
\operatorname{Fix}(H) \times_{N_G(H)/H} G/H \arrow[r, "\cong"'] \arrow[d] & M \arrow[d] \\
M/G \arrow[r, equal] & M/G.
\end{tikzcd}$$

7.3. Another fiber bundle construction. Let M again be a proper Riemannian G-manifold with only one orbit type. Then we can "partition" M into the totally geodesic submanifolds $\operatorname{Fix}(gHg^{-1}) := \{x \in M : ghg^{-1}.x = x \text{ for all } h \in H\}$ where $H = G_{x_0}$ ($x_0 \in M$ arbitrary) is fixed and g varies. This is not a proper partitioning in the sense that if $g \neq e$ commutes with H, for instance, then $\operatorname{Fix}(gHg^{-1}) = \operatorname{Fix}(eHe^{-1})$. We want to find out just which g give the same sets $\operatorname{Fix}(gHg^{-1})$.

Claim.

$$\operatorname{Fix}(gHg^{-1}) = \operatorname{Fix}\left(g'Hg'^{-1}\right) \quad \Longleftrightarrow \quad gN(H) = g'N(H)$$

where $N(H)$ denotes the normalizer of H in G.

Proof. First let us show the following identity:

$$N(H) = \{g \in G : g\,\mathrm{Fix}(H) \subseteq \mathrm{Fix}(H)\}$$

($\subseteq$) Let $n \in N(H)$ and $y \in \mathrm{Fix}(H)$. Then $n.y$ is H-invariant:

$$hn.y = nn^{-1}hn.y = n(n^{-1}hn).y = n.y$$

($\supseteq$) $g\,\mathrm{Fix}(H) \subseteq \mathrm{Fix}(H)$ implies that $hg.y = g.y$, or equivalently $g^{-1}hg.y = y$, for any $y \in \mathrm{Fix}(H)$ and $h \in H$. Recall at this point, that $H = G_{x_0}$ for some $x_0 \in M$. Therefore, we have $g^{-1}hg.x_0 = x_0$ and consequently $g^{-1}hg \in G_{x_0} = H$.

Using this characterization for $N(H)$ and the identity

$$g'\{g \in G : g\,\mathrm{Fix}(H) \subseteq \mathrm{Fix}(H)\} = \{g \in G : g\,\mathrm{Fix}(H) \subseteq g'\,\mathrm{Fix}(H)\},$$

we can convert the righthand side of our equality, $gN(H) = g'N(H)$, to the following:

$$\{a \in G : a\,\mathrm{Fix}(H) \subseteq g.\,\mathrm{Fix}(H)\} = \{a \in G : a\,\mathrm{Fix}(H) \subseteq g'.\,\mathrm{Fix}(H)\}.$$

In particular, this is the case if

$$g.\,\mathrm{Fix}(H) = g'.\,\mathrm{Fix}(H).$$

In fact, let us show that the two equations are equivalent. Suppose indirectly that $g.y \notin g'.\,\mathrm{Fix}(H)$ for some $y \in \mathrm{Fix}(H)$. Then $a = g$ has the property $a.\,\mathrm{Fix}(H) \not\subseteq g'.\,\mathrm{Fix}(H)$, so $\{a \in G : a\,\mathrm{Fix}(H) \subseteq g.\,\mathrm{Fix}(H)\} \neq \{a \in G : a\,\mathrm{Fix}(H) \subseteq g'.\,\mathrm{Fix}(H)\}$.

So far we have shown that $gN(H) = g'N(H) \Leftrightarrow g.\,\mathrm{Fix}(H) = g'.\,\mathrm{Fix}(H)$. To complete the proof it only remains to check whether

$$\mathrm{Fix}(gHg^{-1}) = g\,\mathrm{Fix}(H).$$

This is easily done (as well as plausible, since it resembles strongly the "dual" notion $G_{gx} = gG_xg^{-1}$)

$$\begin{aligned} y \in \mathrm{Fix}(gHg^{-1}) \quad &\Longleftrightarrow \quad ghg^{-1}.y = y \;\text{ for all } h \in H \\ &\Longleftrightarrow hg^{-1}.y = g^{-1}y \;\text{ for all } h \in H \\ &\Longleftrightarrow \quad g^{-1}.y \in \mathrm{Fix}(H) \\ &\Longleftrightarrow \quad y \in g\,\mathrm{Fix}(H) \qquad \square \end{aligned}$$

Claim. The map $\bar{\pi} : M \to G/N(H)$ defined by $\mathrm{Fix}(gHg^{-1}) \ni x \mapsto g.N(H)$ is a fiber bundle with typical fiber $\mathrm{Fix}(H)$.

Proof. To prove this, let us consider the following diagram.

$$\begin{tikzcd}
G \times \mathrm{Fix}(H) \arrow[r, "\ell"] \arrow[d, "q"'] & M \arrow[ddl, "\tilde{\pi}"] \\
G \times_{N(H)} \mathrm{Fix}(H) \arrow[ur, "\tilde{\ell}"] \arrow[d] & \\
G/N(H) &
\end{tikzcd}$$

Here we use the restricted action $\ell : N(H) \times \mathrm{Fix}(H) \to \mathrm{Fix}(H)$ to associate to the principal bundle $G \to G/N(H)$ the bundle $G[\mathrm{Fix}(H), \ell] = G \times_{N(H)} \mathrm{Fix}(H)$. It remains to show that $\tilde{\ell}$ is a diffeomorphism, since then $\tilde{\pi}$ has the desired fiber bundle structure.

$\tilde{\ell}$ is smooth, since $\tilde{\ell} \circ q = \ell$ is smooth and q is a submersion. Now let us show that $\tilde{\ell}$ is bijective.

(1) $\tilde{\ell}$ is surjective: Since H is the only orbit type, for every $x \in M$ there is a $g \in G$, such that $G_x = gHg^{-1}$, which implies $x \in \mathrm{Fix}(gHg^{-1}) = g\,\mathrm{Fix}(H) \subseteq \ell(G \times \mathrm{Fix}(H))$. So ℓ is surjective and, by the commutativity of the diagram, so is $\tilde{\ell}$.

(2) $\tilde{\ell}$ is injective: Suppose $\ell(a, x) = a.x = b.y = \ell(b, y)$, for some $a, b \in G$, $x, y \in \mathrm{Fix}(H)$. Then $b^{-1}a.x = y \in \mathrm{Fix}\, H$ implies $hb^{-1}a.x = y = b^{-1}a.x$ which implies again $(b^{-1}a)^{-1}hb^{-1}a.x = x$. Since there is only one orbit type and all isotropy groups are compact, we know that $x \in \mathrm{Fix}\, H \Rightarrow H = G_x$ (by 4.8). So $(b^{-1}a)^{-1}hb^{-1}a$ is again in H, and $b^{-1}a \in N(H)$. In this case, $q(a, x) = [a, x] = [bb^{-1}a, x] = [b, b^{-1}a.x] = [b, y] = q(b, y)$.

$\tilde{\ell}^{-1}$ is smooth, since ℓ is a submersion. So $\tilde{\ell}$ is a diffeomorphism and $\tilde{\pi}$ a fiber bundle with typical fiber $\mathrm{Fix}(H)$. $\square$

7.4. Construction for more than one orbit type. Let (H) be one particular orbit type ($H = G_x$). To reduce the case at hand to the previous one, we must partition the points in M into sets with common orbit type:

$$M_{(H)} := \{x \in M : (G_x) = (H)\}$$

Claim. For a proper Riemannian G-manifold, the space $M_{(H)}$ as defined above is a smooth G-invariant submanifold.

Proof. $M_{(H)}$ is of course G-invariant as a collection of orbits of a certain type. We only have to prove that it is a smooth submanifold. Take any x in $M_{(H)}$(then, without loss of generality, $H = G_x$), and let S_x be a slice at x. Consider the tubular neighborhood $G.S \cong G \times_H S_x$ (4.15). Then the orbits of type (H) in $G.S$ are just those orbits that meet S_x in S_x^H (where S_x^H shall denote the fixed point set of H in S_x). Or, equivalently, $(G \times_H S_x)_{(H)} = G \times_H S_x^H$:

($\subseteq$) $[g, s] \in (G \times_H S_x)_{(H)} \Rightarrow g.s \in G.S_{(H)} \Rightarrow gHg^{-1} = G_s \subseteq H \Rightarrow G_s = H \Rightarrow s \in S_x^H \Rightarrow [g, s] \in G \times_H S_x^H$

($\supseteq$) $[g, s] \in G \times_H S_x^H \Rightarrow s \in S_x^H \Rightarrow H \subseteq G_s$, but since $s \in S_x$ we have $G_s \subseteq G_x = H$ by 4.13(2), therefore $G_s = H$ and $[g, s] \in (G \times_H S_x)_{(H)}$

Now, let $S_x = \exp_x(\mathrm{Nor}_r(G.x))$ be the normal slice at x. That is, r is chosen so small that $\exp_x$ is a diffeomorphism on $\mathrm{Nor}_r(G.x) =: V$. Notice, that V is not only diffeomorphic to S_x, but G-equivariantly so, if we let G act on $\mathrm{Nor}_x(G.x)$ via the slice representation. Since the slice action is orthogonal, in particular linear, the set of points fixed by the action of H is a linear subspace of $\mathrm{Nor}_x(G.x)$ and its intersection with V, a "linear" submanifold. Therefore S_x^H is also a submanifold of S_x. Now consider the diagram

$$\begin{tikzcd} G\times S_x^H \arrow[rr, "\ell"] \arrow[dr, "p"] & & M \\ & G\times_H S_x^H \arrow[ur, "i"] & \end{tikzcd}$$

The map i is well defined, injective and smooth, since p is a submersion and ℓ is smooth. Furthermore, p is open, and so is ℓ. Just consider any open set of the form $U\times W$ in $G\times S_x^H$. Then $\ell(U\times W)$ is the union of all sets $\ell_u(W)$ for $u\in U$. Since ℓ_u is a diffeomorphism, each one of these is open, so $\ell(U\times W)$ is open as well. Therefore, i must be open, and so i is an embedding and $G.S^H \cong G\times_H S_x^H$ an embedded submanifold of M. $\square$

Let (H) be one particular orbit type ($H = G_x$), then $\mathrm{Fix}(H)$ is again a closed, totally geodesic submanifold of M (see 6.1(3)).

Claim. $\mathrm{Fix}^*(H) := \{x\in M : G_x = H\}$ is an open submanifold of $\mathrm{Fix}(H)$.

Remark. For one orbit type, $x\in \mathrm{Fix}(H)$ implied $H = G_x$, and thus $\mathrm{Fix}^*(H) = \mathrm{Fix}(H)$. For more than one orbit type $\mathrm{Fix}(H)$ is not necessarily contained in $M_{(H)}$. Therefore, it is necessary to study $\mathrm{Fix}^*(H) = \mathrm{Fix}(H)\cap M_{(H)}$.

Proof. In 7.3 we saw that $N(H)$ is the largest subgroup of G acting on $\mathrm{Fix}(H)$. It induces a proper $N(H)/H$-action on $\mathrm{Fix}(H)$. Now, $\mathrm{Fix}^*(H)$ is the set of all points in $\mathrm{Fix}(H)$ with trivial isotropy group with respect to this action. So by 6.19 it is simply the set of all regular points. Therefore, by 6.15, $\mathrm{Fix}^*(H)$ is an open, dense submanifold of $\mathrm{Fix}(H)$. $\square$

Now, $M_{(H)}$ can be turned into a fiber bundle over $G/N(H)$ with typical fiber $\mathrm{Fix}^*(H)$ just as before ($\mathrm{Fix}^*(H)$ is really the fixed point space of H in $M_{(H)}$). And, on the other hand, $M_{(H)}$ is a fiber bundle over $M_{(H)}/G$ with typical fiber G/H. The partition of M into submanifolds $M_{(H)}$ and that of M/G into the different orbit types is locally finite by 6.16. So M and M/G are in a sense stratified, and $\pi: M\to M/G$ is a stratified Riemannian submersion (see also [13]).

7.5. Definition. *Let $p: E\to B$ be a Riemannian submersion.*
A vector field $\xi\in\mathfrak{X}(E)$ is called **vertical**, *if $\xi(x)\in V_x(p)$ for all x (i.e. if $Tp\,\xi(x) = 0$).*
$\xi\in\mathfrak{X}(E)$ is called **horizontal**, *if $\xi(x)\in \mathrm{Hor}_x(p)$ for all x, that is, if $\xi(x)\perp V_x(p)$ for all x.*
$\xi\in\mathfrak{X}(E)$ is called **projectable**, *if there is an $\eta\in\mathfrak{X}(B)$, such that $Tp.\xi = \eta\circ p$*
$\xi\in\mathfrak{X}(E)$ is called **basic**, *if it is horizontal and projectable.*

Remark. The orthogonal projection $\phi : TE \to V(E)$ with respect to the Riemann metric is a (generalized) connection on the bundle (E,p) and defines a local parallel transport over each curve in B (denoted by $Pt^\phi(c,.)$) as well as the horizontal lift:

$$C : TB \underset{B}{\times} E \to TE : (X_b, e) \mapsto Y_e, \text{ where } Y_e \in \operatorname{Hor}_e(p) \text{ with } T_e p.Y_e = X_b$$

This map also gives us an isomorphism $C_* : \mathfrak{X}(B) \to \mathfrak{X}_{\text{basic}}$ between the vector fields on B and the basic vector fields.

7.6. Lemma. *Consider a Riemannian submersion $p : (E, \gamma_E) \to (B, \gamma_B)$ with connection $\phi : TE \to V(p)$, and $c : [0,1] \to B$, a geodesic. Let $L_a^b(c)$ denote the arc length of c from $c(a)$ to $c(b)$ in B. Then:*

(1) $L_0^t(c) = L_0^t Pt^\phi(c,.,u)$, *where $u \in E_{c(0)}$ is the starting point of the parallel transport.*
(2) $Pt^\phi(c,.,u) \perp E_{c(t)}$ *for all t*
(3) *If c is a geodesic of minimal length in B, then we have* $L_0^1(Pt^\phi(c,.,u)) = \operatorname{dist}\big(E_{c(0)}, E_{c(1)}\big)$.
(4) $t \mapsto Pt^\phi(c,t,u)$ *is a geodesic in E (again for any geodesic c in B).*

Proof.

