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FOUNDATIONS OF DIFFERENTIAL GEOMETRY

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These notes are from a lecture course

Differentialgeometrie und Lie Gruppen

which has been held at the University of Vienna during the academic year 1990/91, again in 1994/95, and in WS 1997. It is not yet complete and will be enlarged during the year.

In this lecture course I give complete definitions of manifolds in the beginning, but (beside spheres) examples are treated extensively only later when the theory is developed enough. I advise every novice to the field to read the excellent lecture notes

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1. Differentiable Manifolds

1.1. Manifolds. A *topological manifold* is a separable metrizable space M which is locally homeomorphic to $\mathbb{R}^n$. So for any $x \in M$ there is some homeomorphism $u : U \to u(U) \subseteq \mathbb{R}^n$, where U is an open neighborhood of x in M and $u(U)$ is an open subset in $\mathbb{R}^n$. The pair (U, u) is called a *chart* on M.

From algebraic topology it follows that the number n is locally constant on M; if n is constant, M is sometimes called a *pure* manifold. We will only consider pure manifolds and consequently we will omit the prefix pure.

A family $(U_\alpha, u_\alpha)_{\alpha \in A}$ of charts on M such that the U_α form a cover of M is called an *atlas.* The mappings $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ are called the chart changings for the atlas (U_α), where $U_{\alpha\beta} := U_\alpha \cap U_\beta$.

An atlas $(U_\alpha, u_\alpha)_{\alpha \in A}$ for a manifold M is said to be a C^k-*atlas*, if all chart changings $u_{\alpha\beta} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ are differentiable of class C^k. Two C^k-atlases are called C^k-*equivalent*, if their union is again a C^k-atlas for M. An equivalence class of C^k-atlases is called a C^k-*structure* on M. From differential topology we know that if M has a C^1-structure, then it also has a C^1-equivalent C^∞-structure and even a C^1-equivalent C^ω-structure, where C^ω is shorthand for real analytic, see [Hirsch, 1976]. By a C^k-manifold M we mean a topological manifold together with a C^k-structure and a chart on M will be a chart belonging to some atlas of the C^k-structure.

But there are topological manifolds which do not admit differentiable structures. For example, every 4-dimensional manifold is smooth off some point, but there are such which are not smooth, see [Quinn, 1982], [Freedman, 1982]. There are also topological manifolds which admit several inequivalent smooth structures. The spheres from dimension 7 on have finitely many, see [Milnor, 1956]. But the most surprising result is that on $\mathbb{R}^4$ there are uncountably many pairwise inequivalent (exotic) differentiable structures. This follows from the results of [Donaldson, 1983] and [Freedman, 1982], see [Gompf, 1983] or [Mattes, Diplomarbeit, Wien, 1990] for an overview.

Note that for a Hausdorff C^∞-manifold in a more general sense the following properties are equivalent:

(1) It is paracompact.
(2) It is metrizable.
(3) It admits a Riemannian metric.
(4) Each connected component is separable.

In this book a manifold will usually mean a C^∞-manifold, and smooth is used synonymously for C^∞, it will be Hausdorff, separable, finite dimensional, to state it precisely.

Note finally that any manifold M admits a finite atlas consisting of $\dim M+1$ (not connected) charts. This is a consequence of topological dimension theory [Nagata, 1965], a proof for manifolds may be found in [Greub-Halperin-Vanstone, Vol. I].

1.2. Example: Spheres. We consider the space $\mathbb{R}^{n+1}$, equipped with the standard inner product $\langle x,y\rangle = \sum x^i y^i$. The *n-sphere* S^n is then the subset $\{x\in\mathbb{R}^{n+1} : \langle x,x\rangle = 1\}$. Since $f(x)=\langle x,x\rangle$, $f:\mathbb{R}^{n+1}\to\mathbb{R}$, satisfies $df(x)y = 2\langle x,y\rangle$, it is of rank 1 off 0 and by 1.12 the sphere S^n is a submanifold of $\mathbb{R}^{n+1}$.

In order to get some feeling for the sphere we will describe an explicit atlas for S^n, the *stereographic atlas*. Choose $a\in S^n$ ('south pole'). Let

$$\begin{aligned}
U_+ &:= S^n\setminus\{a\}, &\qquad u_+ &: U_+\to\{a\}^\perp, &\qquad u_+(x) &= \tfrac{x-\langle x,a\rangle a}{1-\langle x,a\rangle},\\
U_- &:= S^n\setminus\{-a\}, &\qquad u_- &: U_-\to\{a\}^\perp, &\qquad u_-(x) &= \tfrac{x-\langle x,a\rangle a}{1+\langle x,a\rangle}.
\end{aligned}$$

From an obvious drawing in the 2-plane through 0, x, and a it is easily seen that u_+ is the usual stereographic projection. We also get

$$u_+^{-1}(y) = \tfrac{|y|^2-1}{|y|^2+1}a + \tfrac{2}{|y|^2+1}y \qquad \text{for } y\in\{a\}^\perp\setminus\{0\}$$

and $(u_-\circ u_+^{-1})(y) = \frac{y}{|y|^2}$. The latter equation can directly be seen from the drawing using 'Strahlensatz'.

1.3. Smooth mappings. A mapping $f: M\to N$ between manifolds is said to be C^k if for each $x\in M$ and one (equivalently: any) chart (V,v) on N with $f(x)\in V$ there is a chart (U,u) on M with $x\in U$, $f(U)\subseteq V$, and $v\circ f\circ u^{-1}$ is C^k. We will denote by $C^k(M,N)$ the space of all C^k-mappings from M to N.

A C^k-mapping $f:M\to N$ is called a C^k-*diffeomorphism* if $f^{-1}:N\to M$ exists and is also C^k. Two manifolds are called *diffeomorphic* if there exists a diffeomorphism between them. From differential topology we know that if there is a C^1-diffeomorphism between M and N, then there is also a C^∞-diffeomorphism.

There are manifolds which are homeomorphic but not diffeomorphic: on $\mathbb{R}^4$ there are uncountably many pairwise non-diffeomorphic differentiable structures; on every other $\mathbb{R}^n$ the differentiable structure is unique. There are finitely many different differentiable structures on the spheres S^n for $n\geq 7$.

A mapping $f:M\to N$ between manifolds of the same dimension is called a *local diffeomorphism*, if each $x\in M$ has an open neighborhood U such that $f|U:U\to f(U)\subset N$ is a diffeomorphism. Note that a local diffeomorphism need not be surjective.

1.4. Smooth functions. The set of smooth real valued functions on a manifold M will be denoted by $C^\infty(M,\mathbb{R})$, in order to distinguish it clearly from spaces of sections which will appear later. $C^\infty(M,\mathbb{R})$ is a real commutative algebra.

The *support* of a smooth function f is the closure of the set, where it does not vanish, $supp(f) = \overline{\{x \in M : f(x) \neq 0\}}$. The *zero set* of f is the set where f vanishes, $Z(f) = \{x \in M : f(x) = 0\}$.

1.5. Theorem. *Any manifold admits smooth partitions of unity: Let $(U_\alpha)_{\alpha\in A}$ be an open cover of M. Then there is a family $(\varphi_\alpha)_{\alpha\in A}$ of smooth functions on M, such that $supp(\varphi_\alpha) \subset U_\alpha$, $(supp(\varphi_\alpha))$ is a locally finite family, and $\sum_\alpha \varphi_\alpha = 1$ (locally this is a finite sum).*

Proof. Any manifold is a *"Lindelöf space"*, i. e. each open cover admits a countable subcover. This can be seen as follows:

Let $\mathcal{U}$ be an open cover of M. Since M is separable there is a countable dense subset S in M. Choose a metric on M. For each $U \in \mathcal{U}$ and each $x \in U$ there is an $y \in S$ and $n \in \mathbb{N}$ such that the ball $B_{1/n}(y)$ with respect to that metric with center y and radius $\frac{1}{n}$ contains x and is contained in U. But there are only countably many of these balls; for each of them we choose an open set $U \in \mathcal{U}$ containing it. This is then a countable subcover of $\mathcal{U}$.

Now let $(U_\alpha)_{\alpha\in A}$ be the given cover. Let us fix first α and $x \in U_\alpha$. We choose a chart (U,u) centered at x (i. e. $u(x) = 0$) and $\varepsilon > 0$ such that $\varepsilon\mathbb{D}^n \subset u(U \cap U_\alpha)$, where $\mathbb{D}^n = \{y \in \mathbb{R}^n : |y| \leq 1\}$ is the closed unit ball. Let

$$h(t) := \begin{cases} e^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

a smooth function on $\mathbb{R}$. Then

$$f_{\alpha,x}(z) := \begin{cases} h(\varepsilon^2 - |u(z)|^2) & \text{for } z \in U, \\ 0 & \text{for } z \notin U \end{cases}$$

is a non negative smooth function on M with support in U_α which is positive at x.

We choose such a function $f_{\alpha,x}$ for each α and $x \in U_\alpha$. The interiors of the supports of these smooth functions form an open cover of M which refines (U_α), so by the argument at the beginning of the proof there is a countable subcover with corresponding functions $f_1, f_2, \dots$. Let

$$W_n = \{x \in M : f_n(x) > 0 \text{ and } f_i(x) < \tfrac{1}{n} \text{ for } 1 \leq i < n\},$$

and denote by $\overline{W}$ the closure. We claim that $(\overline{W}_n)$ is a locally finite open cover of M: Let $x \in M$. Then there is a smallest n such that $x \in W_n$. Let

$V := \{y \in M : f_n(y) > \frac{1}{2} f_n(x)\}$. If $y \in V \cap \overline{W}_k$ then we have $f_n(y) > \frac{1}{2} f_n(x)$ and $f_i(y) \le \frac{1}{k}$ for $i < k$, which is possible for finitely many k only.

Now we define for each n a non negative smooth function g_n by $g_n(x) = h(f_n(x))h(\frac{1}{n} - f_1(x)) \dots h(\frac{1}{n} - f_{n-1}(x))$. Then obviously $\operatorname{supp}(g_n) = \overline{W}_n$. So $g := \sum_n g_n$ is smooth, since it is locally only a finite sum, and everywhere positive, thus $(g_n/g)_{n\in\mathbb{N}}$ is a smooth partition of unity on M. Since $\operatorname{supp}(g_n) = \overline{W}_n$ is contained in some $U_{\alpha(n)}$ we may put $\varphi_\alpha = \sum_{\{n:\alpha(n)=\alpha\}} \frac{g_n}{g}$ to get the required partition of unity which is subordinated to (U_α). $\square$

1.6. Germs. Let M be a manifold and $x \in M$. We consider all smooth functions $f : U_f \to \mathbb{R}$, where U_f is some open neighborhood of x in M, and we put $f \underset{x}{\sim} g$ if there is some open neighborhood V of x with $f|V = g|V$. This is an equivalence relation on the set of functions we consider. The equivalence class of a function f is called the *germ of f at x*, sometimes denoted by $\operatorname{germ}_x f$. We may add and multiply germs, so we get the real commutative algebra of germs of smooth functions at x, sometimes denoted by $C^\infty_x(M,\mathbb{R})$. This construction works also for other types of functions like real analytic or holomorphic ones, if M has a real analytic or complex structure.

Using smooth partitions of unity (1.4) it is easily seen that each germ of a smooth function has a representative which is defined on the whole of M. For germs of real analytic or holomorphic functions this is not true. So $C^\infty_x(M,\mathbb{R})$ is the quotient of the algebra $C^\infty(M,\mathbb{R})$ by the ideal of all smooth functions $f : M \to \mathbb{R}$ which vanish on some neighborhood (depending on f) of x.

1.7. The tangent space of $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$. A *tangent vector* with foot point a is simply a pair (a, X) with $X \in \mathbb{R}^n$, also denoted by X_a. It induces a *derivation* $X_a : C^\infty(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}$ by $X_a(f) = df(a)(X_a)$. The value depends only on the germ of f at a and we have $X_a(f \cdot g) = X_a(f) \cdot g(a) + f(a) \cdot X_a(g)$ (the derivation property).

If conversely $D : C^\infty(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}$ is linear and satisfies $D(f \cdot g) = D(f) \cdot g(a) + f(a) \cdot D(g)$ (a derivation at a), then D is given by the action of a tangent vector with foot point a. This can be seen as follows. For $f \in C^\infty(\mathbb{R}^n,\mathbb{R})$ we have

$$\begin{aligned}
f(x) &= f(a) + \int_0^1 \tfrac{d}{dt} f(a + t(x-a))\,dt \\
&= f(a) + \sum_{i=1}^n \int_0^1 \tfrac{\partial f}{\partial x^i}(a + t(x-a))\,dt\; (x^i - a^i) \\
&= f(a) + \sum_{i=1}^n h_i(x)(x^i - a^i).
\end{aligned}$$

$$
\begin{aligned}
&D(1) = D(1\cdot 1) = 2D(1), \text{ so } D(\text{constant}) = 0. \text{ Thus}\\
&D(f) = D(f(a) + \sum_{i=1}^n h_i(x^i - a^i))\\
&\quad = 0 + \sum_{i=1}^n D(h_i)(a^i - a^i) + \sum_{i=1}^n h_i(a)(D(x^i) - 0)\\
&\quad = \sum_{i=1}^n \tfrac{\partial f}{\partial x^i}(a) D(x^i),
\end{aligned}
$$

where x^i is the i-th coordinate function on $\mathbb{R}^n$. So we have

$$D(f) = \sum_{i=1}^n D(x^i)\tfrac{\partial}{\partial x^i}|_a(f), \qquad D = \sum_{i=1}^n D(x^i)\tfrac{\partial}{\partial x^i}|_a.$$

Thus D is induced by the tangent vector $(a, \sum_{i=1}^n D(x^i)e_i)$, where (e_i) is the standard basis of $\mathbb{R}^n$.

1.8. The tangent space of a manifold. Let M be a manifold and let $x \in M$ and $\dim M = n$. Let T_xM be the vector space of all derivations at x of $C^\infty_x(M,\mathbb{R})$, the algebra of germs of smooth functions on M at x. (Using 1.5 it may easily be seen that a derivation of $C^\infty(M,\mathbb{R})$ at x factors to a derivation of $C^\infty_x(M,\mathbb{R})$.)

So T_xM consists of all linear mappings $X_x : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ with the property $X_x(f\cdot g) = X_x(f)\cdot g(x) + f(x)\cdot X_x(g)$. The space T_xM is called the *tangent space of M at x*.

If (U,u) is a chart on M with $x \in U$, then $u^* : f \mapsto f\circ u$ induces an isomorphism of algebras $C^\infty_{u(x)}(\mathbb{R}^n,\mathbb{R}) \cong C^\infty_x(M,\mathbb{R})$, and thus also an isomorphism $T_xu : T_xM \to T_{u(x)}\mathbb{R}^n$, given by $(T_xu.X_x)(f) = X_x(f\circ u)$. So T_xM is an n-dimensional vector space.

We will use the following notation: $u = (u^1,\dots,u^n)$, so u^i denotes the i-th coordinate function on U, and

$$\tfrac{\partial}{\partial u^i}|_x := (T_xu)^{-1}(\tfrac{\partial}{\partial x^i}|_{u(x)}) = (T_xu)^{-1}(u(x), e_i).$$

So $\frac{\partial}{\partial u^i}|_x \in T_xM$ is the derivation given by

$$\tfrac{\partial}{\partial u^i}|_x(f) = \frac{\partial(f\circ u^{-1})}{\partial x^i}(u(x)).$$

From 1.7 we have now

$$\begin{aligned}
T_xu.X_x &= \sum_{i=1}^n (T_xu.X_x)(x^i)\tfrac{\partial}{\partial x^i}|_{u(x)} = \sum_{i=1}^n X_x(x^i\circ u)\tfrac{\partial}{\partial x^i}|_{u(x)}\\
&= \sum_{i=1}^n X_x(u^i)\tfrac{\partial}{\partial x^i}|_{u(x)},
\end{aligned}$$

$$X_x = (T_xu)^{-1}.T_xu.X_x = \sum_{i=1}^n X_x(u^i)\tfrac{\partial}{\partial u^i}|_x.$$

1.9. The tangent bundle. For a manifold M of dimension n we put $TM := \bigsqcup_{x\in M} T_xM$, the disjoint union of all tangent spaces. This is a family of vector spaces parameterized by M, with projection $\pi_M : TM \to M$ given by $\pi_M(T_xM) = x$.

For any chart (U_α, u_α) of M consider the chart $(\pi_M^{-1}(U_\alpha), Tu_\alpha)$ on TM, where $Tu_\alpha : \pi_M^{-1}(U_\alpha) \to u_\alpha(U_\alpha)\times\mathbb{R}^n$ is given by the formula $Tu_\alpha.X = (u_\alpha(\pi_M(X)), T_{\pi_M(X)}u_\alpha.X)$. Then the chart changings look as follows:

$$\begin{aligned}
Tu_\beta\circ(Tu_\alpha)^{-1} :&Tu_\alpha(\pi_M^{-1}(U_{\alpha\beta})) = u_\alpha(U_{\alpha\beta})\times\mathbb{R}^n \to\\
&\to u_\beta(U_{\alpha\beta})\times\mathbb{R}^n = Tu_\beta(\pi_M^{-1}(U_{\alpha\beta})),\\
((Tu_\beta\circ(Tu_\alpha)^{-1})(y,Y))(f) &= ((Tu_\alpha)^{-1}(y,Y))(f\circ u_\beta)\\
&= (y,Y)(f\circ u_\beta\circ u_\alpha^{-1}) = d(f\circ u_\beta\circ u_\alpha^{-1})(y).Y\\
&= df(u_\beta\circ u_\alpha^{-1}(y)).d(u_\beta\circ u_\alpha^{-1})(y).Y\\
&= (u_\beta\circ u_\alpha^{-1}(y), d(u_\beta\circ u_\alpha^{-1})(y).Y)(f).
\end{aligned}$$

So the chart changings are smooth. We choose the topology on TM in such a way that all Tu_α become homeomorphisms. This is a Hausdorff topology, since X, $Y\in TM$ may be separated in M if $\pi(X)\ne\pi(Y)$, and in one chart if $\pi(X)=\pi(Y)$. So TM is again a smooth manifold in a canonical way; the triple (TM,π_M,M) is called the *tangent bundle* of M.