(1) Since $\frac{d}{ds}Pt^\phi(c,s,u)$ is a horizontal vector and by the property of p as Riemannian submersion, we have

$$\begin{aligned} L_0^t Pt^\phi(c,.,u) &= \int_0^t \gamma_E\left(\frac{d}{ds}Pt^\phi(c,s,u), \frac{d}{ds}Pt^\phi(c,s.u)\right)^{\frac12} ds \\ &= \int_0^t \gamma_B(c'(s),c'(s))^{\frac12} ds = L_0^t(c). \end{aligned}$$

(2) This is due to our choice of ϕ as orthogonal projection onto the vertical bundle in terms of the given metric on E. By this choice, the parallel transport is the unique horizontal curve covering c, so it is orthogonal to each fiber $E_{c(t)}$ it meets.
(3) Consider a (piecewise) smooth curve $e : [0,1] \to E$ from $E_{c(0)}$ to $E_{c(1)}$, then $p \circ e$ is a (piecewise) smooth curve from $c(0)$ to $c(1)$. Since c is a minimal geodesic, we have $L_0^1 c \le L_0^1(p \circ e)$. Furthermore, we can decompose the vectors tangent to e into horizontal and vertical components and use the fact that Tp is an isometry on horizontal vectors to show: $L_0^1 e \ge L_0^1(p \circ e)$ (in more detail: $e'(t) = h(t) + v(t) \in HE \oplus_E VE$, and since p is a Riemannian submersion $\gamma_B(Tp.h(t), Tp.h(t)) = \gamma_E(h(t), h(t))$ and $Tp.v(t) = 0$. Therefore $|\,Tp.e'(t)\,| = |\,Tp.h(t)\,| = |\,h(t)\,| \le |\,h(t) + v(t)\,| = |\,e'(t)\,|$, and $L_0^1 p \circ e \le L_0^1 e$.) Now with (1) we can conclude: $L_0^1 Pt^\phi(c,.,u) = L_0^1 c \le L_0^1 e$ for all (piecewise) smooth curves e from $E_{c(0)}$ to $E_{c(1)}$. Therefore, $L_0^1(Pt^\phi(c,.,u)) = \operatorname{dist}\big(E_{c(0)}, E_{c(1)}\big)$.
(4) This is a consequence of (3) and the observation that every curve which minimizes length locally is a geodesic. $\square$

7.7. Corollary. *Consider a Riemannian submersion $p : E \to B$, and let $c : [0,1] \to E$ be a geodesic in E with the property $c'(t_0) \perp E_{p(c(t_0))}$ for some t_0. Then $c'(t) \perp E_{p(c(t))}$ for all $t \in [0,1]$.*

Proof. Consider the curve $d : t \mapsto \exp^B_{p(c(t_0))}(tT_{c(t_0)}p.c'(t_0))$. It is a geodesic in B and therefore lifts to a geodesic $e(t) = Pt^\phi(d, t - t_0, c(t_0))$ in E (by 7.6(4)). Furthermore $e(t_0) = c(t_0)$ and $e'(t_0) = C(T_{c(t_0)}p.c'(t_0), c(t_0)) = c'(t_0)$ since $c'(t_0) \perp E_{p(c(t_0))}$ is horizontal. But geodesics are uniquely determined by their starting point and starting vector. Therefore $e = c$, and e is orthogonal to each fiber it meets by 7.6(2). □

7.8. Corollary. *Let $p : E \to B$ be a Riemannian submersion*

(1) *If* $\operatorname{Hor}(E)$ *is integrable, then every leaf is totally geodesic.*
(2) *If* $\operatorname{Hor}(E)$ *is integrable and S is a leaf, then $p_S : S \to B$ is a local isometry.*

Proof. (1) follows from corollary 7.7, while (2) is just a direct consequence of the definitions. □

7.9. Remark. If $p : E \to B$ is a Riemannian submersion, then $\operatorname{Hor}(E)|_{E_b} = \operatorname{Nor}(E_b)$ for all $b \in B$ and p defines a global parallelism as follows. A section $\tilde v \in C^\infty(\operatorname{Nor}(E_b))$ is called p-parallel, if $T_e p.\tilde v(e) = v \in T_b B$ is the same point for all $e \in E_b$. There is also a second parallelism. It is given by the induced covariant derivative: A section $\tilde v \in C^\infty(\operatorname{Nor}(E_b))$ is called parallel if $\nabla^{\operatorname{Nor}}\tilde v = 0$. The p-parallelism is always flat and with trivial holonomy which is not generally true for $\nabla^{\operatorname{Nor}}$. Yet we will see later on that if $\operatorname{Hor}(E)$ is integrable then the two parallelisms coincide.

7.10. Remark. Let M be a connected Riemannian G-manifold and (H) the principal orbit type, then we saw in 7.4 that $\pi : M_{(H)} \to M_{(H)}/G$ is a Riemannian submersion. Now we can prove:

Claim. $\xi \in C^\infty(\operatorname{Nor}(G.x))$ $\big(x \in M_{\text{reg}} = M_{(H)}\big)$ is π-parallel iff ξ is G-equivariant.

($\Longleftarrow$) $\xi(g.x) = T_x\ell_g.\xi(x)$ implies $T_{g.x}\pi.\xi(g.x) = T_{g.x}\pi \circ T_x\ell_g.\xi(x) = T_x\pi.\xi(x)$ for all $g \in G$. Therefore ξ is π-parallel.

($\Longrightarrow$) $\xi(g.x)$ and $T_x\ell_g\xi(x)$ are both in $\operatorname{Nor}_{g.x}(G.x)$, and since ξ is π-parallel we have:
$T_{g.x}\pi.\xi(g.x) = T_x\pi.\xi(x) = T_{g.x}\pi \circ T_x\ell_g.\xi(x)$. So $\xi(g.x)$ and $T_x\ell_g.\xi(x)$ both have the same image under $T_{g.x}\pi$. Because $T_{g.x}\pi$ restricted to $\operatorname{Nor}_{g.x}(G.x)$ is an isomorphism, $\xi(g.x) = T_x\ell_g.\xi(x)$. □

7.11. Definition. *A Riemannian submersion $p : E \to B$ is called integrable, if* $\operatorname{Hor}(E) = (\operatorname{Ker} Tp)^\perp$ *is an integrable distribution.*

7.12. Local Theory of Riemannian Submersions. Let $p : (E, \gamma_E) \to (B, \gamma_B)$ be a Riemannian submersion. Choose for an open neighborhood U in E an orthonormal frame field

$$s = (s_1, \dots, s_{n+k}) \in C^\infty(TE|U)^{n+k}$$

in such a way that $s_1, \dots, s_n$ are vertical and $s_{n+1}, \dots, s_{n+k}$ are basic. That way, if we "project" $s_{n+1}, \dots, s_{n+k}$ onto $TB|p(U)$ we get another orthonormal frame field, $\bar{s} = (\bar{s}_{n+1}, \dots, \bar{s}_{n+k}) \in C^\infty(TB|p(U))^k$, since p, as Riemannian submersion, is isometric on horizontal vectors.

In the following, $\sum$ will always refer to the sum over all indices occurring twice unless otherwise specified. Furthermore, we adopt the following index convention. The listed indices will always run in the domain indicated:

$$\begin{gathered} 1 \leq i, j, k \leq n \\ n+1 \leq \alpha, \beta, \gamma \leq n+k \\ 1 \leq A, B, C \leq n+k \end{gathered}$$

In this spirit, the orthogonal coframe corresponding to s is defined by the relation

$$\sigma^A(s_B) = \delta^A_B$$

We will write its components in the form of a column vector and in general adhere to the conventions of linear algebra so that, wherever possible, we can use matrix multiplication to avoid having to write down indices.

$$\sigma = \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^{n+k} \end{pmatrix} \in \Omega^1(U)^{n+k}$$

Analogously, we have the orthonormal coframe $\bar{\sigma}^\alpha \in \Omega^1(p(U))$ on $p(U) \subseteq B$, with

$$\bar{\sigma}^\alpha(\bar{s}_\beta) = \delta^\alpha_\beta.$$

It is related to σ^α by $p^*\bar{\sigma}^\alpha = \sigma^\alpha$. In terms of these, the Riemannian metrics γ_E and γ_B take on the form

$$\begin{aligned} \gamma_E|_U &= \sum_A \sigma^A \otimes \sigma^A \\ \gamma_B|_{p(U)} &= \sum_\alpha \bar{\sigma}^\alpha \otimes \bar{\sigma}^\alpha. \end{aligned}$$

Now let ∇ denote the Levi-Civita covariant derivative on (E, γ_E)

$$\nabla : \mathfrak{X}(E) \times \mathfrak{X}(E) \to \mathfrak{X}(E), \qquad (X, Y) \mapsto \nabla_X Y.$$

In terms of the frame field we will write the covariant derivative as

$$\nabla s_A = \sum_B s_B \omega^B_A, \qquad \omega^B_A \in \Omega^1(U).$$

If we view ω as the matrix of 1-forms (ω^B_A), then the above equation can be written in terms of matrix multiplication:

$$\nabla s = s.\omega$$

We get the following relation for ω.

$$0 = d\gamma_E(s_A, s_B) = \gamma_E(\nabla s_A, s_B) + \gamma_E(s_A, \nabla s_B) =$$
$$\gamma_E(\sum s_C\omega_A^C, s_B) + \gamma_E(s_A, \sum s_C\omega_B^C) = \omega_A^B + \omega_B^A$$

Therefore $\omega(X)$ is a real skewsymmetric matrix for all $X \in \mathfrak{X}(U)$, and we have

$$\omega \in \Omega^1(U, \mathfrak{so}(n+k)).$$

An arbitrary vector field X on U can be written as $X = \sum s_i u^i$ where $u^i \in C^\infty(U, \mathbb{R})$ can be regarded as the components of a column-vector-valued function u so that we can write $X = s.u$. Its covariant derivative can be calculated directly using the derivation property.

$$\nabla(s.u) = \nabla s.u + s.du = s.\omega.u + s.du$$

Now let us calculate the curvature tensor in this setting.

$$R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Let $R(X,Y)s$ denote the row of vector fields $R(X,Y)s_A$. Then we can go on to calculate:

$$\begin{aligned}
R(X,Y)s &= \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s = \\
&= \nabla_X(s.\omega(Y)) - \nabla_Y(s.\omega(X)) - s.\omega\,([X,Y]) = \\
&= (\nabla_X s).\omega(Y) + s.X.\omega(Y) - (\nabla_Y s).\omega(X) - s.Y.\omega(X) - s.\omega\,([X,Y]) = \\
&= s.\omega(X).\omega(Y) - s.\omega(Y).\omega(X) + s.\left(X.\omega(Y) - Y.\omega(X) - \omega\,([X,Y])\right) = \\
&= s.\omega\wedge\omega(X,Y) + s.d\omega(X,Y) = s.(d\omega + \omega\wedge\omega)(X,Y)
\end{aligned}$$

The notation $\omega \wedge \omega$ stands for $(\sum \omega_C^A \wedge \omega_B^C)_B^A$, which has the form of a standard matrix multiplication, only with the usual product on the components replaced by the exterior product. This leads to the definition $\Omega := d\omega + \omega\wedge\omega = d\omega + \frac{1}{2}[\omega,\omega]^\wedge$. Like with ω, the orthonormality of s implies $\Omega_i^j = -\Omega_j^i$, so $\Omega^2(U, \mathfrak{so}(n+k))$. The **second Bianchi identity** follows directly:

(2. Bianchi identity) $$d\Omega + \omega\wedge\Omega - \Omega\wedge\omega = d\Omega + [\omega, \Omega]^\wedge = 0$$

Using the property that the Levi-Civita connection is free of torsion, we can derive the so-called **structure equation** on ω. It determines the Levi-Civita connection completely.

$$\begin{aligned}
0 = \operatorname{Tor}(X,Y) &= \nabla_X(s.\sigma(Y)) - \nabla_Y(s.\sigma(X)) - s.\sigma\,([X,Y]) = \\
&= s.\omega(X).\sigma(Y) + s.X(\sigma(Y)) - s.\omega(Y)\sigma(X) - s.Y(\sigma(X)) - s.\sigma\,([X,Y]) = \\
&= s.(\omega(X).\sigma(Y) - \omega(Y).\sigma(X)) + s.(X(\sigma(Y)) - Y(\sigma(X)) - \sigma\,([X,Y])) = \\
&= s.(\omega\wedge\sigma(X,Y)) + s.d\sigma(X,Y) = s.(\omega\wedge\sigma + d\sigma)(X,Y)
\end{aligned}$$

"structure equation"
$$\omega\wedge\sigma+d\sigma=0$$
$$\text{or}\quad \sum_B \omega^A_B\wedge\sigma^B+d\sigma^A=0$$

As a direct consequence, the **first Bianchi identity** takes on the following form.

(1. Bianchi identity)
$$\Omega\wedge\sigma=0$$

If we pull back the structure equation $d\bar\sigma+\bar\omega\wedge\bar\sigma=0$ from B to E, we can derive some relations between the components ω^α_A of ω:

$$\begin{aligned}0=p^*\left(d\bar\sigma^\alpha+\sum\bar\omega^\alpha_\beta\wedge\bar\sigma^\beta\right)=\\ =dp^*\bar\sigma^\alpha+\sum\left(p^*\bar\omega^\alpha_\beta\right)\wedge\left(p^*\bar\sigma^\beta\right)=d\sigma^\alpha+\sum\left(p^*\bar\omega^\alpha_\beta\right)\wedge\sigma^\beta\end{aligned}$$

Together with the α-component of the structure equation on E, $d\sigma^\alpha+\sum\omega^\alpha_\beta\wedge\sigma^\beta+\sum\omega^\alpha_i\wedge\sigma^i=0$, this gives us:

$$(*)\qquad \sum\left(p^*\bar\omega^\alpha_\beta\right)\wedge\sigma^\beta=\sum\omega^\alpha_\beta\wedge\sigma^\beta+\sum\omega^\alpha_i\wedge\sigma^i$$

The lefthand side of this equation contains no $\sigma^i\wedge\sigma^\alpha$- or $\sigma^i\wedge\sigma^j$-terms. Let us write out ω^α_β and ω^α_i in this basis.

$$\begin{aligned}\omega^\alpha_\beta&=-\omega^\beta_\alpha=:\sum q^\alpha_{\beta\gamma}\sigma^\gamma+\sum b^\alpha_{\beta i}\sigma^i\\ \omega^\alpha_i&=-\omega^i_\alpha=:\sum a^\alpha_{i\beta}\sigma^\beta+\sum r^\alpha_{ij}\sigma^j\end{aligned}$$

This gives us for the righthand side of (*)

$$\begin{aligned}&\sum q^\alpha_{\beta\gamma}\sigma^\gamma\wedge\sigma^\beta+\sum b^\alpha_{\beta i}\sigma^i\wedge\sigma^\beta+\sum a^\alpha_{i\beta}\sigma^\beta\wedge\sigma^i+\sum r^\alpha_{ij}\sigma^j\wedge\sigma^i=\\ &\sum q^\alpha_{\beta\gamma}\sigma^\gamma\wedge\sigma^\beta+\sum\left(b^\alpha_{\beta i}-a^\alpha_{i\beta}\right)\sigma^i\wedge\sigma^\beta+\frac12\sum\left(r^\alpha_{ij}-r^\alpha_{ji}\right)\sigma^j\wedge\sigma^i\end{aligned}$$