1.10. Kinematic definition of the tangent space. Let $C_0^\infty(\mathbb{R},M)$ denote the space of germs at 0 of smooth curves $\mathbb{R}\to M$. We put the following equivalence relation on $C_0^\infty(\mathbb{R},M)$: the germ of c is equivalent to the germ of e if and only if $c(0)=e(0)$ and in one (equivalently each) chart (U,u) with $c(0)=e(0)\in U$ we have $\frac{d}{dt}|_0(u\circ c)(t) = \frac{d}{dt}|_0(u\circ e)(t)$. The equivalence classes are also called velocity vectors of curves in M. We have the following mappings

$$\begin{tikzcd} C^\infty_0(\mathbb{R},M)/\sim \arrow[d, "\alpha"'] & C^\infty_0(\mathbb{R},M) \arrow[l] \arrow[d, "\mathrm{ev}_0"] \\ TM \arrow[ur, "\beta"'] \arrow[r, "\pi_M"'] & M, \end{tikzcd}$$

where $\alpha(c)(\mathrm{germ}_{c(0)} f) = \frac{d}{dt}|_0 f(c(t))$ and $\beta : TM \to C^\infty_0(\mathbb{R},M)$ is given by: $\beta((Tu)^{-1}(y,Y))$ is the germ at 0 of $t \mapsto u^{-1}(y+tY)$. So TM is canonically identified with the set of all possible velocity vectors of curves in M.

1.11. Tangent mappings. Let $f : M \to N$ be a smooth mapping between manifolds. Then f induces a linear mapping $T_x f : T_x M \to T_{f(x)}N$ for each $x \in M$ by $(T_x f.X_x)(h) = X_x(h \circ f)$ for $h \in C^\infty_{f(x)}(N,\mathbb{R})$. This mapping is well defined and linear since $f^* : C^\infty_{f(x)}(N,\mathbb{R}) \to C^\infty_x(M,\mathbb{R})$, given by $h \mapsto h \circ f$, is linear and an algebra homomorphism, and $T_x f$ is its adjoint, restricted to the subspace of derivations.

If (U,u) is a chart around x and (V,v) is one around $f(x)$, then

$$\begin{aligned} (T_x f.\tfrac{\partial}{\partial u^i}|_x)(v^j) &= \tfrac{\partial}{\partial u^i}|_x(v^j \circ f) = \tfrac{\partial}{\partial x^i}(v^j \circ f \circ u^{-1})(u(x)), \\ T_x f.\tfrac{\partial}{\partial u^i}|_x &= \textstyle\sum_j (T_x f.\tfrac{\partial}{\partial u^i}|_x)(v^j)\tfrac{\partial}{\partial v^j}|_{f(x)} \qquad \text{by 1.9} \\ &= \textstyle\sum_j \tfrac{\partial(v^j\circ f\circ u^{-1})}{\partial x^i}(u(x))\tfrac{\partial}{\partial v^j}|_{f(x)}. \end{aligned}$$

So the matrix of $T_x f : T_x M \to T_{f(x)}N$ in the bases $(\frac{\partial}{\partial u^i}|_x)$ and $(\frac{\partial}{\partial v^j}|_{f(x)})$ is just the Jacobi matrix $d(v \circ f \circ u^{-1})(u(x))$ of the mapping $v \circ f \circ u^{-1}$ at $u(x)$, so $T_{f(x)}v \circ T_x f \circ (T_x u)^{-1} = d(v \circ f \circ u^{-1})(u(x))$.

Let us denote by $Tf : TM \to TN$ the total mapping, given by $Tf|T_x M := T_x f$. Then the composition $Tv \circ Tf \circ (Tu)^{-1} : u(U) \times \mathbb{R}^m \to v(V) \times \mathbb{R}^n$ is given by $(y,Y) \mapsto ((v \circ f \circ u^{-1})(y), d(v \circ f \circ u^{-1})(y)Y)$, and thus $Tf : TM \to TN$ is again smooth.

If $f : M \to N$ and $g : N \to P$ are smooth mappings, then we have $T(g \circ f) = Tg \circ Tf$. This is a direct consequence of $(g \circ f)^* = f^* \circ g^*$, and it is the global version of the chain rule. Furthermore we have $T(Id_M) = Id_{TM}$.

If $f \in C^\infty(M,\mathbb{R})$, then $Tf : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$. We then define the *differential* of f by $df := pr_2 \circ Tf : TM \to \mathbb{R}$. Let t denote the identity function on $\mathbb{R}$, then $(Tf.X_x)(t) = X_x(t \circ f) = X_x(f)$, so we have $df(X_x) = X_x(f)$.

1.12. Submanifolds. A subset N of a manifold M is called a *submanifold*, if for each $x \in N$ there is a chart (U,u) of M such that $u(U \cap N) = u(U) \cap (\mathbb{R}^k \times 0)$, where $\mathbb{R}^k \times 0 \hookrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$. Then clearly N is itself a manifold with $(U \cap N, u|U \cap N)$ as charts, where (U, u) runs through all *submanifold charts* as above.

If $f : \mathbb{R}^n \to \mathbb{R}^q$ is smooth and the rank of f (more exactly: the rank of its derivative) is q at each point y of $f^{-1}(0)$, say, then $f^{-1}(0)$ is a submanifold of $\mathbb{R}^n$ of dimension $n - q$ (or empty). This is an immediate consequence of the implicit function theorem, as follows: Permute the coordinates $(x^1, \dots, x^n)$ on $\mathbb{R}^n$ such that the Jacobi matrix

$$df(y) = \left(\begin{array}{ccc|ccc} \frac{\partial f^1}{\partial x^1}(y) & \dots & \frac{\partial f^1}{\partial x^q}(y) & \frac{\partial f^1}{\partial x^{q+1}}(y) & \dots & \frac{\partial f^1}{\partial x^n}(y) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f^q}{\partial x^1}(y) & \dots & \frac{\partial f^q}{\partial x^q}(y) & \frac{\partial f^q}{\partial x^{q+1}}(y) & \dots & \frac{\partial f^q}{\partial x^n}(y) \end{array}\right)$$

has the left part invertible. Then $(f, \mathrm{pr}_{n-q}) : \mathbb{R}^n \to \mathbb{R}^q \times \mathbb{R}^{n-q}$ has invertible differential at y, so $u := f^{-1}$ exists in locally near y and we have $f \circ u^{-1}(z^1, \dots, z^n) = (z^1, \dots, z^q)$, so $u(f^{-1}(0)) = u(U) \cap (0 \times \mathbb{R}^{n-q})$ as required.

The following theorem needs three applications of the implicit function theorem for its proof, which is sketched in execise 1.21 below, or can be found in [Dieudonné, I, 10.3.1].

Constant rank theorem. *Let $f : W \to \mathbb{R}^q$ be a smooth mapping, where W is an open subset of $\mathbb{R}^n$. If the derivative $df(x)$ has constant rank k for each $x \in W$, then for each $a \in W$ there are charts (U, u) of W centered at a and (V, v) of $\mathbb{R}^q$ centered at $f(a)$ such that $v \circ f \circ u^{-1} : u(U) \to v(V)$ has the following form:*

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k, 0, \dots, 0).$$

So $f^{-1}(b)$ is a submanifold of W of dimension $n - k$ for each $b \in f(W)$.

1.13. Products. Let M and N be smooth manifolds described by smooth atlases $(U_\alpha, u_\alpha)_{\alpha \in A}$ and $(V_\beta, v_\beta)_{\beta \in B}$, respectively. Then the family $(U_\alpha \times V_\beta, u_\alpha \times v_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n)_{(\alpha,\beta) \in A \times B}$ is a smooth atlas for the cartesian product $M \times N$. Clearly the projections

$$M \xleftarrow{pr_1} M \times N \xrightarrow{pr_2} N$$

are also smooth. The *product* $(M \times N, pr_1, pr_2)$ has the following universal property:

For any smooth manifold P and smooth mappings $f : P \to M$ and $g : P \to N$ the mapping $(f, g) : P \to M \times N$, $(f, g)(x) = (f(x), g(x))$, is the unique smooth mapping with $pr_1 \circ (f, g) = f$, $pr_2 \circ (f, g) = g$.

From the construction of the tangent bundle in 1.9 it is immediately clear that

$$TM \xleftarrow{T(pr_1)} T(M \times N) \xrightarrow{T(pr_2)} TN$$

is again a product, so that $T(M \times N) = TM \times TN$ in a canonical way.

Clearly we can form products of finitely many manifolds.

1.14. Theorem. *Let M be a connected manifold and suppose that $f : M \to M$ is smooth with $f \circ f = f$. Then the image $f(M)$ of f is a submanifold of M.*

This result can also be expressed as: 'smooth retracts' of manifolds are manifolds. If we do not suppose that M is connected, then $f(M)$ will not be a pure manifold in general, it will have different dimension in different connected components.

Proof. We claim that there is an open neighborhood U of $f(M)$ in M such that the rank of $T_y f$ is constant for $y \in U$. Then by theorem 1.12 the result follows.

For $x \in f(M)$ we have $T_x f \circ T_x f = T_x f$, thus $\operatorname{im} T_x f = \ker(Id - T_x f)$ and $\operatorname{rank} T_x f + \operatorname{rank}(Id - T_x f) = \dim M$. Since $\operatorname{rank} T_x f$ and $\operatorname{rank}(Id - T_x f)$ cannot fall locally, $\operatorname{rank} T_x f$ is locally constant for $x \in f(M)$, and since $f(M)$ is connected, $\operatorname{rank} T_x f = r$ for all $x \in f(M)$.

But then for each $x \in f(M)$ there is an open neighborhood U_x in M with $\operatorname{rank} T_y f \geq r$ for all $y \in U_x$. On the other hand $\operatorname{rank} T_y f = \operatorname{rank} T_y(f \circ f) = \operatorname{rank} T_{f(y)} f \circ T_y f \leq \operatorname{rank} T_{f(y)} f = r$. So the neighborhood we need is given by $U = \bigcup_{x \in f(M)} U_x$. $\square$

1.15. Corollary. *1. The (separable) connected smooth manifolds are exactly the smooth retracts of connected open subsets of $\mathbb{R}^n$'s.*

2. $f : M \to N$ is an embedding of a submanifold if and only if there is an open neighborhood U of $f(M)$ in N and a smooth mapping $r : U \to M$ with $r \circ f = Id_M$.

Proof. Any manifold M may be embedded into some $\mathbb{R}^n$, see 1.16 below. Then there exists a tubular neighborhood of M in R^n (see later or [Hirsch, 1976, pp. 109–118]), and M is clearly a retract of such a tubular neighborhood. The converse follows from 1.14.

For the second assertion repeat the argument for N instead of $\mathbb{R}^n$. $\square$

1.16. Embeddings into $\mathbb{R}^n$'s. Let M be a smooth manifold of dimension m. Then M can be embedded into $\mathbb{R}^n$, if

(1) $n = 2m + 1$ (see [Hirsch, 1976, p 55] or [Bröcker-Jänich, 1973, p 73]),
(2) $n = 2m$ (see [Whitney, 1944]).
(3) Conjecture (still unproved): The minimal n is $n = 2m - \alpha(m) + 1$, where $\alpha(m)$ is the number of 1's in the dyadic expansion of m.

There exists an immersion (see section 2) $M \to \mathbb{R}^n$, if

(1) $n = 2m$ (see [Hirsch, 1976]),
(2) $n = 2m - \alpha(m)$ (see [Cohen, 1982]).

Examples and Exercises

1.17. Discuss the following submanifolds of $\mathbb{R}^n$, in particular make drawings of them:

The unit sphere $S^{n-1} = \{x \in \mathbb{R}^n :< x, x >= 1\} \subset \mathbb{R}^n$.

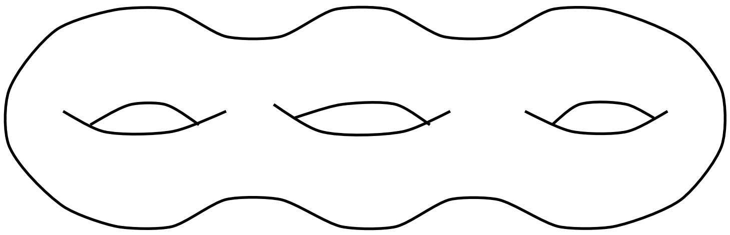
The *ellipsoid* $\{x \in \mathbb{R}^n : f(x) := \sum_{i=1}^n \frac{x_i^2}{a_i^2} = 1\}$, $a_i \neq 0$ with principal axis $a_1, \dots, a_n$.

The *hyperboloid* $\{x \in \mathbb{R}^n : f(x) := \sum_{i=1}^n \varepsilon_i \frac{x_i^2}{a_i^2} = 1\}$, $\varepsilon_i = \pm 1$, $a_i \neq 0$ with principal axis a_i and index $= \sum \varepsilon_i$.

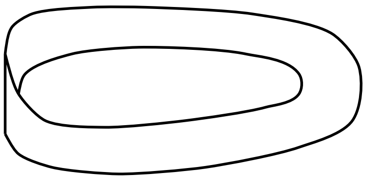
The *saddle* $\{x \in \mathbb{R}^3 : x_3 = x_1 x_2\}$.

The *torus*: the rotation surface generated by rotation of $(y - R)^2 + z^2 = r^2$, $0 < r < R$ with center the z–axis, i.e. $\{(x, y, z) : (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}$.

1.18. A compact surface of genus g. Let $f(x) := x(x-1)^2(x-2)^2 \dots (x-(g-1))^2(x-g)$. For small $r > 0$ the set $\{(x, y, z) : (y^2 + f(x))^2 + z^2 = r^2\}$ describes a surface of genus g (topologically a sphere with g handles) in $\mathbb{R}^3$. Prove this.



1.19. The Moebius strip.



It is not the set of zeros of a regular function on an open neighborhood of $\mathbb{R}^n$. Why not? But it may be represented by the following parametrization:

$$f(r,\varphi) := \begin{pmatrix} \cos\varphi(R + r\cos(\varphi/2)) \\ \sin\varphi(R + r\cos(\varphi/2)) \\ r\sin(\varphi/2) \end{pmatrix}, \qquad (r,\varphi) \in (-1,1)\times[0,2\pi),$$

where R is quite big.

1.20. Describe an atlas for the real projective plane which consists of three charts (homogeneous coordinates) and compute the chart changings.

Then describe an atlas for the n-dimensional real projective space $P^n(\mathbb{R})$ and compute the chart changes.

1.21. Proof of the constant rank theorem 1.12. Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^m$ be a C^∞–mapping. If the Jacobi matrix df has constant rank k on U, we have:
For each $a \in U$ there exists an open neighborhood U_a of a in U, a diffeomorphism $\varphi : U_a \to \varphi(U_a)$ onto an open subset of $\mathbb{R}^n$ with $\varphi(a) = 0$, an open subset $V_{f(a)}$ of $f(a)$ in $\mathbb{R}^m$, and a diffeomorphism $\psi : V_{f(a)} \to \psi(V_{f(a)})$ onto an open subset of $\mathbb{R}^m$ with $\psi(f(a)) = 0$, such that $\psi\circ f\circ\varphi^{-1} : \varphi(U_a) \to \psi(V_{f(a)})$ has the following form: $(x_1,\dots,x_n) \mapsto (x_1,\dots,x_k,0,\dots,0)$.
(Hints: Use the inverse function theorem 3 times. 1. step: $df(a)$ has rank $k \le n,m$, without loss we may assume that the upper left $k\times k$ submatrix of $df(a)$ is invertible. Moreover, let $a = 0$ and $f(a) = 0$. Choose a suitable neighborhood U of 0 and consider $\varphi : U \to \mathbb{R}^n$, $\varphi(x_1,\dots,x_n) := (f_1(x_1),\dots,f_k(x_k),x_{k+1},\dots,x_n)$. Then φ is a diffeomorphism locally near 0. Consider $g = f\circ\varphi^{-1}$. What can you tell about g? Why is $g(z_1,\dots,z_n) = (z_1,\dots,z_k,g_{k+1}(z),\dots,g_n(z))$? What is the form of $dg(z)$? Deduce further properties of g from the rank of $dg(z)$? Put

$$\psi\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} := \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} - g_{k+1}(y_1,\dots,y_k,0,\dots,0) \\ \vdots \\ y_n - g_n(y_1,\dots,y_k,0,\dots,0) \end{pmatrix}$$

Then ψ is locally a diffeomorphism and $\psi\circ f\circ\varphi^{-1}$ has the desired form.)
Prove also the following **Corollary**: Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be C^∞ with df of constant rank k. Then for each $b \in f(U)$ the set $f^{-1}(b) \subset \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$.

1.22. Let $f : L(\mathbb{R}^n, \mathbb{R}^n) \to L(\mathbb{R}^n, \mathbb{R}^n)$ be given by $f(A) := A^t A$. Where is f of constant rank? What is $f^{-1}(\mathrm{Id})$?

1.23. Let $f : L(\mathbb{R}^n, \mathbb{R}^m) \to L(\mathbb{R}^n, \mathbb{R}^n)$, $n < m$ be given by $f(A) := A^t A$. Where is f of constant rank? What is $f^{-1}(Id_{\mathbb{R}^n})$?

1.24. Let S be a symmetric a symmetric matrix, i.e., $S(x, y) := x^t S y$ is a symmetric bilinear form on $\mathbb{R}^n$. Let $f : L(\mathbb{R}^n, \mathbb{R}^n) \to L(\mathbb{R}^n, \mathbb{R}^n)$ be given by $f(A) := A^t S A$. Where is f of constant rank? What is $f^{-1}(S)$?

1.25. Describe $TS^2 \subset \mathbb{R}^6$.

2. Submersions and Immersions

2.1. Definition. A mapping $f : M \to N$ between manifolds is called a *submersion* at $x \in M$, if the rank of $T_xf : T_xM \to T_{f(x)}N$ equals $\dim N$. Since the rank cannot fall locally (the determinant of a submatrix of the Jacobi matrix is not 0), f is then a submersion in a whole neighborhood of x. The mapping f is said to be a *submersion*, if it is a submersion at each $x \in M$.