So we have found

$$\begin{aligned}a^\alpha_{i\beta}&=b^\alpha_{\beta i}\\ r^\alpha_{ij}&=r^\alpha_{ji},\end{aligned}$$

or, in other words,

$$\begin{aligned}\omega^\alpha_i(s_\beta)&=\omega^\alpha_\beta(s_i)\\ \omega^\alpha_i(s_j)&=\omega^\alpha_j(s_i)\end{aligned}$$

That is: $\omega^\alpha_i(s_A)=\omega^\alpha_A(s_i)$, and this just means that the horizontal part of $[s_A,s_i]$ is 0, or $[s_A,s_i]$ is always vertical:

$$0=\sum s_\alpha\omega^\alpha_i(s_A)-\sum s_\alpha\omega^\alpha_A(s_i)=\left(\nabla_{s_A}s_i-\nabla_{s_i}s_A\right)^{\text{hor}}=\left([s_{A],s_i}\right)^{\text{hor}}.$$

Now we will calculate the second fundamental form $S : \mathfrak{X}E_b \times_{E_b} \mathfrak{X}E_b \to \mathfrak{X}^{\text{hor}}(E|\,E_b)$ of $E_b := p^{-1}(b)$ in E. Let $\tilde{\nabla}$ denote the Levi-Civita covariant derivative on E_b corresponding to the induced metric i^*g (where $i : E_b \hookrightarrow E$ is the inclusion). Since every vector field on E_b can be extended to a vertical vector field on E (do it in charts, patch it up with a partition of unity and then compose with the connection ϕ to make it vertical), we can determine ∇ for vector fields defined only on E_b by extending them onto E. We will denote the restriction of ∇ onto E_b again by ∇. It can easily be checked that this definition is independent of the extension chosen. Now the second fundamental form is defined as:

$$S(X^{\text{ver}}, Y^{\text{ver}}) := \nabla_{X^{\text{ver}}} Y^{\text{ver}} - \tilde{\nabla}_{X^{\text{ver}}} Y^{\text{ver}}$$

If we express $\tilde{\nabla}$ in terms of ∇, we get

$$S(X^{\text{ver}}, Y^{\text{ver}}) = \nabla_{X^{\text{ver}}} Y^{\text{ver}} - (\nabla_{X^{\text{ver}}} Y^{\text{ver}})^{\text{ver}} = (\nabla_{X^{\text{ver}}} Y^{\text{ver}})^{\text{hor}}$$

Expressed in the local frame, it is:

$$\begin{aligned}
(\nabla_{X^{\text{ver}}} Y^{\text{ver}})^{\text{hor}} &= \Big(\nabla_{X^{\text{ver}}}(\sum s_i\sigma^i(Y^{\text{ver}}))\Big)^{\text{hor}} = \\
&= \Big(\sum(\nabla_{X^{\text{ver}}} s_i)\sigma^i\,(Y^{\text{ver}}) + \sum s_i d(\sigma^i(Y^{\text{ver}})).X^{\text{ver}}\Big)^{\text{hor}} = \\
&= (\sum s_A\omega_i^A(X^{\text{ver}})\sigma^i(Y^{\text{ver}}))^{\text{hor}} + 0 = \sum s_\alpha\omega_i^\alpha(X^{\text{ver}})\sigma^i(Y^{\text{ver}}) = \\
&= \sum r_{ij}^\alpha s_\alpha \otimes \sigma^j \otimes \sigma^i\,(X^{\text{ver}}, Y^{\text{ver}})
\end{aligned}$$

So

$$\sum s_\alpha \sigma^\alpha(S) = \sum r_{ij}^\alpha\, s_\alpha \otimes \sigma^j \otimes \sigma^i.$$

S is a symmetric tensor field as indeed the second fundamental form must always be. But in our special case we have already shown that $r_{ij}^\alpha = r_{ji}^\alpha$ and thereby proved this result directly.

Similarly to the covariant derivative on the vertical bundle, which was obtained by taking the vertical part of the covariant derivative $\nabla_{X^{\text{ver}}} Y^{\text{ver}}$ of two vertical vector fields, we can define a covariant derivative on the the normal bundle $\operatorname{Nor}(E_b) \to E_b$ by taking the horizontal part of the covariant derivative $\nabla_{X^{\text{ver}}} Y^{\text{hor}}$ of a horizontal vector field along a vertical vector field:

$$\begin{gathered}
\nabla^{\text{Nor}} : \mathfrak{X}(E_b) \times C^\infty(\operatorname{Nor}(E_b)) \to C^\infty(\operatorname{Nor}(E_b)) \\
\nabla^{\text{Nor}}_{X^{\text{ver}}} Y^{\text{hor}} := \big(\nabla_{X^{\text{ver}}} Y^{\text{hor}}\big)^{\text{hor}}.
\end{gathered}$$

In our frame field:

$$\begin{aligned}
\nabla^{\text{Nor}}_{X^{\text{ver}}} Y^{\text{hor}} &= \Big(\nabla_{X^{\text{ver}}}\Big(\sum s_\beta \sigma^\beta(Y^{\text{hor}})\Big)\Big)^{\text{hor}} = \\
&= \Big(\sum (\nabla_{X^{\text{ver}}} s_\beta)\,\sigma^\beta(Y^{\text{hor}})\Big)^{\text{hor}} + \sum s_\beta d\sigma^\beta(Y^{\text{hor}}).X^{\text{ver}} = \\
&= \sum s_\alpha\omega_\beta^\alpha(X^{\text{ver}})\sigma^\beta(Y^{\text{hor}}) + \sum s_\beta d\sigma^\beta(Y^{\text{hor}}).X^{\text{ver}} = \\
&= \sum b_{\beta i}^\alpha s_\alpha \otimes \sigma^i \otimes \sigma^\beta(X^{\text{ver}}, Y^{\text{hor}}) + \sum s_\alpha \otimes d\sigma^\beta(Y^{\text{hor}})(X^{\text{ver}})
\end{aligned}$$

or

$$\nabla^{\mathrm{Nor}} Y^{\mathrm{hor}} = \sum \left(b^{\alpha}_{\beta i} \sigma^{\beta}(Y^{\mathrm{hor}}) \sigma^i + d\sigma^{\alpha}(Y^{\mathrm{hor}}) \right) \otimes s_{\alpha}.$$

Like ∇ itself, ∇^{Nor} is not a tensor field. Yet in the decomposition

$$\nabla_X Y = \left(\nabla_{X^{\mathrm{ver}} + X^{\mathrm{hor}}} (Y^{\mathrm{ver}} + Y^{\mathrm{hor}}) \right)^{\mathrm{ver} + \mathrm{hor}}$$

we can find two more tensor fields (besides S), the so called fundamental (or O'Neill-) tensor fields. (see [28])

$$\begin{aligned}
&X, Y \in \mathfrak{X}(E) \\
&T(X,Y) := (\nabla_{X^{\mathrm{ver}}} Y^{\mathrm{ver}})^{\mathrm{hor}} + \left(\nabla_{X^{\mathrm{ver}}} Y^{\mathrm{hor}} \right)^{\mathrm{ver}} \\
&A(X,Y) := \left(\nabla_{X^{\mathrm{hor}}} Y^{\mathrm{hor}} \right)^{\mathrm{ver}} + (\nabla_{X^{\mathrm{hor}}} Y^{\mathrm{ver}})^{\mathrm{hor}}
\end{aligned}$$

In fact each of of these four summands which make up A and T are tensor fields by themselves - the first one restricting to S on E_b. Why they are combined to two tensors in just this way we will see once we have expressed them in our local frame. At the same time, we will see that they really are tensor fields.

$$\begin{aligned}
A(X,Y) &= \left(\nabla_{X^{\mathrm{hor}}} \left(\sum s_{\alpha} \sigma^{\alpha}(Y) \right) \right)^{\mathrm{ver}} + \left(\nabla_{X^{\mathrm{hor}}} (\sum s_i \sigma^i(Y)) \right)^{\mathrm{hor}} = \\
&= \sum s_i \omega^i_{\alpha}(X^{\mathrm{hor}}) \sigma^{\alpha}(Y) + 0 + \sum s_{\alpha} \omega^{\alpha}_i (X^{\mathrm{hor}}) \sigma^i(Y) + 0 = \\
&= \sum s_i \left(-a^{\alpha}_{i\beta} \right) \sigma^{\beta}(X) \sigma^{\alpha}(Y) + \sum s_{\alpha} a^{\alpha}_{i\beta} \sigma^{\beta}(X) \sigma^i(Y) = \\
&= \left(\sum a^{\alpha}_{i\beta} \left(\sigma^{\beta} \otimes \sigma^i \otimes s_{\alpha} - \sigma^{\beta} \otimes \sigma^{\alpha} \otimes s_i \right) (X,Y) \right)
\end{aligned}$$

Analogously:

$$T = \sum r^{\alpha}_{ij} \left(\sigma^j \otimes \sigma^i \otimes s_{\alpha} - \sigma^i \otimes \sigma^{\alpha} \otimes s_i \right)$$

If $\mathrm{Hor}(E)$ is integrable, then every leaf L is totally geodesic by 7.8(1), and the $s_{\alpha}|_L$ are a local orthonormal frame field on L. L being totally geodesic is equivalent to its second fundamental form vanishing. Now, in the same way we found S, the second fundamental form of L is

$$S_L(X^{\mathrm{hor}}, Y^{\mathrm{hor}}) := \left(\nabla_{X^{\mathrm{hor}}} Y^{\mathrm{hor}} \right)^{\mathrm{ver}}$$

So it is a necessary condition for the integrability of $\mathrm{Hor}(E)$ that $S_L = 0$, that is

$$\begin{aligned}
0 = S_L(s_{\alpha}, s_{\beta}) &= (\nabla_{s_{\alpha}} s_{\beta})^{\mathrm{ver}} = \\
= \sum s_i \omega^i_{\beta}(s_{\alpha}) &= \sum s_i \left(-a^{\beta}_{i\gamma} \right) \sigma^{\gamma}(s_{\alpha}).
\end{aligned}$$

This is equivalent to the condition

$$a^{\alpha}_{i\beta} = 0 \qquad \text{for all } {}^{\alpha}_{i\beta}$$

or

$$A = 0.$$

Let us now prove the converse: If A vanishes, then the the horizontal distribution on E is integrable. In this case, we have $0 = A(s_{\alpha}, s_{\beta}) = (\nabla_{s_{\alpha}} s_{\beta})^{\mathrm{ver}} + 0$, as well as $0 = A(s_{\beta}, s_{\alpha}) = \left(\nabla_{s_{\beta}} s_{\alpha} \right)^{\mathrm{ver}} + 0$. Therefore, $[s_{\alpha}]_{, s_{\beta}} = \nabla_{s_{\alpha}} s_{\beta} - \nabla_{s_{\beta}} s_{\alpha}$ is horizontal, and the horizontal distribution is integrable.

7.13. Theorem. *Let $p : E \to B$ be a Riemannian submersion, then the following conditions are equivalent.*

(1) *p is integrable (that is* $\operatorname{Hor}(p)$ *is integrable).*
(2) *Every p-parallel normal field along E_b is ∇^{Nor}-parallel.*
(3) *The O'Neill tensor A is zero.*

Proof. We already saw (1) $\iff$ (3) above.

(3) $\implies$ (2) Take s_α for a p-parallel normal field X along E_b. $A = 0$ implies $A(s_\alpha, s_i) = 0 + (\nabla_{s_\alpha} s_i)^{\mathrm{hor}} = 0$. Recall that, as we showed above, $[s_i, s_\alpha]$ is vertical. Therefore,

$$\nabla^{\mathrm{Nor}}_{s_i} s_\alpha = (\nabla_{s_i} s_\alpha)^{\mathrm{hor}} = \big([s_i, s_\alpha] + \nabla_{s_\alpha} s_i\big)^{\mathrm{hor}} = 0$$

Since for any $e \in E_b$, $T_e p|_{\mathrm{Nor}_b(E_b)}$ is an isometric isomorphism, a p-parallel normal field X along E_b is determined completely by the equation $X(e) = \sum X^\alpha(e) s_\alpha(e)$. Therefore it is always a linear combination of the s_α with constant coefficients and we are done.

(2) $\implies$ (3) By (2) $\nabla^{\mathrm{Nor}}_{s_i} s_\alpha = (\nabla_{s_i} s_\alpha)^{\mathrm{hor}} = 0$. Therefore, as above, we have that $\big([s_i, s_\alpha] + \nabla_{s_\alpha} s_i\big)^{\mathrm{hor}} = 0 + (\nabla_{s_\alpha} s_i)^{\mathrm{hor}} = A(s_\alpha, s_i) = 0$. Thus $\sigma^\beta A(s_\alpha, s_i) = a^\beta_{\alpha i} = 0$, so A vanishes completely. $\square$

8. Sections

In this chapter, let (M,γ) always denote a connected, complete Riemannian G-manifold, and assume that the action of G on M is effective and isometric.

8.1. Lemma. *Consider $X \in \mathfrak{g}$, the Lie algebra of G, ζ_X, the associated fundamental vector field to X, and c, a geodesic in M. Then $\gamma(c'(t),\zeta_X(c(t)))$ is constant in t.*

Proof. Let ∇ be the Levi-Civita covariant derivative on M. Then

$$\partial_t.\gamma(c'(t),\zeta_X(c(t))) = \gamma\left(\nabla_{\partial_t}c'(t),\zeta_X(c(t))\right) + \gamma\left(c'(t),\nabla_{\partial_t}(\zeta_X\circ c)\right).$$

Since c is a geodesic, $\nabla_{\partial_t}c'(t)=0$, and so is the entire first summand. So it remains to show that $\gamma\left(c'(t),\nabla_{\partial_t}(\zeta_X\circ c)\right)$ vanishes as well.