2.2. Lemma. *If $f : M \to N$ is a submersion at $x \in M$, then for any chart (V, v) centered at $f(x)$ on N there is chart (U, u) centered at x on M such that $v \circ f \circ u^{-1}$ looks as follows:*

$$(y^1, \dots, y^n, y^{n+1}, \dots, y^m) \mapsto (y^1, \dots, y^n)$$

Proof. Use the inverse function theorem. □

2.3. Corollary. *Any submersion $f : M \to N$ is open: for each open $U \subset M$ the set $f(U)$ is open in N.* □

2.4. Definition. A triple (M, p, N), where $p : M \to N$ is a surjective submersion, is called a *fibered manifold*. M is called the *total space*, N is called the *base*.

A fibered manifold admits local sections: For each $x \in M$ there is an open neighborhood U of $p(x)$ in N and a smooth mapping $s : U \to M$ with $p \circ s = Id_U$ and $s(p(x)) = x$.

The existence of local sections in turn implies the following universal property:

$$\begin{tikzcd} M \arrow[d, "p"'] \arrow[dr] & \\ N \arrow[r, "f"] & P \end{tikzcd}$$

If (M, p, N) is a fibered manifold and $f : N \to P$ is a mapping into some further manifold, such that $f \circ p : M \to P$ is smooth, then f is smooth.

2.5. Definition. A smooth mapping $f : M \to N$ is called an *immersion at* $x \in M$ if the rank of $T_xf : T_xM \to T_{f(x)}N$ equals $\dim M$. Since the rank is maximal at x and cannot fall locally, f is an immersion on a whole neighborhood of x. f is called an immersion if it is so at every $x \in M$.

2.6. Lemma. *If $f : M \to N$ is an immersion, then for any chart (U,u) centered at $x \in M$ there is a chart (V,v) centered at $f(x)$ on N such that $v \circ f \circ u^{-1}$ has the form:*

$$(y^1, \dots , y^m) \mapsto (y^1, \dots , y^m, 0, \dots , 0)$$

Proof. Use the inverse function theorem. □

2.7. Corollary. *If $f : M \to N$ is an immersion, then for any $x \in M$ there is an open neighborhood U of $x \in M$ such that $f(U)$ is a submanifold of N and $f \restriction U : U \to f(U)$ is a diffeomorphism.* □

2.8. Definition. If $i : M \to N$ is an injective immersion, then (M, i) is called an *immersed submanifold* of N.

A submanifold is an immersed submanifold, but the converse is wrong in general. The structure of an immersed submanifold (M, i) is in general not determined by the subset $i(M) \subset N$. All this is illustrated by the following example. Consider the curve $\gamma(t) = (\sin^3 t, \sin t. \cos t)$ in $\mathbb{R}^2$. Then $((-\pi, \pi), \gamma \restriction (-\pi, \pi))$ and $((0, 2\pi), \gamma \restriction (0, 2\pi))$ are two different immersed submanifolds, but the image of the embedding is in both cases just the figure eight.

2.9. Let M be a submanifold of N. Then the embedding $i : M \to N$ is an injective immersion with the following property:

(1) *For any manifold Z a mapping $f : Z \to M$ is smooth if and only if $i \circ f : Z \to N$ is smooth.*

The example in 2.8 shows that there are injective immersions without property (1).

2.10. We want to determine all injective immersions $i : M \to N$ with property 2.9.1. To require that i is a homeomorphism onto its image is too strong as 2.11 and 2.12 below show. To look for all smooth mappings $i : M \to N$ with property 2.9.1 (initial mappings in categorical terms) is too difficult as remark 2.13 below shows.

2.11. Lemma. *If an injective immersion $i : M \to N$ is a homeomorphism onto its image, then $i(M)$ is a submanifold of N.*

Proof. Use 2.7. □

2.12. Example. We consider the 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then the quotient mapping $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is a covering map, so locally a diffeomorphism. Let us also consider the mapping $f : \mathbb{R} \to \mathbb{R}^2$, $f(t) = (t, \alpha.t)$, where α is irrational. Then $\pi \circ f : \mathbb{R} \to \mathbb{T}^2$ is an injective immersion with dense image, and it is obviously not a homeomorphism onto its image. But $\pi \circ f$ has property 2.9.1, which follows from the fact that π is a covering map.

2.13. Remark. If $f:\mathbb{R}\to\mathbb{R}$ is a function such that f^p and f^q are smooth for some p, q which are relatively prime in $\mathbb{N}$, then f itself turns out to be smooth, see [Joris, 1982]. So the mapping $i: t\mapsto \binom{t^p}{t^q}$, $\mathbb{R}\to\mathbb{R}^2$, has property 2.9.1, but i is not an immersion at 0.

2.14. Definition. For an arbitrary subset A of a manifold N and $x_0\in A$ let $C_{x_0}(A)$ denote the set of all $x\in A$ which can be joined to x_0 by a smooth curve in M lying in A.

A subset M in a manifold N is called *initial submanifold* of dimension m, if the following property is true:

(1) *For each $x\in M$ there exists a chart (U,u) centered at x on N such that $u(C_x(U\cap M)) = u(U)\cap(\mathbb{R}^m\times 0)$.*

The following three lemmas explain the name initial submanifold.

2.15. Lemma. *Let $f: M\to N$ be an injective immersion between manifolds with property 2.9.1. Then $f(M)$ is an initial submanifold of N.*

Proof. Let $x\in M$. By 2.6 we may choose a chart (V,v) centered at $f(x)$ on N and another chart (W,w) centered at x on M such that $(v\circ f\circ w^{-1})(y^1,\dots,y^m) = (y^1,\dots,y^m,0,\dots,0)$. Let $r>0$ be so small that $\{y\in\mathbb{R}^m : |y|<r\}\subset w(W)$ and $\{z\in\mathbb{R}^n : |z|<2r\}\subset v(V)$. Put

$$\begin{aligned} U &:= v^{-1}(\{z\in\mathbb{R}^n : |z|<r\})\subset N,\\ W_1 &:= w^{-1}(\{y\in\mathbb{R}^m : |y|<r\})\subset M. \end{aligned}$$

We claim that $(U, u = v\restriction U)$ satisfies the condition of 2.14.1.

$$\begin{aligned} u^{-1}(u(U)\cap(\mathbb{R}^m\times 0)) &= u^{-1}(\{(y^1,\dots,y^m,0\dots,0) : |y|<r\}) =\\ &= f\circ w^{-1}\circ(u\circ f\circ w^{-1})^{-1}(\{(y^1,\dots,y^m,0\dots,0) : |y|<r\}) =\\ &= f\circ w^{-1}(\{y\in\mathbb{R}^m : |y|<r\}) = f(W_1)\subseteq C_{f(x)}(U\cap f(M)), \end{aligned}$$

since $f(W_1)\subseteq U\cap f(M)$ and $f(W_1)$ is C^∞-contractible.

Now let conversely $z\in C_{f(x)}(U\cap f(M))$. Then by definition there is a smooth curve $c:[0,1]\to N$ with $c(0)=f(x)$, $c(1)=z$, and $c([0,1])\subseteq U\cap f(M)$. By property 2.9.1 the unique curve $\bar c:[0,1]\to M$ with $f\circ\bar c = c$, is smooth.

We claim that $\bar c([0,1])\subseteq W_1$. If not then there is some $t\in[0,1]$ with $\bar c(t)\in w^{-1}(\{y\in\mathbb{R}^m : r\le|y|<2r\})$ since $\bar c$ is smooth and thus continuous. But then we have

$$\begin{aligned} (v\circ f)(\bar c(t)) &\in (v\circ f\circ w^{-1})(\{y\in\mathbb{R}^m : r\le|y|<2r\}) =\\ &= \{(y,0)\in\mathbb{R}^m\times 0 : r\le|y|<2r\}\subseteq\{z\in\mathbb{R}^n : r\le|z|<2r\}. \end{aligned}$$

This means $(v \circ f \circ \bar{c})(t) = (v \circ c)(t) \in \{z \in \mathbb{R}^n : r \le |z| < 2r\}$, so $c(t) \notin U$, a contradiction.

So $\bar{c}([0,1]) \subseteq W_1$, thus $\bar{c}(1) = f^{-1}(z) \in W_1$ and $z \in f(W_1)$. Consequently we have $C_{f(x)}(U \cap f(M)) = f(W_1)$ and finally $f(W_1) = u^{-1}(u(U) \cap (\mathbb{R}^m \times 0))$ by the first part of the proof. □

2.16. Lemma. *Let M be an initial submanifold of a manifold N. Then there is a unique C^∞-manifold structure on M such that the injection $i : M \to N$ is an injective immersion with property 2.9.(1):*

(1) *For any manifold Z a mapping $f : Z \to M$ is smooth if and only if $i \circ f : Z \to N$ is smooth.*

The connected components of M are separable (but there may be uncountably many of them).

Proof. We use the sets $C_x(U_x \cap M)$ as charts for M, where $x \in M$ and (U_x, u_x) is a chart for N centered at x with the property required in 2.14.1. Then the chart changings are smooth since they are just restrictions of the chart changings on N. But the sets $C_x(U_x \cap M)$ are not open in the induced topology on M in general. So the identification topology with respect to the charts $(C_x(U_x \cap M), u_x)_{x \in M}$ yields a topology on M which is finer than the induced topology, so it is Hausdorff. Clearly $i : M \to N$ is then an injective immersion. Uniqueness of the smooth structure follows from the universal property (1) which we prove now: For $z \in Z$ we choose a chart (U, u) on N, centered at $f(z)$, such that $u(C_{f(z)}(U \cap M)) = u(U) \cap (\mathbb{R}^m \times 0)$. Then $f^{-1}(U)$ is open in Z and contains a chart (V, v) centered at z on Z with $v(V)$ a ball. Then $f(V)$ is C^∞-contractible in $U \cap M$, so $f(V) \subseteq C_{f(z)}(U \cap M)$, and $(u \restriction C_{f(z)}(U \cap M)) \circ f \circ v^{-1} = u \circ f \circ v^{-1}$ is smooth.

Finally note that N admits a Riemannian metric (see ??) which can be induced on M, so each connected component of M is separable. □

2.18. Transversal mappings. Let M_1, M_2, and N be manifolds and let $f_i : M_i \to N$ be smooth mappings for $i = 1, 2$. We say that f_1 and f_2 are *transversal* at $y \in N$, if

$$\operatorname{im} T_{x_1} f_1 + \operatorname{im} T_{x_2} f_2 = T_y N \quad \text{whenever} \quad f_1(x_1) = f_2(x_2) = y.$$

Note that they are transversal at any y which is not in $f_1(M_1)$ or not in $f_2(M_2)$. The mappings f_1 and f_2 are simply said to be *transversal*, if they are transversal at every $y \in N$.

If P is an initial submanifold of N with embedding $i : P \to N$, then $f : M \to N$ is said to be transversal to P, if i and f are transversal.

Lemma. *In this case $f^{-1}(P)$ is an initial submanifold of M with the same codimension in M as P has in N, or the empty set. If P is a submanifold, then also $f^{-1}(P)$ is a submanifold.*

Proof. Let $x \in f^{-1}(P)$ and let (U, u) be an initial submanifold chart for P centered at $f(x)$ on N, i.e. $u(C_{f(x)}(U \cap P)) = u(U) \cap (\mathbb{R}^p \times 0)$. Then the mapping

$$M \supseteq f^{-1}(U) \xrightarrow{f} U \xrightarrow{u} u(U) \subseteq \mathbb{R}^p \times \mathbb{R}^{n-p} \xrightarrow{pr_2} \mathbb{R}^{n-p}$$

is a submersion at x since f is transversal to P. So by lemma 2.2 there is a chart (V, v) on M centered at x such that we have

$$(pr_2 \circ u \circ f \circ v^{-1})(y^1, \dots, y^{n-p}, \dots, y^m) = (y^1, \dots, y^{n-p}).$$

But then $z \in C_x(f^{-1}(P) \cap V)$ if and only if $v(z) \in v(V) \cap (0 \times \mathbb{R}^{m-n+p})$, so $v(C_x(f^{-1}(P) \cap V)) = v(V) \cap (0 \times \mathbb{R}^{m-n+p})$. □

2.19. Corollary. *If $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are smooth and transversal, then the topological pullback*

$$M_1 \underset{(f_1,N,f_2)}{\times} M_2 = M_1 \times_N M_2 := \{(x_1, x_2) \in M_1 \times M_2 : f_1(x_1) = f_2(x_2)\}$$

is a submanifold of $M_1 \times M_2$, and it has the following universal property.

For any smooth mappings $g_1 : P \to M_1$ and $g_2 : P \to M_2$ with $f_1 \circ g_1 = f_2 \circ g_2$ there is a unique smooth mapping $(g_1, g_2) : P \to M_1 \times_N M_2$ with $pr_1 \circ (g_1, g_2) = g_1$ and $pr_2 \circ (g_1, g_2) = g_2$.

$$\begin{tikzcd}
P \arrow[drr, bend left, "g_2"] \arrow[dr, "(g_1,g_2)"] \arrow[ddr, bend right, "g_1"'] & & \\
& M_1 \times_N M_2 \arrow[r, "pr_2"'] \arrow[d, "pr_1"] & M_2 \arrow[d, "f_2"] \\
& M_1 \arrow[r, "f_1"'] & N.
\end{tikzcd}$$

This is also called the pullback property in the category $\mathcal{M}f$ of smooth manifolds and smooth mappings. So one may say, that transversal pullbacks exist in the category $\mathcal{M}f$. But there also exist pullbacks which are not transversal.

Proof. $M_1 \times_N M_2 = (f_1 \times f_2)^{-1}(\Delta)$, where $f_1 \times f_2 : M_1 \times M_2 \to N \times N$ and where Δ is the diagonal of $N \times N$, and $f_1 \times f_2$ is transversal to Δ if and only if f_1 and f_2 are transversal. □

3. Vector Fields and Flows

3.1. Definition. A *vector field* X on a manifold M is a smooth section of the tangent bundle; so $X : M \to TM$ is smooth and $\pi_M \circ X = Id_M$. A *local vector field* is a smooth section, which is defined on an open subset only. We denote the set of all vector fields by $\mathfrak{X}(M)$. With point wise addition and scalar multiplication $\mathfrak{X}(M)$ becomes a vector space.

Example. Let (U, u) be a chart on M. Then the $\frac{\partial}{\partial u^i} : U \to TM \restriction U$, $x \mapsto \frac{\partial}{\partial u^i}|_x$, described in 1.8, are local vector fields defined on U.

Lemma. *If X is a vector field on M and (U, u) is a chart on M and $x \in U$, then we have $X(x) = \sum_{i=0}^m X(x)(u^i) \frac{\partial}{\partial u^i}|_x$. We write $X \restriction U = \sum_{i=1}^m X(u^i) \frac{\partial}{\partial u^i}$.* $\square$

3.2. The vector fields $(\frac{\partial}{\partial u^i})_{i=1}^m$ on U, where (U, u) is a chart on M, form a *holonomic frame field*. By a *frame field* on some open set $V \subset M$ we mean $m = \dim M$ vector fields $s_i \in \mathfrak{X}(U)$ such that $s_1(x), \dots, s_m(x)$ is a linear basis of $T_x M$ for each $x \in V$. A frame field is said to be *holonomic*, if $s_i = \frac{\partial}{\partial v^i}$ for some chart (V, v). If no such chart may be found locally, the frame field is called *anholonomic*.

With the help of partitions of unity and holonomic frame fields one may construct 'many' vector fields on M. In particular the values of a vector field can be arbitrarily preassigned on a discrete set $\{x_i\} \subset M$.

3.3. Lemma. *The space $\mathfrak{X}(M)$ of vector fields on M coincides canonically with the space of all derivations of the algebra $C^\infty(M, \mathbb{R})$ of smooth functions, i.e. those $\mathbb{R}$-linear operators $D : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ with $D(fg) = D(f)g + fD(g)$.*

Proof. Clearly each vector field $X \in \mathfrak{X}(M)$ defines a derivation (again called X, later sometimes called $\mathcal{L}_X$) of the algebra $C^\infty(M, \mathbb{R})$ by the prescription $X(f)(x) := X(x)(f) = df(X(x))$.

If conversely a derivation D of $C^\infty(M, \mathbb{R})$ is given, for any $x \in M$ we consider $D_x : C^\infty(M, \mathbb{R}) \to \mathbb{R}$, $D_x(f) = D(f)(x)$. Then D_x is a derivation at x of $C^\infty(M, \mathbb{R})$ in the sense of 1.7, so $D_x = X_x$ for some $X_x \in T_x M$. In this way we get a section $X : M \to TM$. If (U, u) is a chart on M, we have $D_x = \sum_{i=1}^m X(x)(u^i) \frac{\partial}{\partial u^i}|_x$ by 1.7. Choose V open in M, $V \subset \overline{V} \subset U$, and $\varphi \in C^\infty(M, \mathbb{R})$ such that $\operatorname{supp}(\varphi) \subset U$ and $\varphi \restriction V = 1$. Then $\varphi \cdot u^i \in C^\infty(M, \mathbb{R})$ and $(\varphi u^i) \restriction V = u^i \restriction V$. So $D(\varphi u^i)(x) = X(x)(\varphi u^i) = X(x)(u^i)$ and $X \restriction V = \sum_{i=1}^m D(\varphi u^i) \restriction V \cdot \frac{\partial}{\partial u^i} \restriction V$ is smooth. $\square$

3.4. The Lie bracket. By lemma 3.3 we can identify $\mathfrak{X}(M)$ with the vector space of all derivations of the algebra $C^\infty(M,\mathbb{R})$, which we will do without any notational change in the following.

If X, Y are two vector fields on M, then the mapping $f \mapsto X(Y(f)) - Y(X(f))$ is again a derivation of $C^\infty(M,\mathbb{R})$, as a simple computation shows. Thus there is a unique vector field $[X,Y] \in \mathfrak{X}(M)$ such that $[X,Y](f) = X(Y(f)) - Y(X(f))$ holds for all $f \in C^\infty(M,\mathbb{R})$.