Let $s_1,\dots,s_n$ be a local orthonormal frame field on an open neighborhood U of $c(t)$, and $\sigma^1,\dots,\sigma^n$ the orthonormal coframe. Then $\gamma=\sum\sigma^i\otimes\sigma^i$. Let us use the notation

$$\begin{aligned}\zeta_X|_U &=: \sum s_i X^i\\ \nabla\zeta_X|_U &=: \sum X_i^j s_j\otimes\sigma^i.\end{aligned}$$

Then we have

$$\nabla_{\partial_t}(\zeta_X\circ c) = \sum X_i^j(c(t))s_j(c(t))\sigma^i(c'(t)).$$

So

$$\begin{aligned}\gamma\left(c'(t),\nabla_{\partial_t}(\zeta_X\circ c)\right) &= \sum\sigma^j(c'(t))\sigma^j\left(\nabla_{\partial_t}(\zeta_X\circ c)\right) =\\ &= \sum X_i^j(c(t))\sigma^j(c'(t))\sigma^i(c'(t)).\end{aligned}$$

If we now show that $X_i^j+X_j^i=0$, then $\gamma\left(c'(t),\nabla_{\partial_t}(\zeta_X\circ c)\right)$ will be zero, and the proof will be complete. Since the action of G is isometric, ζ_X is a Killing vector field; that is $\mathcal{L}_{\zeta_X}\gamma=0$. So we have

$$\sum\mathcal{L}_{\zeta_X}\sigma^i\otimes\sigma^i+\sum\sigma^i\otimes\mathcal{L}_{\zeta_X}\sigma^i=0.$$

Now we must try to express $\mathcal{L}_{\zeta_X}\sigma^i$ in terms of X_i^j. For this, recall the structure equation: $d\sigma^k+\sum\omega_j^k\wedge\sigma^j=0$. Now we have

$$\begin{aligned}\mathcal{L}_{\zeta_X}\sigma^i &= i_{\zeta_X}d\sigma^i+d\left(i_{\zeta_X}\sigma^i\right) = -i_{\zeta_X}\left(\sum\omega_j^i\wedge\sigma^j\right)+d(\sigma^i(\zeta_X)) =\\ &= -i_{\zeta_X}\sum\omega_j^i\wedge\sigma^j+dX^i = \sum\omega_j^i.X^j-\sum\omega_j^i(\zeta_X)\sigma^j+dX^i.\end{aligned}$$

Since

$$\nabla\zeta_X|_U = \nabla(\sum s_j X^j) = \sum s_i.\omega^i_j.X^j + \sum s_i \otimes dX^i = \sum X^i_j s_i \otimes \sigma^j,$$

we can replace $\sum \omega^i_j.X^j$ by $\sum X^i_j\sigma^j - dX^i$. Therefore,

$$\mathcal{L}_{\zeta_X}\sigma^i = \sum (X^i_j\sigma^j - \omega^i_j(\zeta_X)\sigma^j) = \sum (X^i_j - \omega^i_j(\zeta_X))\sigma^j$$

Now, let us insert this into $0 = \mathcal{L}_{\zeta_X}\gamma$:

$$\begin{aligned} 0 &= \sum \mathcal{L}_{\zeta_X}\sigma^i \otimes \sigma^i + \sum \sigma^i \otimes \mathcal{L}_{\zeta_X}\sigma^i = \\ &= \sum (X^i_j - \omega^i_j(\zeta_X))\sigma^j \otimes \sigma^i + \sum (X^i_j - \omega^i_j(\zeta_X))\sigma^i \otimes \sigma^j = \\ &= \sum (X^i_j + X^j_i)\sigma^j \otimes \sigma^i - \sum (\omega^i_j(\zeta_X) + \omega^j_i(\zeta_X))\sigma^j \otimes \sigma^i = \\ &= \sum (X^i_j + X^j_i)\sigma^j \otimes \sigma^i - 0 \end{aligned}$$

since $\omega(Y)$ is skew symmetric. This implies $X^i_j + X^j_i = 0$, and we are done. □

8.2. Definition. *For any x in M_{reg} we define:*

$$\begin{aligned} E(x) &:= \exp^\gamma_x(\operatorname{Nor}_x(G.x)) \subseteq M \\ E_{\text{reg}}(x) &:= E(x) \cap M_{\text{reg}} \end{aligned}$$

In a neighborhood of x, $E(x)$ is a manifold; globally, it can intersect itself.

8.3. Lemma. *Let $x \in M_{\text{reg}}$ then*

(1) $g.E(x) = E(g.x)$, $g.E_{\text{reg}}(x) = E_{\text{reg}}(g.x)$.
(2) *For $X_x \in \operatorname{Nor}(G.x)$ the geodesic $c : t \mapsto \exp(t.X_x)$ is orthogonal to every orbit it meets.*
(3) *If G is compact, then $E(x)$ meets every orbit in M.*

Proof.

(1) This is a direct consequence of 6.1(1): $g.\exp_x(t.X) = \exp_{g.x}(t.T_x\ell_g.X)$.
(2) By choice of starting vector X_x, the geodesic c is orthogonal to the orbit $G.x$, which it meets at $t = 0$. Therefore it intersects every orbit it meets orthogonally, by Lemma 8.1.
(3) For arbitrary $x, y \in M$, we will prove that $E(x)$ intersects $G.y$. Since G is compact, by continuity of $\ell^y : G \to M$ the orbit $G.y$ is compact as well. Therefore we can choose $g \in G$ in such a way, that $\operatorname{dist}(x, G.y) = \operatorname{dist}(x, g.y)$. Let $c(t) := \exp_x(t.X_x)$ be a minimal geodesic connecting $x = c(0)$ with $g.y = c(1)$. We now have to show, that $X_x \in \operatorname{Nor}_x(G.x)$: Take a point $p = c(t)$ on the geodesic very close to $g.y$—close enough so that $\exp_p$ is a diffeomorphism into a neighborhood U_p of p containing $g.y$ (it shall have domain $V \subseteq T_pM$). In this situation the lemma of Gauss states, that all geodesics through p are orthogonal to the "geodesic spheres": $\exp_p\left(k.S^{m-1}\right)$ (where $S^{m-1} := \{X_p \in T_pM : \gamma(X_p, X_p) = 1\}$, and $k > 0$ is

small enough for $k.S^{m-1} \subseteq V$ to hold). From this it can be concluded that c is orthogonal to $G.y$: Take the smallest geodesic sphere around p touching $G.y$. By the minimality of c, c must leave the geodesic sphere at a touching point, and by Gauss' lemma, it must leave at a right angle to the geodesic sphere. Clearly, the touching point is just $g.y = c(1)$, and there c also meets $G.y$ at a right angle. By (2), c encloses a right angle with every other orbit it meets as well. In particular, c starts orthogonally to $G.x$. Therefore, X_x is in $\operatorname{Nor}_x(G.x)$, and $g.y = c(1) \in E(x)$. $\square$

8.4. Remark. Let $x \in M$ be a regular point and S_x the normal slice at x. If S_x is orthogonal to every orbit it meets, then so are all $g.S_x$ ($g \in G$ arbitrary). So the submanifolds $g.S_x$ can be considered as leaves of the horizontal foliation (local solutions of the horizontal distribution—which has constant rank in a neighborhood of a regular point), and the Riemannian submersion $\pi : M_{\text{reg}} \to M_{\text{reg}}/G$ is integrable. Since this is not always the case (the horizontal distribution is not generally integrable), it must also be false, in general, that the normal slice is orthogonal to every orbit it meets. But it does always meet orbits transversally.

Example. Consider the isometric action of the circle group S^1 on $\mathbb{C} \times \mathbb{C}$ (as real vector spaces) defined by $e^{it}.(z_1, z_2) := \big(e^{it}.z_1, e^{it}.z_2\big)$. Then $p = (0,1)$ is a regular point: $G_p = \{1\}$. The subspace $\operatorname{Nor}_p(S^1.p)$ of $T_p\mathbb{C} \times \mathbb{C}$ takes on the following form: $\operatorname{Nor}_p(S^1.p) = \langle (1,0), (i,0), (0,1) \rangle_{\mathbb{R}} = \mathbb{C} \times \mathbb{R}$. Therefore, we get: $E(0,1) = \{(u, 1{+}r) : u \in \mathbb{C}, r \in \mathbb{R}\}$. In particular, $y = (1,1) \in E(0,1)$, but $S^1.y = \{\big(e^{it}, e^{it}\big) : t \in \mathbb{R}\}$ is not orthogonal to $E(0,1)$. Its tangent space, $T_y(S^1.y) = \langle (i,i) \rangle_{\mathbb{R}}$, is not orthogonal to $\mathbb{C} \times \mathbb{R}$.

8.5. Definition. *A connected closed complete submanifold $\Sigma \subset M$ is called a* **section** *for the G-action if*

(1) *Σ meets every orbit, or equivalently: $G.\Sigma = M$.*
(2) *Where Σ meets an orbit, it meets it orthogonally.*

The second condition can be replaced by the equivalent

(2') $x \in \Sigma \Rightarrow T_x\Sigma \subseteq \operatorname{Nor}_x(G.x)$ *or*
(2") $x \in \Sigma, X \in \mathfrak{g} \Rightarrow \zeta_X(x) \perp T_x\Sigma$.

Remark. If Σ is a section, then so is $g.\Sigma$ for all g in G. Since $G.\Sigma = M$, there is a section through every point in M. We say "M admits sections".

The notion of a section was introduced by Szenthe [42], [43], in slightly different form by Palais and Terng in [32], [33]. The case of linear representations was considered by Bott and Samelson [4], Conlon [10], and then by Dadok [11] who called representations admitting sections polar representations (see 8.20) and completely classified all polar representations of connected Lie groups. Conlon [9] considered Riemannian manifolds admitting flat sections. We follow here the notion of Palais and Terng.

8.6. Example. For the standard action of $O(n)$ on $\mathbb{R}^n$ the orbits are spheres, and every line through 0 is a section.

8.7. Example. If G is a compact, connected Lie group with biinvariant metric, then $\operatorname{conj} : G \times G \to G$, $\operatorname{conj}_g(h) = ghg^{-1}$ is an isometric action on G. The orbits are just the conjugacy classes of elements.

Proposition. *Every maximal torus H of a compact connected Lie group G is a section.*

A torus is a product of circle groups or equivalently a compact connected abelian Lie group; a maximal torus of a compact Lie group is a toral subgroup which is not properly contained in any larger toral subgroup (cf. [5], chapter 6).)

Proof. (1) $\operatorname{conj}(G).H = G$: This states that any $g \in G$ can be found in some to H conjugate subgroup, $g \in aHa^{-1}$. This is equivalent to $ga \in aH$ or $gaH = aH$. So the conjecture now presents itself as a fixed point problem: does the map $\ell_g : G/H \to G/H : aH \mapsto gaH$ have a fixed point. It is solved in the following way:

The **fixed point theorem of Lefschetz** (see [41], 11.6.2, p.297) says that

> *a smooth mapping $f : M \to M$ from a connected compact manifold to itself has no fixed point if and only if*
>
> $$\sum_{i=0}^{\dim M} (-1)^i \operatorname{Trace}(H^i(f) : H^i(M) \to H^i(M)) = 0.$$

Since G is connected, ℓ_g is homotopic to the identity, so

$$\sum_{i=0}^{\dim G/H} (-1)^i \operatorname{Trace}(H^i(\ell_g) : H^i(G/H) \to H^i(G/H)) =$$
$$= \sum_{i=0}^{\dim G/H} (-1)^i \operatorname{Trace}(H^i(\mathrm{Id})) = \sum_{i=0}^{\dim G/H} (-1)^i \dim H^i(G/H) = \chi(G/H),$$

the Euler characteristic of G/H. This is given by the following **theorem** ([30], Sec. 13, Theorem 2, p.217)

> *If G is a connected compact Lie group and H is a connected compact subgroup then the Euler characteristic $\chi(G/H) \geq 0$. Moreover $\chi(G/H) > 0$ if and only if the rank of G equals the rank of H. In case when $\chi(G/H) > 0$ then $\chi(G/H) = |W_G|/|W_H|$, the quotient of the respective Weyl groups.*

Since the Weyl group of a torus is trivial, in our case we have $\chi(G/H) = |W_G| > 0$, and thus there exists a fixed point.

(2") $h \in H, X \in \mathfrak{g} \Rightarrow \zeta_X(h) \perp T_h H$:
$\zeta_X(h) = \frac{d}{dt}\big|_{t=0} \exp(tX)h\exp(-tX) = T_e\mu^h.X - T_e\mu_h.X$. Now choose $Y \in \mathfrak{h}$. Then we have $T_e\mu_h.Y \in T_hH$, and

$$\gamma_h(T_e\mu_h.Y, T_e\mu^h.X - T_e\mu_h.X) = \gamma_e(Y, Ad(h).X - X) =$$
$$= \gamma_e(Y, Ad(h).X) - \gamma_e(Y,X) = \gamma_e(Ad(h).Y, Ad(h).X) - \gamma_e(Y,X) = 0$$

by the right, left and therefore Ad-invariance of γ and by the commutativity of H. $\square$

8.8. Example. Let G be a compact semisimple Lie group acting on its Lie algebra by the adjoint action $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$. Then every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a section.

Proof. Every element of a semisimple Lie algebra $\mathfrak{g}$ is contained in a Cartan subalgebra, and any two Cartan subalgebras are conjugated by an element $g \in G$, since G is compact. This is a consequence of 8.7 above, since the subgroup in G corresponding to a Cartan subalgebra is a maximal torus. Thus every Ad_G-orbit meets the Cartan subalgebra $\mathfrak{h}$. It meets orthogonally with respect to the Cartan Killing form B: Let $H_1, H_2 \in \mathfrak{h}$ and $X \in \mathfrak{g}$. Then $\frac{d}{dt}|_0 \mathrm{Ad}(\exp(tX)).H_1 = \mathrm{ad}(X)H_1$ is a typical vector tangent to the orbit through $H_1 \in \mathfrak{h}$, and H_2 is tangent to $\mathfrak{h}$. Then

$$B(\mathrm{ad}(X)H_1, H_2) = B([X, H_1], H_2) = B(X, [H_1, H_2]) = 0$$

since $\mathfrak{h}$ is commutative. $\square$

8.9. Example. In Theorem 1.1 we showed that for the $O(n)$-action on $S(n)$ by conjugation the space Σ of all diagonal matrices is a section.

8.10. Example. Similarly as in 8.9, when the $SU(n)$ act on the Hermitian matrices by conjugation, the (real) diagonal matrices turn out to be a section.

8.11. Definition. *The* **principal horizontal distribution** *on a Riemannian G-manifold M is the horizontal distribution on $M_{\mathrm{reg}} \xrightarrow{\pi} M_{\mathrm{reg}}/G$.*

8.12. Theorem. *If a connected, complete Riemannian G-manifold M has a section Σ, then*

(1) *The principal horizontal distribution is integrable.*
(2) *Every connected component of Σ_{reg} is a leaf for the principal horizontal distribution.*
(3) *If L is the leaf of* $\mathrm{Hor}\,(M_{\mathrm{reg}})$ *through $x \in M_{\mathrm{reg}}$, then $\pi|_L : L \to M_{\mathrm{reg}}/G$ is an isometric covering map.*
(4) *Σ is totally geodesic.*
(5) *Through every regular point $x \in M$ there is a unique section: $E(x) = \exp_x^\gamma(\mathrm{Nor}_x(G.x))$*
(6) *A G-equivariant normal field along a principal orbit is parallel in terms of the induced covariant derivative ∇^{Nor}.*

Proof.