In a local chart (U,u) on M one immediately verifies that for $X \restriction U = \sum X^i \frac{\partial}{\partial u^i}$ and $Y \restriction U = \sum Y^i \frac{\partial}{\partial u^i}$ we have

$$\Big[\sum_i X^i \tfrac{\partial}{\partial u^i}, \sum_j Y^j \tfrac{\partial}{\partial u^j}\Big] = \sum_{i,j} \big(X^i(\tfrac{\partial}{\partial u^i} Y^j) - Y^i(\tfrac{\partial}{\partial u^i} X^j)\big)\, \tfrac{\partial}{\partial u^j},$$

since second partial derivatives commute. The $\mathbb{R}$-bilinear mapping

$$[\quad,\quad] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is called the *Lie bracket*. Note also that $\mathfrak{X}(M)$ is a module over the algebra $C^\infty(M,\mathbb{R})$ by point wise multiplication $(f,X) \mapsto fX$.

Theorem. *The Lie bracket* $[\quad,\quad] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *has the following properties:*

$$\begin{aligned}
&[X,Y] = -[Y,X],\\
&[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]], \quad \textit{the Jacobi identity,}\\
&[fX,Y] = f[X,Y] - (Yf)X,\\
&[X,fY] = f[X,Y] + (Xf)Y.
\end{aligned}$$

The form of the Jacobi identity we have chosen says that $ad(X) = [X,\quad]$ is a derivation for the Lie algebra $(\mathfrak{X}(M),[\quad,\quad])$.

The pair $(\mathfrak{X}(M),[\quad,\quad])$ is the prototype of a *Lie algebra*. The concept of a Lie algebra is one of the most important notions of modern mathematics.

Proof. All these properties are checked easily for the commutator $[X,Y] = X \circ Y - Y \circ X$ in the space of derivations of the algebra $C^\infty(M,\mathbb{R})$. $\square$

3.5. Integral curves. Let $c : J \to M$ be a smooth curve in a manifold M defined on an interval J. We will use the following notations: $c'(t) = \dot c(t) = \frac{d}{dt}c(t) := T_t c.1$. Clearly $c' : J \to TM$ is smooth. We call c' a vector field along c since we have $\pi_M \circ c' = c$.

$$\begin{tikzcd}
 & TM \arrow[d, "\pi_M"] \\
J \arrow[ur, "\dot c"] \arrow[r, "c"'] & M
\end{tikzcd}$$

A smooth curve $c : J \to M$ will be called an *integral curve* or *flow line* of a vector field $X \in \mathfrak{X}(M)$ if $c'(t) = X(c(t))$ holds for all $t \in J$.

3.6. Lemma. *Let X be a vector field on M. Then for any $x \in M$ there is an open interval J_x containing 0 and an integral curve $c_x : J_x \to M$ for X (i.e. $c_x' = X \circ c_x$) with $c_x(0) = x$. If J_x is maximal, then c_x is unique.*

Proof. In a chart (U, u) on M with $x \in U$ the equation $c'(t) = X(c(t))$ is an ordinary differential equation with initial condition $c(0) = x$. Since X is smooth there is a unique local solution by the theorem of Picard-Lindelöf, which even depends smoothly on the initial values, [Dieudonné I, 1969, 10.7.4]. So on M there are always local integral curves. If $J_x = (a, b)$ and $\lim_{t \to b-} c_x(t) =: c_x(b)$ exists in M, there is a unique local solution c_1 defined in an open interval containing b with $c_1(b) = c_x(b)$. By uniqueness of the solution on the intersection of the two intervals, c_1 prolongs c_x to a larger interval. This may be repeated (also on the left hand side of J_x) as long as the limit exists. So if we suppose J_x to be maximal, J_x either equals $\mathbb{R}$ or the integral curve leaves the manifold in finite (parameter-) time in the past or future or both. $\square$

3.7. The flow of a vector field. Let $X \in \mathfrak{X}(M)$ be a vector field. Let us write $\mathrm{Fl}^X_t(x) = \mathrm{Fl}^X(t, x) := c_x(t)$, where $c_x : J_x \to M$ is the maximally defined integral curve of X with $c_x(0) = x$, constructed in lemma 3.6.

Theorem. *For each vector field X on M, the mapping $\mathrm{Fl}^X : \mathcal{D}(X) \to M$ is smooth, where $\mathcal{D}(X) = \bigcup_{x \in M} J_x \times \{x\}$ is an open neighborhood of $0 \times M$ in $\mathbb{R} \times M$. We have*

$$\mathrm{Fl}^X(t + s, x) = \mathrm{Fl}^X(t, \mathrm{Fl}^X(s, x))$$

in the following sense. If the right hand side exists, then the left hand side exists and we have equality. If both t, $s \geq 0$ or both are ≤ 0, and if the left hand side exists, then also the right hand side exists and we have equality.

Proof. As mentioned in the proof of 3.6, $\mathrm{Fl}^X(t, x)$ is smooth in (t, x) for small t, and if it is defined for (t, x), then it is also defined for (s, y) nearby. These are local properties which follow from the theory of ordinary differential equations.

Now let us treat the equation $\mathrm{Fl}^X(t + s, x) = \mathrm{Fl}^X(t, \mathrm{Fl}^X(s, x))$. If the right hand side exists, then we consider the equation

$$\begin{cases} \frac{d}{dt} \mathrm{Fl}^X(t + s, x) = \frac{d}{du} \mathrm{Fl}^X(u, x)|_{u = t+s} = X(\mathrm{Fl}^X(t + s, x)), \\ \mathrm{Fl}^X(t + s, x)|_{t=0} = \mathrm{Fl}^X(s, x). \end{cases}$$

But the unique solution of this is $\mathrm{Fl}^X(t, \mathrm{Fl}^X(s, x))$. So the left hand side exists and equals the right hand side.

If the left hand side exists, let us suppose that $t, s \geq 0$. We put

$$c_x(u) = \begin{cases} \operatorname{Fl}^X(u,x) & \text{if } u \leq s \\ \operatorname{Fl}^X(u-s, \operatorname{Fl}^X(s,x)) & \text{if } u \geq s. \end{cases}$$

$$\begin{aligned} \tfrac{d}{du} c_x(u) &= \left.\begin{cases} \tfrac{d}{du} \operatorname{Fl}^X(u,x) = X(\operatorname{Fl}^X(u,x)) & \text{for } u \leq s \\ \tfrac{d}{du} \operatorname{Fl}^X(u-s, \operatorname{Fl}^X(s,x)) = X(\operatorname{Fl}^X(u-s, \operatorname{Fl}^X(s,x))) \end{cases}\right\} = \\ &= X(c_x(u)) \quad \text{for } 0 \leq u \leq t+s. \end{aligned}$$

Also $c_x(0) = x$ and on the overlap both definitions coincide by the first part of the proof, thus we conclude that $c_x(u) = \operatorname{Fl}^X(u,x)$ for $0 \leq u \leq t+s$ and we have $\operatorname{Fl}^X(t, \operatorname{Fl}^X(s,x)) = c_x(t+s) = \operatorname{Fl}^X(t+s,x)$.

Now we show that $\mathcal{D}(X)$ is open and Fl^X is smooth on $\mathcal{D}(X)$. We know already that $\mathcal{D}(X)$ is a neighborhood of $0 \times M$ in $\mathbb{R} \times M$ and that Fl^X is smooth near $0 \times M$.

For $x \in M$ let J'_x be the set of all $t \in \mathbb{R}$ such that Fl^X is defined and smooth on an open neighborhood of $[0,t] \times \{x\}$ (respectively on $[t,0] \times \{x\}$ for $t < 0$) in $\mathbb{R} \times M$. We claim that $J'_x = J_x$, which finishes the proof. It suffices to show that J'_x is not empty, open and closed in J_x. It is open by construction, and not empty, since $0 \in J'_x$. If J'_x is not closed in J_x, let $t_0 \in J_x \cap (\overline{J'_x} \setminus J'_x)$ and suppose that $t_0 > 0$, say. By the local existence and smoothness Fl^X exists and is smooth near $[-\varepsilon, \varepsilon] \times \{y := \operatorname{Fl}^X(t_0, x)\}$ for some $\varepsilon > 0$, and by construction Fl^X exists and is smooth near $[0, t_0 - \varepsilon] \times \{x\}$. Since $\operatorname{Fl}^X(-\varepsilon, y) = \operatorname{Fl}^X(t_0 - \varepsilon, x)$ we conclude for t near $[0, t_0 - \varepsilon]$, x' near x, and t' near $[-\varepsilon, \varepsilon]$, that $\operatorname{Fl}^X(t + t', x') = \operatorname{Fl}^X(t', \operatorname{Fl}^X(t, x'))$ exists and is smooth. So $t_0 \in J'_x$, a contradiction. $\square$

3.8. Let $X \in \mathfrak{X}(M)$ be a vector field. Its flow Fl^X is called *global* or *complete*, if its domain of definition $\mathcal{D}(X)$ equals $\mathbb{R} \times M$. Then the vector field X itself will be called a *"complete vector field"*. In this case Fl^X_t is also sometimes called $\exp tX$; it is a diffeomorphism of M.