(1) The submanifolds $g.\Sigma_{\mathrm{reg}}$ of M_{reg} are integral manifolds to the horizontal distribution, since they are orthogonal to each orbit and by an argument of dimension.
(2) clear.
(3) see 7.8(2).
(4) see 7.8(1).
(5) This is a consequence of (4). Namely, for $x \in M$ choose $g \in G$ such that $g.x \in \Sigma \cap G.x$, then $g^{-1}.\Sigma$ is a section through x. By (2) and (4) we have $E(x) \subseteq g^{-1}.\Sigma$. The converse can be seen as follows: Let $y \in g^{-1}.\Sigma$ and choose a minimal geodesic from x to y. By the argument given in the proof

of 8.3.(3) this gedesic is orthogonal to the orbit through x and thus lies in $E(x)$. So $y \in E(x)$.

(6) see 7.13 (1) $\iff$ (2) and recall that by remark 7.10 a normal field is G-equivariant iff it is π-parallel, where $\pi : M \to M/G$ is the orbit map. $\square$

8.13. Remark. The converse of 8.12(1) is not true. Namely, an integral manifold of $\mathrm{Hor}(M_{\mathrm{reg}})$ is not, in general, a section.

Example. Consider the Lie group $G = S^1 \times \{1\}$, and let it act on $M := S^1 \times S^1$ by translation. Let $\xi = (1,0)$ denote the fundamental vector field of the action, and choose any $\eta \in \mathrm{Lie}(S^1 \times S^1) = \mathbb{R} \times \mathbb{R}$ which generates a one-parameter subgroup c which is dense in $S^1 \times S^1$ (irrational ascent). Now, endow $S^1 \times S^1$ with a Riemannian metric making ξ and η an orthonormal frame field. Any section of M would then have to be a coset of c, and therefore dense. This contradicts the assumption that a section is a closed embedded submanifold.

8.14. Definition. *A* **symmetric space** *is a complete, connected Riemannian manifold M such that for each $x \in M$ there is an isometry S_x (defined globally) which locally around x takes on the form:*

$$\exp_x tX \mapsto \exp_x(-tX)$$

In particular, x is an isolated fixed point.

Remark. Equivalent to this definition is the following one: A symmetric space is a quotient space $M = G/H$ of a Lie group G with a subgroup H together with an automorphism $\sigma : G \to G$ which satisfies two conditions

(1) $\sigma \circ \sigma = id$

(2) $(G^\sigma)_o \subseteq H \subseteq G^\sigma := \{g \in G : \sigma(g) = g\}$

An indication for this is that the first definition of a symmetric space implies that the group of isometries must act transitively. For any $x, y \in M$, take a geodesic joining the two, then the reflection S_c at the central point between x and y on the geodesic carries x into y. Now if we identify $G := \mathrm{Isom}(M)$ and let $H := G_{x_0}$ for some point x_0 in M, then $M = G/H$, and σ can be defined as $\sigma(g) := S_{x_0} \circ g \circ S_{x_0}$. It clearly fulfills $\sigma \circ \sigma = id$. Let us check (2). Take any $h \in H$. Since $T_{x_0} S_{x_0} = -Id_{T_{x_0}M}$ and $h.x_0 = x_0$, we get $T_{x_0}\sigma(h) = T_{x_0}h$ by the chain rule. This suffices to prove that $\sigma(h) = h$ (cf. [17], Lemma 4 p.254). So we have $H \in G_\sigma$. To see $(G_\sigma)_o \subseteq H$, take a one-parameter subgroup g_t of G_σ with $g_0 = id$. Then $\sigma(g_t) = g_t$ implies that $S_{x_0} \circ g_t = g_t \circ S_{x_0}$. So $S_{x_0} \circ g_t(x_0) = g_t(x_0)$, and since $g_0(x_0) = x_0$ and x_0 is an isolated fixed point of S_{x_0}, $g_t(x_0) = x_0$ for the other t as well, so $g_t \in H$.

8.15. Theorem. *([18], Ch.XI, 4.3) If $(G/H, \sigma)$ is a symmetric space, then the totally geodesic connected submanifolds N of G/H through $e \in G/H$ correspond exactly to the linear subspaces $T_eN = \mathfrak{m}' \subseteq \mathfrak{m} := T_eG/H \cong \{X \in \mathfrak{g} : \sigma'(X) = -X\}$ which fulfill $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subseteq \mathfrak{m}'$.*

Remark. This implies that a locally totally geodesic submanifold of a simply connected symmetric space can be extended uniquely to a complete, totally geodesic submanifold. Here we mean by locally geodesic submanifold that a geodesic can leave the submanifold only at its "boundary". In other words, the second fundamental form must be zero.

8.16. Corollary. *Let $M = G/H$ be a simply connected, complete symmetric space, $K \subseteq G$, a subgroup. Then the action of K on G/H admits sections iff* $\operatorname{Hor}(M_{\text{reg}})$ *is integrable. In particular, if the principal K-orbits have codimension* 1*, there are always sections.*

8.17. Theorem. *Consider any Riemannian G-manifold M. Then the following statements are equivalent.*

(1) $\operatorname{Hor}(M_{\text{reg}})$ *is integrable.*
(2) *Every G-equivariant normal field along a principal orbit is ∇^{Nor}-parallel.*
(3) *For $x \in M_{\text{reg}}$, S the normal slice at x and $X \in \mathfrak{g}$ and $s \in S$ arbitrary, $\zeta_X(s) \perp T_s(S)$.*

Proof. The equivalence of (1) and (2) is a direct consequence of 7.13 and remark 7.10. Furthermore, suppose (1), then there is an integral submanifold H of the horizontal distribution going through x. H is totally geodesic by 7.8(1), and so $S = \exp_x(\operatorname{Nor}_r(G.x))$ is contained in H. Therefore, (3) holds: The fundamental vector field ζ_X is tangent to the orbit $G.s$ and with that perpendicular to the horizontal distribution and to $T_s(S)$. Now if we suppose (3), then S is an integral submanifold of $\operatorname{Hor}(M_{\text{reg}})$, and (1) holds. $\square$

8.18. Remark. We already saw in 6.10 that $\operatorname{Nor} G.x$ is a trivial bundle. Now we even have a parallel global frame field. So the normal bundle to a regular orbit is flat.

8.19. Corollary. *Consider an orthogonal representation V of G, $G \to O(V)$. Let $x \in V$ be any regular point and Σ the linear subspace of V that is orthogonal to the orbit through x. Then the following statements are equivalent:*

(1) *V admits sections*
(2) *Σ is a section*
(3) *for all $y \in \Sigma$ and $X \in \mathfrak{g}$, $\zeta_X(y) \perp \Sigma$*

Proof. (3) implies that the horizontal bundle is integrable (8.17). In this case 8.15 implies (1). (1) $\Rightarrow$ (2) is clear with 8.12(5). (2) $\Rightarrow$ (3) is trivial. $\square$

8.20. Definition. *An orthogonal representation of G is called* **polar representation** *if it admits sections.*

8.21.. Corollary. *Let $\pi : G \to O(V)$ be a polar representation, and let $v \in V$ be a regular point. Then*

$$\Sigma := \{w \in V : \zeta_{\mathfrak{g}}(w) \subseteq \zeta_{\mathfrak{g}}(v)\}$$

is the section through v, where $\zeta_{\mathfrak{g}}(w) := \{\zeta_X(w) : X \in \mathfrak{g}\} \subseteq V$.

Proof. Since $\zeta_{\mathfrak{g}}(v) = T_v(G.v)$ and by 8.19, a section through v is given by $\Sigma' := \zeta_{\mathfrak{g}}(v)^\perp$. If $z \in \Sigma'$, then $\zeta_{\mathfrak{g}}(z) \subseteq (\Sigma')^\perp$, which in our case implies that $\zeta_{\mathfrak{g}}(z) \subseteq \zeta_{\mathfrak{g}}(v)$. So $z \in \Sigma$.

Conversely, suppose z is a regular point in Σ. Consider the section $\Sigma'' = \zeta_{\mathfrak{g}}(z)^\perp$ through z. Then, since $\zeta_{\mathfrak{g}}(z) \subseteq \zeta_{\mathfrak{g}}(v)$, we also have that $\Sigma' = \zeta_{\mathfrak{g}}(v)^\perp \subseteq \zeta_{\mathfrak{g}}(z)^\perp = \Sigma''$. Therefore $\Sigma' = \Sigma''$ and, in particular, $z \in \Sigma'$. $\square$

9. The Generalized Weyl Group of a Section

Consider a complete Riemannian G-manifold M which admits sections. For any closed subset S of M we define the largest subgroup of G which induces an action on S:

$$N(S) := \{g \in G : \ell_g(S) = S\}$$

and the subgroup consisting of all $g \in G$ which act trivially on S:

$$Z(S) := \{g \in G : \ell_g(s) = s, \text{ for all } s \in S\}.$$

Then, since S is closed, $N(S)$ is closed, hence a Lie subgroup of G. $Z(S) = \bigcap_{s\in S} G_s$ is closed as well and is a normal subgroup of $N(S)$. Therefore, $N(S)/Z(S)$ is a Lie group, and it acts on S effectively.

If we take for S a section Σ, then the above constructed group is called the **generalized Weyl group** of Σ and is denoted by

$$W(\Sigma) = N(\Sigma)/Z(\Sigma).$$

9.1. Remark. For any regular point $x \in \Sigma$, G_x acts trivially on the normal slice S_x at x (by 6.7). Since $\Sigma = \exp_x \operatorname{Nor}_x(G.x)$ by 8.12(5), S_x is an open subset of Σ, and we see that G_x acts trivially on all of Σ. So we have $G_x \subseteq Z(\Sigma)$. On the other hand, $Z(\Sigma) \subseteq G_x$ is obvious, therefore

$$Z(\Sigma) = G_x \qquad \text{for } x \in \Sigma \cap M_{\text{reg}}.$$

Now, since $Z(\Sigma)$ is a normal subgroup of $N(\Sigma)$, we have $N(\Sigma) \subseteq N(G_x)$ where the second N stands for the normalizer in G. So we have

$$W(\Sigma) \subseteq N(G_x)/G_x \qquad \text{for } x \in \Sigma \cap M_{\text{reg}}.$$

9.2. Proposition. *Let M be a proper Riemannian G-manifold and Σ a section, then the associated Weyl group $W(\Sigma)$ is discrete. If Σ' is a different section, then there is an isomorphism $W(\Sigma) \to W(\Sigma')$ induced by an inner automorphism of G. It is uniquely determined up to an inner automorphism of $W(\Sigma)$.*

Proof. Take a regular point $x \in \Sigma$ and consider the normal slice S_x. Then $S_x \subseteq \Sigma$ open. Therefore, any g in $N(\Sigma)$ close to the identity element maps x back into S_x. By 4.12(2) g then lies in $G_x = Z(\Sigma)$. So $Z(\Sigma)$ is an open subset of $N(\Sigma)$, and the quotient $W(\Sigma)$ is discrete.

If Σ' is another section, then $\Sigma' = g.\Sigma$ where $g \in G$ is uniquely determined up to $N(\Sigma)$. Clearly, $\operatorname{conj}_g : G \to G$ induces isomorphisms

$$\begin{aligned} \operatorname{conj}_g : N(\Sigma) &\xrightarrow{\cong} N(\Sigma') \\ Z(\Sigma) &\xrightarrow{\cong} Z(\Sigma') \end{aligned}$$

and therefore it factors to an isomorphism $W(\Sigma) \xrightarrow{\cong} W(\Sigma')$. $\square$

9.3. Example. Any finite group is a generalized Weyl group in the appropriate setting. That is, to an arbitrary finite group W we will now construct a setting in which it occurs as a Weyl group. Let G be a compact Lie group and H a closed subgroup such that $W \subseteq N(H)/H$ (this is always possible since any finite group can be regarded as a subgroup of $O(V) =: G$ so we need only choose $H = \{e\}$). Next, take a smooth manifold Σ on which W acts effectively. Consider the inverse image of W under the quotient map $\pi : N(H) \to N(H)/H$, $K := \pi^{-1}(W)$. Then the action of W induces a K-action on Σ as well. The smooth manifold $M := G \times_K \Sigma$ has a left G-action. Let $-B$ denote the G-invariant Riemann metric on G induced by the Cartan-Killing form on the semisimple part and any inner product on the center, and let γ_Σ be a W-invariant Riemann metric on Σ. Then the Riemann metric $-B \times \gamma_\Sigma$ on $G \times \Sigma$ induces a G-invariant Riemann metric on the quotient space $G \times_K \Sigma$. With this, $G \times_K \Sigma$ is a Riemannian G-manifold, and if $q : G \times \Sigma \to G \times_K \Sigma$ is the quotient map, then $q(\{e\} \times \Sigma) \cong \Sigma$ meets every G-orbit orthogonally. So it is a section. The largest subgroup of G acting on Σ is K and the largest acting trivially on Σ is H. Therefore, $W(\Sigma) = K/H = W$ is the Weyl group associated to the section Σ.

9.4. Theorem. *Let M be a proper Riemannian G-manifold with sections. Then, for any $x \in M$, the slice representation $G_x \to O(\operatorname{Nor}_x(G.x))$ is a polar representation. If Σ is a section through x in M, then $T_x\Sigma$ is a section in $\operatorname{Nor}_x(G.x)$ for the slice representation. Furthermore,*

$$W(T_x\Sigma) = W(\Sigma)_x.$$

Proof. Clearly $T_x\Sigma \subseteq \operatorname{Nor}_x(G.x)$. We begin by showing that it has the right codimension. Take a $\xi \in \operatorname{Nor}_x(G.x)$ close to 0_x, then $(G_x)_\xi = G_y$ for $y = \exp^\gamma_x \xi$, since $\exp_x$ is a G_x-equivariant diffeomorphism in a neighborhood of 0_x. So $G_x.\xi \cong G_x/(G_x)_\xi = G_x/G_y$. Let us now calculate the codimension of $G_x.\xi$ in $\operatorname{Nor}_x(G.x)$:

$$\begin{aligned} &\dim \operatorname{Nor}_x(G.x) - \dim G_x.\xi = \dim \operatorname{Nor}_x(G.x) - \dim G_x + \dim G_y = \\ &= \underbrace{\dim \operatorname{Nor}_x(G.x) + \dim G/G_x}_{=\dim M} - \underbrace{(\dim G - \dim G_y)}_{=\dim G/G_y} = \operatorname{codim}_M G.y. \end{aligned}$$

Since the regular points lie dense, we can choose $\xi \in T_x\Sigma$ regular by assuming that $y = \exp^\gamma_x(X)$ is regular in Σ. Then y is regular as well and we get:

$$\operatorname{codim}_{\operatorname{Nor}_x(G.x)} G_x.\xi = \operatorname{codim}_M G.y = \dim \Sigma = \dim T_x\Sigma.$$

So $T_x\Sigma$ is a linear subspace of $\operatorname{Nor}_x G.x$ with the right codimension for a section. Therefore, if we show that $T_x\Sigma$ is orthogonal to each orbit it meets, then it is already the entire orthogonal complement of a regular orbit, and by corollary 8.19 (3) $\implies$ (2), we know that it meets every orbit.

Denote the G-action on M by $\ell : G \to \operatorname{Isom}(M)$. If $\xi \in T_x\Sigma$ is arbitrary, then it remains to prove that for all $\eta \in T_x\Sigma$ and $X \in \mathfrak{g}_x$:

$$\gamma_x\left(\eta, \zeta_X^{T\ell|_{G_x}}(\xi)\right) = 0.$$

To do this, choose a smooth one-parameter family $\eta(t) \in T_{\exp(t\xi)}\Sigma$ such that $\eta(0) = \eta$ and $\nabla_{\partial_t}\eta = 0$. Since Σ is a section in M we know for each single t that

$$\gamma_{\exp(t\xi)}\left(\zeta_X^\ell(\exp^\gamma(t\xi)), \eta(t)\right) = 0.$$

If we derive this equation we get

$$0 = \left.\frac{d}{ds}\right|_{s=0} \gamma\left(\zeta_X^\ell(\exp^\gamma(s\xi)), \eta(s)\right) = \gamma\left(\nabla_{\partial_s}\zeta_X^\ell(\exp^\gamma(s\xi)), \eta(0)\right).$$

So it remains to show that $\nabla_{\partial_s}\zeta_X^\ell(\exp^\gamma(s\xi))$ is the fundamental vector field of X at ξ for the slice representation.

$$\begin{aligned}
\nabla_{\partial_s}\zeta_X^\ell(\exp^\gamma(s\xi)) &= \nabla_\xi\zeta_X^\ell = K \circ T\zeta_X^\ell.\xi = \\
&= K \circ T(\partial_t|_0 \ell_{\exp^G(tX)}).\partial_s|_0 \exp_x^\gamma(s\xi)) \\
&= K.\partial_s|_0.\partial_t|_0 \ell_{\exp^G(tX)}(\exp_x^\gamma(s\xi)) \\
&= K.\kappa_M.\partial_t|_0.\partial_s|_0 \ell_{\exp^G(tX)}(\exp_x^\gamma(s\xi)) \\
&= K.\kappa_M.\partial_t|_0.T(\ell_{\exp^G(tX)})(\xi)
\end{aligned}$$

Here, K denotes the connector and κ_M the canonical flip between the two structures of TTM, and we use the identity $K \circ \kappa = K$, which is a consequence of the symmetry of the Levi-Civita connection. The argument of K in the last expression is vertical already since $X \in \mathfrak{g}_x$. Therefore we can replace K by the vertical projection and get

$$\nabla_{\partial_s}\zeta_X^\ell(\exp^\gamma(s\xi)) = \operatorname{vpr} \left.\frac{d}{dt}\right|_{t=0} T_x(\ell_{\exp^G(tX)}).\xi = \zeta_X^{T_2\ell|_{G_x}}(\xi).$$

So $\zeta_X^{T_2\ell|_{G_x}}(\xi)$ intersects $T_x\Sigma$ orthogonally, and therefore $T_x\Sigma$ is a section.

Now consider $N_{G_x}(T_x(\Sigma)) = \{g \in G_x : T_x(\ell_g).T_x\Sigma = T_x\Sigma\}$. Clearly, $N_G(\Sigma) \cap G_x \subseteq N_{G_x}(T_x(\Sigma))$. On the other hand, any $g \in N_{G_x}(T_x(\Sigma))$ leaves Σ invariant as the following argument shows.

For any regular $y \in \Sigma$ we have $\Sigma = \exp_y \operatorname{Nor}(G.y)$. Therefore $x = \exp_y \eta$ for a suitable $\eta \in T_y\Sigma$, and conversely, y can be written as $y = \exp_x \xi$ for $\xi = -\frac{d}{dt}|_{t=1} \exp_y t\eta \in T_x\Sigma$. Now $g.y = g.\exp_x \xi = \exp_x T_x\ell_g.\xi$ lies in Σ, since $T_x\ell_g.\xi$ lies in $T_x\Sigma$. So g maps all regular points in Σ back into Σ. Since these form a dense subset and since ℓ_g is continuous, we get $g \in N_G(\Sigma)$.

We have now shown that

$$N_{G_x}(T_x\Sigma) = N_G(\Sigma) \cap G_x.$$

Analogous arguments used on $Z_{G_x}(T_x\Sigma)$ give

$$Z_{G_x}(T_x\Sigma) = Z_G(\Sigma),$$

and we see that

$$W_{G_x}(T_x\Sigma) = (N(\Sigma) \cap G_x)/Z(\Sigma) = W(\Sigma)_x. \qquad \square$$

9.5.. Corollary. *Let M be a Riemannian G-manifold admitting sections and let $x \in M$. Then for any section Σ through x we have*

$$\operatorname{Nor}_x(G.x)^{G_x^0} \subseteq T_x\Sigma,$$

where G_x^0 is the connected component of the isotropy group G_x at x.

Proof. By theorem 9.4 the tangent space $T_x\Sigma$ is a section for the slice representation $G_x \to O(\operatorname{Nor}_x(G.x))$. Let $\xi \in T_x\Sigma$ be a regular vector for the slice representation. By corollary 8.21 we have $T_x\Sigma = \{\eta \in \operatorname{Nor}_x(G.x) : \zeta_{\mathfrak{g}_x}(\eta) \subset \zeta_{\mathfrak{g}_x}(\xi)\}$. Since $\operatorname{Nor}_x(G.x)^{G_x^0}$ consists of all η in $\operatorname{Nor}_x(G.x)$ with $\zeta_{\mathfrak{g}_x}(\eta) = 0$, the result follows. $\square$

9.6. Corollary. *Let M be a proper Riemannian G-manifold with sections and $x \in M$. Then G_x acts transitively on the set of all sections through x.*

Proof. Consider two arbitrary sections Σ_1 and Σ_2 through x and a normal slice S_x at x. By theorem 9.4, $T_x\Sigma_2$ is a section for the slice representation. Since $\exp_x$ can be restricted to a G_x-equivariant diffeomorphism onto S_x, $\Sigma_2 \cap S_x$ is a section for the G_x-action on S_x. Next, choose a regular point $y \in \Sigma_1 \cap S_x$. Its G_x-orbit meets the section $\Sigma_2 \cap S_x$, that is we can find a $g \in G_x$ such that $g.y \in \Sigma_2$. Now Σ_2 and $g.\Sigma_1$ are both sections containing the regular point $g.y$. Therefore they are equal. $\square$

9.7. Corollary. *Let M be a proper G-manifold with sections, Σ a section of M and $x \in \Sigma$. Then*

$$G.x \cap \Sigma = W(\Sigma).x$$

Proof. The inclusion $(\supseteq)$ is clear. Now we have

$$y \in G.x \cap \Sigma \quad \Longleftrightarrow \quad y = g.x \in \Sigma \text{ for some } g \in G.$$

Take this g and consider the section $\Sigma' := g.\Sigma$. Then Σ and Σ' are both sections through y, and by 9.6 there is a $g' \in G_y$ which carries Σ' back into Σ. Now $g'g.\Sigma = \Sigma$, that is $g'g \in N(\Sigma)$, and $g'g.x = g'.y = y$. So $y \in N(\Sigma).x = W(\Sigma).x$. $\square$

9.8. Corollary. *If M is a proper G-manifold with section Σ, then the inclusion of Σ into M induces a homeomorphism j between the orbit spaces.*

$$\begin{tikzcd}
\Sigma \arrow[r, hook, "i"] \arrow[d, "\pi_\Sigma"'] & M \arrow[d, "\pi_M"'] \\
\Sigma/W(\Sigma) \arrow[r, "j"] & M/G
\end{tikzcd}$$

(but it does not necessarily preserve orbit types, see remark 4.14).

Proof. By the preceding corollary there is a one to one correspondence between the G-orbits in M and the $W(G)$-orbits in Σ, so j is well defined and bijective. Since $j \circ \pi_\Sigma = \pi_M \circ i$ and π_Σ is open, j is continuous.

Consider any open set $U \subseteq \Sigma/W(\Sigma)$. We now have to show that

$$\pi_M^{-1} j(U) = G.\pi_\Sigma^{-1}(U)$$

is an open subset of M (since then $j(U)$ is open and j^{-1} continuous). Take any $x \in \pi_M^{-1} j(U)$. We assume $x \in \Sigma$ (otherwise it can be replaced by a suitable $g.x \in \Sigma$). So $x \in \pi_\Sigma^{-1}(U)$. Let S_x be a normal slice at x, then $\Sigma \cap S_x$ is a submanifold of S_x of dimension $\dim \Sigma$. In S_x, x has arbitrarily small G_x-invariant neighborhoods, since the slice action is orthogonal and S_x G-equivariantly diffeomorphic to an open ball in $\operatorname{Nor}_x(G.x)$. Let V_x be such an open neighborhood of x, small enough for $V_x \cap \Sigma$ to be contained in $\pi_\Sigma^{-1}(U)$. V_x is again a slice, therefore $G.V_x$ is open in M (4.12(3)). Now we have to check whether $G.V_x$ is really a subset of $\pi_M^{-1} j(U)$. Using corollary 9.6 we get

$$G.(V_x \cap \Sigma) = G.G_x(V_x \cap \Sigma) = G.(V_x \cap G_x.\Sigma) = G.V_x.$$

Therefore, $G.V_x \subseteq G.\pi_\Sigma^{-1}(U) = \pi_M^{-1} j(U)$ where it is an open neighborhood of x. So $\pi_M^{-1} j(U)$ is an open subset of M, $j(U)$ is open in M/G, and j^{-1} is continuous. □

9.9. Corollary. *Let M be a proper Riemannian G-manifold and $\Sigma \subseteq M$ a section with Weyl group W. Then the inclusion $i : \Sigma \hookrightarrow M$ induces an isomorphism*

$$C^0(M)^G \xrightarrow{i^*} C^0(\Sigma)^W.$$

Proof. By corollary 9.7 we see that every $f \in C^0(\Sigma)^W$ has a unique G-equivariant extension $\tilde f$ onto V. If we consider once more the diagram

$$\begin{tikzcd}
\Sigma \arrow[r, hook, "i"] \arrow[d, "\pi_\Sigma"'] & M \arrow[d, "\pi_M"'] \\
\Sigma/W(\Sigma) \arrow[r, "j"] & M/G
\end{tikzcd}$$

we see that f factors over π_Σ to a map $f' \in C^0(\Sigma/W(\Sigma))$, and since j is a homeomorphism (9.8) we get for the G-invariant extension $\tilde f$ of f:

$$\tilde f = f' \circ j^{-1} \circ \pi_M \in C^0(M)^G. \qquad \square$$

9.10. Theorem. [32], 4.12, or [44], theorem D. *Let $G \to GL(V)$ be a polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$.*

Then the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^W$ of W-invariant polynomials on the section Σ, via the restriction mapping $f \mapsto f|\Sigma$.

9.11. Remark. This seemingly very algebraic theorem is actually a consequence of the geometry of the orbits. This already becomes evident in the case of a first degree homogeneous polynomial. To see that the G-invariant extension of $p \in \mathbb{R}[\Sigma]_1^W$ to V is again a polynomial (and again of first degree), we we must assume the following convexity result of Terng.

Under the conditions of the theorem, for every regular orbit $G.x$ the orthogonal projection onto Σ, $\mathrm{pr}(G.x)$, is contained in the convex hull of $G.x \cap \Sigma$ (this is a finite subset of Σ by 9.7 since G is compact and $W(\Sigma)$ discrete).

Let us make this assumption. Denote by $\tilde{p}$ the unique G-invariant extension of p, then clearly $\tilde{p}$ is homogeneous. Now, notice that for any orbit $G.x$, p is constant on the convex hull of $G.x \cap \Sigma =: \{g_1.x, g_2.x, \dots, g_k.x\}$. Just take any $s = \sum \lambda_i g_i.x$ with $\sum \lambda_i = 1$, then

$$p(s) = \sum \lambda_i p(g_i.x) = p(g_1.x) \sum \lambda_i = p(g_1.x).$$

With this and with our assumption we can show that for regular points $u, v \in M$, $\tilde{p}(u+v) = \tilde{p}(u) + \tilde{p}(v)$. Suppose without loss of generality that $u + v \in \Sigma$, then

$$p(u+v) = p(\mathrm{pr}(u) + \mathrm{pr}(v)) = p(\mathrm{pr}(u)) + p(\mathrm{pr}(v))$$

At this point, the convexity theorem asserts that $\mathrm{pr}(u)$ and $\mathrm{pr}(v)$ can be written as convex combinations of elements of $G.u \cap \Sigma$, respectively $G.v \cap \Sigma$. If we fix an arbitrary g_u (resp. g_v) in G such that $g_u.u$ (resp. $g_v.v$) lie in Σ, then by the above argument we get

$$p(\mathrm{pr}(u)) = p(g_u.u) \quad \text{and} \quad p(\mathrm{pr}(v)) = p(g_v.v).$$

So we have

$$p(u+v) = p(g_u.u) + p(g_v.v) = \tilde{p}(u) + \tilde{p}(v),$$

and $\tilde{p}$ is linear on V_{reg}. Since the regular points are a dense subset of V, and $\tilde{p}$ is continuous by 9.9, $\tilde{p}$ is linear altogether.

A proof of the convexity theorem can be found in [45] or again in [33], pp. 168–170. For a proof of theorem 9.10 we refer to [44]. In both sources the assertions are shown for the more general case where the principal orbits are replaced by **isoparametric submanifolds** (i.e. submanifolds of a space form with flat normal bundle and whose principal curvatures along any parallel normal field are constant; compare 6.13 and 8.18). To any isoparametric submanifold there is a singular foliation which generalizes the orbit foliation of a polar action but retains many of its fascinating properties (cf. [33]).

9.12. Remark. In connection with the example we studied in chapter 1, the convexity theorem from above yields the following classical result of Schur [39], 1923:

Let $M \subseteq S(n)$ be the subset of all matrices with fixed distinct eigenvalues $a_1, \dots, a_n$ and $\mathrm{pr} : S(n) \to \mathbb{R}^n$ defined by

$$\mathrm{pr}\left((x_{ij})\right) := (x_{11}, x_{22}, \dots, x_{nn})$$

then $\mathrm{pr}(M)$ is contained in the convex hull of $\mathfrak{S}_n.a$ where $a = (a_1, \dots, a_n)$.

9.13. Theorem. *Let M be a proper Riemannian G-manifold with section Σ and Weyl group W. Then the inclusion $i : \Sigma \hookrightarrow M$ induces an isomorphism*

$$C^\infty(M)^G \xrightarrow{i^*} C^\infty(\Sigma)^{W(\Sigma)}.$$

Proof. Clearly $f \in C^\infty(M)^G$ implies $i^* f \in C^\infty(\Sigma)^W$. By 9.9 we know that every $f \in C^\infty(\Sigma)^W$ has a unique continuous G-invariant extension $\tilde f$. We now have to show that $\tilde f \in C^\infty(M)^G$.

Let us take an $x \in M$ and show that $\tilde f$ is smooth at x. Actually, we can assume $x \in \Sigma$, because if $\tilde f$ is smooth at x then $\tilde f \circ \ell_{g^{-1}}$ is smooth at $g.x$, so $\tilde f$ is smooth at $g.x$ as well. Now let S_x denote a normal slice at x. Then we have

$$\begin{tikzcd}
G.S_x \arrow[dr] & & G \times_{G_x} S_x \arrow[ll, "I"'] \arrow[dl] \arrow[dr, "\tilde f \circ I"'] & & G \times S_x \arrow[ll, two heads, "q"'] \arrow[dl, "\tilde f|_{S_x} \circ \mathrm{pr}_2"] \\
& G/G_x & & \mathbb{R} &
\end{tikzcd}$$

Since in the above diagram I is an isomorphism and q a submersion, it is sufficient to show that $\tilde f|_{S_x} \circ \mathrm{pr}_2$ or equivalently $\tilde f|_{S_x}$ is smooth at x. Let $B \subseteq T_x S_x$ be a ball around 0_x such that $B \cong S_x$ and $T_x\Sigma \cap B \cong \Sigma \cap S_x$. Then, by theorem 9.4, the G_x-action on S_x is basically a polar representation (up to diffeomorphism). So it remains to show the following:

Claim: If Σ is a section of a polar representation $G_x \to O(V)$ with Weyl group W_x and f is a smooth W_x-invariant function on Σ, then f extends to a smooth G_x-invariant function $\tilde f$ on V.

In order to show this, let $\rho_1, \dots, \rho_k$ be a system of homogeneous Hilbert generators for $\mathbb{R}[\Sigma]^{W_x}$. Then, by Schwarz' theorem, there is an $f' \in C^\infty\left(\mathbb{R}^k\right)$ such that $f = f' \circ (\rho_1, \dots, \rho_k)$. By theorem 9.10, each ρ_i extends to a polynomial $\tilde\rho_i \in \mathbb{R}[V]^{G_x}$. Therefore we get

$$\tilde f := f' \circ (\tilde\rho_1, \dots, \tilde\rho_k) : V \to \mathbb{R}$$

is a smooth G_x-invariant extension of f. $\square$

10. Basic Differential Forms

Our aim in this section is to show that pullback along the embedding $\Sigma \to M$ induces an isomorphism $\Omega^p_{\text{hor}}(M)^G \cong \Omega^p(\Sigma)^{W(\Sigma)}$ for each p, where a differential form ω on M is called *horizontal* if it kills each vector tangent to some orbit. For each point x in M, the slice representation of the isotropy group G_x on the normal space $T_x(G.x)^\perp$ to the tangent space to the orbit through x is a polar representation. The first step is to show that the result holds for polar representations. This is done in theorem 10.6 for polar representations whose generalized Weyl group is really a Coxeter group, is generated by reflections. Every polar representation of a connected Lie group has this property. The method used there is inspired by Solomon [40]. Then the general result is proved under the assumption that each slice representation has a Coxeter group as a generalized Weyl group. This result is from [24].

10.1. Basic differential forms. Let G be a Lie group with Lie algebra $\mathfrak{g}$, multiplication $\mu : G \times G \to G$, and for $g \in G$ let $\mu_g, \mu^g : G \to G$ denote the left and right translation.

Let $\ell : G \times M \to M$ be a left action of the Lie group G on a smooth manifold M. We consider the partial mappings $\ell_g : M \to M$ for $g \in G$ and $\ell^x : G \to M$ for $x \in M$ and the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ given by $\zeta_X(x) = T_e(\ell^x)X$. Since ℓ is a left action, the negative $-\zeta$ is a Lie algebra homomorphism.

A differential form $\varphi \in \Omega^p(M)$ is called *G-invariant* if $(\ell_g)^*\varphi = \varphi$ for all $g \in G$ and *horizontal* if φ kills each vector tangent to a G-orbit: $i_{\zeta_X}\varphi = 0$ for all $X \in \mathfrak{g}$. We denote by $\Omega^p_{\text{hor}}(M)^G$ the space of all horizontal G-invariant p-forms on M. They are also called *basic forms.*

10.2. Lemma. *Under the exterior differential $\Omega_{\text{hor}}(M)^G$ is a subcomplex of $\Omega(M)$.*

Proof. If $\varphi \in \Omega_{\text{hor}}(M)^G$ then the exterior derivative $d\varphi$ is clearly G-invariant. For $X \in \mathfrak{g}$ we have

$$i_{\zeta_X} d\varphi = i_{\zeta_X} d\varphi + d i_{\zeta_X}\varphi = L_{\zeta_X}\varphi = 0,$$

so $d\varphi$ is also horizontal. $\square$

10.3. Main Theorem. ([24] and [25]) *Let $M \times G \to M$ be a proper isometric right action of a Lie group G on a smooth Riemannian manifold M, which admits a section Σ.*

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G-invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ.

The proof of this theorem will take up the rest of this section.

Proof of injectivity. Let $i : \Sigma \to M$ be the embedding of the section. It clearly induces a linear mapping $i^* : \Omega^p_{\text{hor}}(M)^G \to \Omega^p(\Sigma)^{W(\Sigma)}$ which is injective by the following argument: Let $\omega \in \Omega^p_{\text{hor}}(M)^G$ with $i^*\omega = 0$. For $x \in \Sigma$ we have $i_X\omega_x = 0$ for $X \in T_x\Sigma$ since $i^*\omega = 0$, and also for $X \in T_x(G.x)$ since ω is horizontal. Let $x \in \Sigma \cap M_{\text{reg}}$ be a regular point, then $T_x\Sigma = (T_x(G.x))^\perp$ and so $\omega_x = 0$. This holds along the whole orbit through x since ω is G-invariant. Thus $\omega|M_{\text{reg}} = 0$, and since M_{reg} is dense in M, $\omega = 0$.

So it remains to show that i^* is surjective. This will be done in 10.10 below. $\square$

10.4. Lemma. *Let $\ell \in V^*$ be a linear functional on a finite dimensional vector space V, and let $f \in C^\infty(V, \mathbb{R})$ be a smooth function which vanishes on the kernel of ℓ, so that $f|\ell^{-1}(0) = 0$. Then there is a unique smooth function g such that $f = \ell.g$*

Proof. Choose coordinates $x^1, \dots, x^n$ on V with $\ell = x^1$. Then $f(0, x^2, \dots, x^n) = 0$ and we have $f(x^1, \dots, x^n) = \int_0^1 \partial_1 f(tx^1, x^2, \dots, x^n)dt.x^1 = g(x^1, \dots, x^n).x^1$. $\square$

10.5. Question. *Let $G \to GL(V)$ be a representation of a compact Lie group in a finite dimensional vector space V. Let $\rho = (\rho_1, \dots, \rho_m) : V \to \mathbb{R}^m$ be the polynomial mapping whose components ρ_i are a minimal set of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials.*

We consider the pullback homomorphism $\rho^* : \Omega^p(\mathbb{R}^m) \to \Omega^p(V)$. Is it surjective onto the space $\Omega^p_{\text{hor}}(V)^G$ of G-invariant horizontal smooth p-forms on V?

See remark 10.7 for a class of representations where the answer is yes.

In general the answer is no. A counterexample is the following: Let the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of order n, viewed as the group of n-th roots of unity, act on $\mathbb{C} = \mathbb{R}^2$ by complex multiplication. A generating system of polynomials consists of $\rho_1 = |z|^2$, $\rho_2 = \text{Re}(z^n)$, $\rho_3 = \text{Im}(z^n)$. But then each $d\rho_i$ vanishes at 0 and there is no chance to have the horizontal invariant volume form $dx \wedge dy$ in $\rho^*\Omega(\mathbb{R}^3)$.

10.6. Theorem. ([24] and [25]) *Let $G \to GL(V)$ be a polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$.*

Then the pullback to Σ of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(V)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}.$$

According to Dadok [11], remark after proposition 6, for any polar representation of a connected Lie group the generalized Weyl group $W(\Sigma)$ is a reflection group. This theorem is true for polynomial differential forms, and also for real analytic differential forms, by essentially the same proof.

Proof. Let $i : \Sigma \to V$ be the embedding. It is proved in 10.3 that the restriction $i^* : \Omega^p_{\text{hor}}(V)^G \to \Omega^p(\Sigma)^{W(G)}$ is injective, so it remains to prove surjectivity.

Let us first suppose that $W = W(\Sigma)$ is generated by reflections (a reflection group or Coxeter group). Let $\rho_1, \dots, \rho_n$ be a minimal set of homogeneous generators of the algebra $\mathbb{R}[\Sigma]^W$ of W-invariant polynomials on Σ. Then this is a set of algebraically independent polynomials, $n = \dim \Sigma$, and their degrees $d_1, \dots, d_n$ are uniquely determined up to order. We even have (see [16])

(1) $d_1 \dots d_n = |W|$, the order of W,
(2) $d_1 + \dots + d_n = n + N$, where N is the number of reflections in W,
(3) $\prod_{i=1}^n (1 + (d_i - 1)t) = a_0 + a_1 t + \dots + a_n t^n$, where a_i is the number of elements in W whose fixed point set has dimension $n - i$.

Let us consider the mapping $\rho = (\rho_1, \dots, \rho_n) : \Sigma \to \mathbb{R}^n$ and its Jacobian $J(x) = \det(d\rho(x))$. Let $x^1, \dots, x^n$ be coordinate functions in Σ. Then for each $\sigma \in W$ we have

$$\begin{aligned} J.dx^1 \wedge \dots \wedge dx^n &= d\rho_1 \wedge \dots \wedge d\rho_n = \sigma^*(d\rho_1 \wedge \dots \wedge d\rho_n) \\ &= (J \circ \sigma)\sigma^*(dx^1 \wedge \dots \wedge dx^n) = (J \circ \sigma)\det(\sigma)(dx^1 \wedge \dots \wedge dx^n), \end{aligned}$$

$$J \circ \sigma = \det(\sigma^{-1})J. \tag{4}$$

If $J(x) \neq 0$, then in a neighborhood of x the mapping ρ is a diffeomorphism by the inverse function theorem, so that the 1-forms $d\rho_1, \dots, d\rho_n$ are a local coframe there. Since the generators $\rho_1, \dots, \rho_n$ are algebraically independent over $\mathbb{R}$, $J \neq 0$. Since J is a polynomial of degree $(d_1 - 1) + \dots + (d_n - 1) = N$ (see (2)), the set $U = \Sigma \setminus J^{-1}(0)$ is open and dense in Σ, and $d\rho_1, \dots, d\rho_n$ form a coframe on U.

Now let $(\sigma_\alpha)_{\alpha=1,\dots,N}$ be the set of reflections in W, with reflection hyperplanes H_α. Let $\ell_\alpha \in \Sigma^*$ be a linear functional with $H_\alpha = \ell^{-1}(0)$. If $x \in H_\alpha$ we have $J(x) = \det(\sigma_\alpha)J(\sigma_\alpha.x) = -J(x)$, so that $J|H_\alpha = 0$ for each α, and by lemma 10.4 we have

$$J = c.\ell_1 \dots \ell_N. \tag{5}$$

Since J is a polynomial of degree N, c must be a constant. Repeating the last argument for an arbitrary function g and using (5), we get:

(6) If $g \in C^\infty(\Sigma, \mathbb{R})$ satisfies $g \circ \sigma = \det(\sigma^{-1})g$ for each $\sigma \in W$, we have $g = J.h$ for $h \in C^\infty(\Sigma, \mathbb{R})^W$.
(7) **Claim.** Let $\omega \in \Omega^p(\Sigma)^W$. Then we have

$$\omega = \sum_{j_1 < \dots < j_p} \omega_{j_1 \dots j_p} d\rho_{j_1} \wedge \dots \wedge d\rho_{j_p},$$

where $\omega_{j_1 \dots j_p} \in C^\infty(\Sigma, \mathbb{R})^W$.

Since $d\rho_1, \dots, d\rho_n$ form a coframe on the W-invariant dense open set $U = \{x : J(x) \neq 0\}$, we have

$$\omega|U = \sum_{j_1 < \dots < j_p} g_{j_1 \dots j_p} d\rho_{j_1}|U \wedge \dots \wedge d\rho_{j_p}|U$$

for $g_{j_1\dots j_p}\in C^\infty(U,\mathbb{R})$. Since ω and all $d\rho_i$ are W-invariant, we may replace $g_{j_1\dots j_p}$ by

$$\tfrac{1}{|W|}\sum_{\sigma\in W} g_{j_1\dots j_p}\circ\sigma\in C^\infty(U,\mathbb{R})^W,$$

or assume without loss that $g_{j_1\dots j_p}\in C^\infty(U,\mathbb{R})^W$.

Let us choose now a form index $i_1<\dots<i_p$ with $\{i_{p+1}<\dots<i_n\}=\{1,\dots,n\}\setminus\{i_1<\dots<i_p\}$. Then for some sign $\varepsilon=\pm1$ we have

$$\begin{aligned}\omega|U\wedge d\rho_{i_{p+1}}\wedge\dots\wedge d\rho_{i_n}&=\varepsilon.g_{i_1\dots i_p}.d\rho_1\wedge\dots\wedge d\rho_n\\&=\varepsilon.g_{i_1\dots i_p}.J.dx^1\wedge\dots\wedge dx^n,\ \text{and}\end{aligned}$$

$$\omega\wedge d\rho_{i_{p+1}}\wedge\dots\wedge d\rho_{i_n}=\varepsilon.k_{i_1\dots i_p}dx^1\wedge\dots\wedge dx^n \tag{8}$$

for a function $k_{i_1\dots i_p}\in C^\infty(\Sigma,\mathbb{R})$. Thus

$$k_{i_1\dots i_p}|U=g_{i_1\dots i_p}.J|U. \tag{9}$$

Since ω and each $d\rho_i$ is W-invariant, from (8) we get $k_{i_1\dots i_p}\circ\sigma=\det(\sigma^{-1})k_{i_1\dots i_p}$ for each $\sigma\in W$. But then by (6) we have $k_{i_1\dots i_p}=\omega_{i_1\dots i_p}.J$ for unique $\omega_{i_1\dots i_p}\in C^\infty(\Sigma,\mathbb{R})^W$, and (9) then implies $\omega_{i_1\dots i_p}|U=g_{i_1\dots i_p}$, so that the claim (7) follows since U is dense.

Now we may finish the proof of the theorem in the case that $W=W(\Sigma)$ is a reflection group. Let $i:\Sigma\to V$ be the embedding. By theorem 9.10 the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^W$ of W-invariant polynomials on the section Σ, via the restriction mapping i^*. Choose polynomials $\tilde\rho_1,\dots\tilde\rho_n\in\mathbb{R}[V]^G$ with $\tilde\rho_i\circ i=\rho_i$ for all i. Put $\tilde\rho=(\tilde\rho_1,\dots,\tilde\rho_n):V\to\mathbb{R}^n$. In the setting of claim (7), use the theorem 3.7 of G. Schwarz to find $h_{i_1,\dots,i_p}\in C^\infty(\mathbb{R}^n,\mathbb{R})$ with $h_{i_1,\dots,i_p}\circ\rho=\omega_{i_1,\dots,i_p}$ and consider

$$\tilde\omega=\sum_{j_1<\dots<j_p}(h_{j_1\dots j_p}\circ\tilde\rho)d\tilde\rho_{j_1}\wedge\dots\wedge d\tilde\rho_{j_p},$$

which is in $\Omega^p_{\text{hor}}(V)^G$ and satifies $i^*\tilde\omega=\omega$.

Thus the mapping $i^*:\Omega^p_{\text{hor}}(V)^G\to\Omega^p_{\text{hor}}(\Sigma)^W$ is surjective in the case that $W=W(\Sigma)$ is a reflection group.

Now we treat the general case. Let G_0 be the connected component of G. From 8.19.(3) one concludes:

> *A subspace Σ of V is a section for G if and only if it is a section for G_0. Thus ρ is a polar representation for G if and only if it is a polar representation for G_0.*

The generalized Weyl groups of Σ with respect to G and to G_0 are related by

$$W(G_0)=N_{G_0}(\Sigma)/Z_{G_0}(\Sigma)\subset W(G)=N_G(\Sigma)/Z_G(\Sigma),$$

since $Z_G(\Sigma)\cap N_{G_0}(\Sigma)=Z_{G_0}(\Sigma)$.

Let $\omega \in \Omega^p(\Sigma)^{W(G)} \subset \Omega^p(\Sigma)^{W(G_0)}$. Since G_0 is connected the generalized Weyl group $W(G_0)$ is generated by reflections (a Coxeter group) by [1], remark after proposition 6. Thus by the first part of the proof

$$i^* : \Omega^p_{\text{hor}}(V)^{G_0} \xrightarrow{\cong} \Omega^p(\Sigma)^{W(G_0)}$$

is an isomorphism, and we get $\varphi \in \Omega^p_{\text{hor}}(M)^{G_0}$ with $i^*\varphi = \omega$. Let us consider

$$\psi := \int_G g^* \varphi \, dg \in \Omega^p_{\text{hor}}(V)^G,$$

where dg denotes Haar measure on G. In order to show that $i^*\psi = \omega$ it suffices to check that $i^* g^* \varphi = \omega$ for each $g \in G$. Now $g(\Sigma)$ is again a section of G, thus also of G_0. Since any two sections are related by an element of the group, there exists $h \in G_0$ such that $hg(\Sigma) = \Sigma$. Then $hg \in N_G(\Sigma)$ and we denote by $[hg]$ the coset in $W(G)$, and we may compute as follows:

$$\begin{aligned}
(i^* g^* \varphi)_x &= (g^*\varphi)_x . \Lambda^p Ti = \varphi_{g(x)} . \Lambda^p Tg . \Lambda^p Ti \\
&= (h^*\varphi)_{g(x)} . \Lambda^p Tg . \Lambda^p Ti, \qquad \text{since } \varphi \in \Omega^p_{\text{hor}}(M)^{G_0} \\
&= \varphi_{hg(x)} . \Lambda^p T(hg) . \Lambda^p Ti = \varphi_{i[hg](x)} . \Lambda^p Ti . \Lambda^p T([hg]) \\
&= \varphi_{i[hg](x)} . \Lambda^p Ti . \Lambda^p T([hg]) = (i^*\varphi)_{[hg](x)} . \Lambda^p T([hg]) \\
&= \omega_{[hg](x)} . \Lambda^p T([hg]) = [hg]^* \omega = \omega. \quad \square
\end{aligned}$$

10.7. Remark. The proof of theorem 10.6 shows that the answer to question 10.5 is yes for the representations treated in 10.6.

10.8. Corollary. *Let $\rho : G \to O(V, \langle \quad , \quad \rangle)$ be an orthogonal polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. Let $B \subset V$ be an open ball centered at 0.*

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(B)^G \xrightarrow{\cong} \Omega^p(\Sigma \cap B)^{W(\Sigma)}.$$

Proof. Check the proof of 10.6 or use the following argument. Suppose that $B = \{v \in V : |v| < 1\}$ and consider a smooth diffeomorphism $f : [0,1) \to [0,\infty)$ with $f(t) = t$ near 0. Then $g(v) := \frac{f(|v|)}{|v|} v$ is a G-equivariant diffeomorphism $B \to V$ and by 10.6 we get:

$$\Omega^p_{\text{hor}}(B)^G \xrightarrow{(g^{-1})^*} \Omega^p_{\text{hor}}(V)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)} \xrightarrow{g^*} \Omega^p(\Sigma \cap B)^{W(\Sigma)}. \quad \square$$

10.9. Let us assume that we are in the situation of the main theorem 10.3, for the rest of this section. For $x \in M$ let S_x be a (normal) slice and G_x the isotropy group, which acts on the slice. Then $G.S_x$ is open in M and G-equivariantly diffeomorphic to the associated bundle $G \to G/G_x$ via

$$\begin{array}{ccccc} G \times S_x & \xrightarrow{q} & G \times_{G_x} S_x & \xrightarrow{\cong} & G.S_x \\ & & \downarrow & & \downarrow r \\ & & G/G_x & \xrightarrow{\cong} & G.x, \end{array}$$

where r is the projection of a tubular neighborhood. Since $q : G\times S_x \to G\times_{G_x} S_x$ is a principal G_x-bundle with principal right action $(g, s).h = (gh, h^{-1}.s)$, we have an isomorphism $q^* : \Omega(G\times_{G_x} S_x) \to \Omega_{G_x-\text{hor}}(G\times S_x)^{G_x}$. Since q is also G-equivariant for the left G-actions, the isomorphism q^* maps the subalgebra $\Omega^p_{\text{hor}}(G.S_x)^G \cong \Omega^p_{\text{hor}}(G\times_{G_x} S_x)^G$ of $\Omega(G\times_{G_x} S_x)$ to the subalgebra $\Omega^p_{G_x-\text{hor}}(S_x)^{G_x}$ of $\Omega_{G_x-\text{hor}}(G\times S_x)^{G_x}$. So we have proved:

Lemma. *In this situation there is a canonical isomorphism*

$$\Omega^p_{\text{hor}}(G.S_x)^G \xrightarrow{\cong} \Omega^p_{G_x-\text{hor}}(S_x)^{G_x}$$

which is given by pullback along the embedding $S_x \to G.S_x$.

10.10. Rest of the proof of theorem 10.6. Let us consider $\omega \in \Omega^p(\Sigma)^{W(\Sigma)}$. We want to construct a form $\tilde\omega \in \Omega^p_{\text{hor}}(M)^G$ with $i^*\tilde\omega = \omega$. This will finish the proof of theorem 10.6.

Choose $x \in \Sigma$ and an open ball B_x with center 0 in T_xM such that the Riemannian exponential mapping $\exp_x : T_xM \to M$ is a diffeomorphism on B_x. We consider now the compact isotropy group G_x and the slice representation $\rho_x : G_x \to O(V_x)$, where $V_x = \text{Nor}_x(G.x) = (T_x(G.x))^\perp \subset T_xM$ is the normal space to the orbit. This is a polar representation with section $T_x\Sigma$, and its generalized Weyl group is given by $W(T_x\Sigma) \cong N_G(\Sigma)\cap G_x/Z_G(\Sigma) = W(\Sigma)_x$ (see 9.4). Then $\exp_x : B_x\cap V_x \to S_x$ is a diffeomorphism onto a slice and $\exp_x : B_x\cap T_x\Sigma \to \Sigma_x \subset \Sigma$ is a diffeomorphism onto an open neighborhood Σ_x of x in the section Σ.

Let us now consider the pullback $(\exp|B_x\cap T_x\Sigma)^*\omega \in \Omega^p(B_x\cap T_x\Sigma)^{W(T_x\Sigma)}$. By corollary 10.8 there exists a unique form $\varphi^x \in \Omega^p_{G_x-\text{hor}}(B_x\cap V_x)^{G_x}$ such that $i^*\varphi^x = (\exp|B_x\cap T_x\Sigma)^*\omega$, where i_x is the embedding. Then we have

$$((\exp|B_x\cap V_x)^{-1}) * \varphi^x \in \Omega^p_{G_x-\text{hor}}(S_x)^{G_x}$$

and by lemma 10.9 this form corresponds uniquely to a differential form $\omega^x \in \Omega^p_{\text{hor}}(G.S_x)^G$ which satisfies $(i|\Sigma_x)^*\omega^x = \omega|\Sigma_x$, since the exponential mapping commutes with the respective restriction mappings. Now the intersection $G.S_x\cap\Sigma$ is the disjoint union of all the open sets $w_j(\Sigma_x)$ where we pick one w_j in each left coset of the subgroup $W(\Sigma)_x$ in $W(\Sigma)$. If we choose $g_j \in N_G(\Sigma)$ projecting on w_j for all j, then

$$\begin{aligned}(i|w_j(\Sigma_x))^*\omega^x &= (\ell_{g_j}\circ i|\Sigma_x\circ w_j^{-1})^*\omega^x \\ &= (w_j^{-1})^*(i|\Sigma_x)^*\ell_{g_j}^*\omega^x \\ &= (w_j^{-1})^*(i|\Sigma_x)^*\omega^x = (w_j^{-1})^*(\omega|\Sigma_x) = \omega|w_j(\Sigma_x),\end{aligned}$$

so that $(i|G.S_x \cap \Sigma)^* \omega^x = \omega|G.S_x \cap \Sigma$. We can do this for each point $x \in \Sigma$.

Using the method of 5.8 and 5.10 we may find a sequence of points $(x_n)_{n\in\mathbb{N}}$ in Σ such that the $\pi(\Sigma_{x_n})$ form a locally finite open cover of the orbit space $M/G \cong \Sigma/W(\Sigma)$, and a smooth partition of unity f_n consisting of G-invariant functions with $\operatorname{supp}(f_n) \subset G.S_{x_n}$. Then $\tilde\omega := \sum_n f_n \omega^{x_n} \in \Omega^p_{\text{hor}}(M)^G$ has the required property $i^*\tilde\omega = \omega$. $\square$

11. Basic versus equivariant cohomology

11.1. Basic cohomology. For a Lie group G and a smooth G-manifold M, by 10.2 we may consider the basic cohomology $H^p_{G-\text{basic}}(M) = H^p(\Omega^*_{\text{hor}}(M)^G, d)$.

11.2. Equivariant cohomology, Borel model. For a topological group and a topological G-space the equivariant cohomology was defined as follows, see [3]: Let $EG \to BG$ be the classifying G-bundle, and consider the associated bundle $EG \times_G M$ with standard fiber the G-space M. Then the equivariant cohomology is given by $H^p(EG \times_G M; \mathbb{R})$.

11.3. Equivariant cohomology, Cartan model. For a Lie group G and a smooth G-manifold M we consider the space

$$(S^k \mathfrak{g}^* \otimes \Omega^p(M))^G$$

of all homogeneous polynomial mappings $\alpha : \mathfrak{g} \to \Omega^p(M)$ of degree k from the Lie algebra $\mathfrak{g}$ of G to the space of k-forms, which are G-equivariant: $\alpha(\mathrm{Ad}(g^{-1})X) = \ell_g^* \alpha(X)$ for all $g \in G$. The mapping

$$\begin{gathered}
d_\mathfrak{g} : A^q_G(M) \to A^{q+1}_G(M) \\
A^q_G(M) := \bigoplus_{2k+p=q} (S^k \mathfrak{g}^* \otimes \Omega^p(M))^G \\
(d_\mathfrak{g} \alpha)(X) := d(\alpha(X)) - i_{\zeta_X} \alpha(X)
\end{gathered}$$

satisfies $d_\mathfrak{g} \circ d_\mathfrak{g} = 0$ and the following result holds.

Theorem. *Let G be a compact connected Lie group and let M be a smooth G-manifold. Then*

$$H^p(EG \times_G M; \mathbb{R}) = H^p(A^*_G(M), d_\mathfrak{g}).$$

This result is stated in [1] together with some arguments, and it is attributed to [6], [7] in chapter 7 of [2]. I was unable to find a satisfactory published proof.

11.4.. Let M be a smooth G-manifold. Then the obvious embedding $j(\omega) = 1 \otimes \omega$ gives a mapping of graded differential algebras

$$j : \Omega^p_{\text{hor}}(M)^G \to (S^0 \mathfrak{g}^* \otimes \Omega^p(M))^G \to \bigoplus_k (S^k \mathfrak{g}^* \otimes \Omega^{p-2k}(M))^G = A^p_G(M).$$

On the other hand evaluation at $0 \in \mathfrak{g}$ defines a homomorphism of graded differential algebras $\mathrm{ev}_0 : A^*_G(M) \to \Omega^*(M)^G$, and $\mathrm{ev}_0 \circ j$ is the embedding $\Omega^*_{\mathrm{hor}}(M)^G \to \Omega^*(M)^G$. Thus we get canonical homomorphisms in cohomology

$$\begin{array}{ccccc}
H^p(\Omega^*_{\mathrm{hor}}(M)^G) & \xrightarrow{J^*} & H^p(A^*_G(M), d_{\mathfrak{g}}) & \longrightarrow & H^p(\Omega^*(M)^G, d) \\
\| & & \| & & \| \\
H^p_{G-\mathrm{basic}}(M) & \longrightarrow & H^p_G(M) & \longrightarrow & H^p(M)^G.
\end{array}$$

If G is compact and connected we have $H^p(M)^G = H^p(M)$, by integration and homotopy invariance.

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