The *support* $\operatorname{supp}(X)$ of a vector field X is the closure of the set $\{x \in M : X(x) \neq 0\}$.

Lemma. *A vector field with compact support on M is complete.*

Proof. Let $K = \operatorname{supp}(X)$ be compact. Then the compact set $0 \times K$ has positive distance to the disjoint closed set $(\mathbb{R} \times M) \setminus \mathcal{D}(X)$ (if it is not empty), so $[-\varepsilon, \varepsilon] \times K \subset \mathcal{D}(X)$ for some $\varepsilon > 0$. If $x \notin K$ then $X(x) = 0$, so $\operatorname{Fl}^X(t,x) = x$ for all t and $\mathbb{R} \times \{x\} \subset \mathcal{D}(X)$. So we have $[-\varepsilon, \varepsilon] \times M \subset \mathcal{D}(X)$. Since $\operatorname{Fl}^X(t + \varepsilon, x) = \operatorname{Fl}^X(t, \operatorname{Fl}^X(\varepsilon, x))$ exists for $|t| \leq \varepsilon$ by theorem 3.7, we have $[-2\varepsilon, 2\varepsilon] \times M \subset \mathcal{D}(X)$ and by repeating this argument we get $\mathbb{R} \times M = \mathcal{D}(X)$. $\square$

So on a compact manifold M each vector field is complete. If M is not compact and of dimension ≥ 2, then in general the set of complete vector fields on M is neither a vector space nor is it closed under the Lie bracket, as the following example on $\mathbb{R}^2$ shows: $X = y\frac{\partial}{\partial x}$ and $Y = \frac{x^2}{2}\frac{\partial}{\partial y}$ are complete, but neither $X+Y$ nor $[X,Y]$ is complete. In general one may embed $\mathbb{R}^2$ as a closed submanifold into M and extend the vector fields X and Y.

3.9. f-related vector fields. If $f : M \to M$ is a diffeomorphism, then for any vector field $X \in \mathfrak{X}(M)$ the mapping $Tf^{-1} \circ X \circ f$ is also a vector field, which we will denote f^*X. Analogously we put $f_*X := Tf \circ X \circ f^{-1} = (f^{-1})^*X$.

But if $f : M \to N$ is a smooth mapping and $Y \in \mathfrak{X}(N)$ is a vector field there may or may not exist a vector field $X \in \mathfrak{X}(M)$ such that the following diagram commutes:

$$
\begin{tikzcd}
TM \arrow[r, "Tf"] & TN \\
M \arrow[u, "X"] \arrow[r, "f"] & N. \arrow[u, "Y"']
\end{tikzcd}
\tag{1}
$$

Definition. Let $f : M \to N$ be a smooth mapping. Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called *f-related*, if $Tf \circ X = Y \circ f$ holds, i.e. if diagram (1) commutes.

Example. If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ and $X \times Y \in \mathfrak{X}(M \times N)$ is given $(X \times Y)(x,y) = (X(x), Y(y))$, then we have:

(2) $X \times Y$ and X are pr_1-related.
(3) $X \times Y$ and Y are pr_2-related.
(4) X and $X \times Y$ are $ins(y)$-related if and only if $Y(y) = 0$, where the mapping $ins(y) : M \to M \times N$ is given by $ins(y)(x) = (x,y)$.

3.10. Lemma. *Consider vector fields $X_i \in \mathfrak{X}(M)$ and $Y_i \in \mathfrak{X}(N)$ for $i = 1,2$, and a smooth mapping $f : M \to N$. If X_i and Y_i are f-related for $i = 1,2$, then also $\lambda_1 X_1 + \lambda_2 X_2$ and $\lambda_1 Y_1 + \lambda_2 Y_2$ are f-related, and also $[X_1, X_2]$ and $[Y_1, Y_2]$ are f-related.*

Proof. The first assertion is immediate. To prove the second we choose $h \in C^\infty(N, \mathbb{R})$. Then by assumption we have $Tf \circ X_i = Y_i \circ f$, thus:

$$
\begin{aligned}
(X_i(h \circ f))(x) &= X_i(x)(h \circ f) = (T_x f . X_i(x))(h) = \\
&= (Tf \circ X_i)(x)(h) = (Y_i \circ f)(x)(h) = Y_i(f(x))(h) = (Y_i(h))(f(x)),
\end{aligned}
$$

so $X_i(h\circ f)=(Y_i(h))\circ f$, and we may continue:

$$\begin{aligned}[X_1,X_2](h\circ f) &= X_1(X_2(h\circ f))-X_2(X_1(h\circ f)) = \\ &= X_1(Y_2(h)\circ f)-X_2(Y_1(h)\circ f) = \\ &= Y_1(Y_2(h))\circ f - Y_2(Y_1(h))\circ f = [Y_1,Y_2](h)\circ f.\end{aligned}$$

But this means $Tf\circ[X_1,X_2]=[Y_1,Y_2]\circ f$. $\square$

3.11. Corollary. *If $f:M\to N$ is a local diffeomorphism (so $(T_xf)^{-1}$ makes sense for each $x\in M$), then for $Y\in\mathfrak{X}(N)$ a vector field $f^*Y\in\mathfrak{X}(M)$ is defined by $(f^*Y)(x)=(T_xf)^{-1}.Y(f(x))$. The linear mapping $f^*:\mathfrak{X}(N)\to\mathfrak{X}(M)$ is then a Lie algebra homomorphism, i.e. $f^*[Y_1,Y_2]=[f^*Y_1,f^*Y_2]$.*

3.12. The Lie derivative of functions. For a vector field $X\in\mathfrak{X}(M)$ and $f\in C^\infty(M,\mathbb{R})$ we define $\mathcal{L}_Xf\in C^\infty(M,\mathbb{R})$ by

$$\begin{aligned}\mathcal{L}_Xf(x) &:= \tfrac{d}{dt}|_0 f(\mathrm{Fl}^X(t,x)) \quad \text{or}\\ \mathcal{L}_Xf &:= \tfrac{d}{dt}|_0(\mathrm{Fl}^X_t)^*f=\tfrac{d}{dt}|_0(f\circ\mathrm{Fl}^X_t).\end{aligned}$$

Since $\mathrm{Fl}^X(t,x)$ is defined for small t, for any $x\in M$, the expressions above make sense.

Lemma. $\frac{d}{dt}(\mathrm{Fl}^X_t)^*f=(\mathrm{Fl}^X_t)^*X(f)=X((\mathrm{Fl}^X_t)^*f)$, *in particular for $t=0$ we have $\mathcal{L}_Xf=X(f)=df(X)$.* $\square$

Proof. We have

$$\tfrac{d}{dt}(\mathrm{Fl}^X_t)^*f(x)=df(\tfrac{d}{dt}\mathrm{Fl}^X(t,x))=df(X(\mathrm{Fl}^X(t,x)))=(\mathrm{Fl}^X_t)^*(Xf)(x).$$

From this we get $\mathcal{L}_Xf=X(f)=df(X)$ and then in turn

$$\tfrac{d}{dt}(\mathrm{Fl}^X_t)^*f=\tfrac{d}{ds}|_0(\mathrm{Fl}^X_t\circ\mathrm{Fl}^X_s)^*f=\tfrac{d}{ds}|_0(\mathrm{Fl}^X_s)^*(\mathrm{Fl}^X_t)^*f=X((\mathrm{Fl}^X_t)^*f).\quad\square$$

3.13. The Lie derivative for vector fields. For $X,Y\in\mathfrak{X}(M)$ we define $\mathcal{L}_XY\in\mathfrak{X}(M)$ by

$$\mathcal{L}_XY:=\tfrac{d}{dt}|_0(\mathrm{Fl}^X_t)^*Y=\tfrac{d}{dt}|_0(T(\mathrm{Fl}^X_{-t})\circ Y\circ\mathrm{Fl}^X_t),$$

and call it the *Lie derivative* of Y along X.

Lemma. *$\mathcal{L}_X Y = [X,Y]$ and $\frac{d}{dt}(\mathrm{Fl}^X_t)^* Y = (\mathrm{Fl}^X_t)^*\mathcal{L}_X Y = (\mathrm{Fl}^X_t)^*[X,Y] = \mathcal{L}_X(\mathrm{Fl}^X_t)^*Y = [X,(\mathrm{Fl}^X_t)^*Y]$.*

Proof. Let $f \in C^\infty(M,\mathbb{R})$ be a testing function and consider the mapping $\alpha(t,s) := Y(\mathrm{Fl}^X(t,x))(f\circ \mathrm{Fl}^X_s)$, which is locally defined near 0. It satisfies

$$\begin{aligned}
\alpha(t,0) &= Y(\mathrm{Fl}^X(t,x))(f),\\
\alpha(0,s) &= Y(x)(f\circ \mathrm{Fl}^X_s),\\
\tfrac{\partial}{\partial t}\alpha(0,0) &= \tfrac{\partial}{\partial t}\big|_0 Y(\mathrm{Fl}^X(t,x))(f) = \tfrac{\partial}{\partial t}\big|_0 (Yf)(\mathrm{Fl}^X(t,x)) = X(x)(Yf),\\
\tfrac{\partial}{\partial s}\alpha(0,0) &= \tfrac{\partial}{\partial s}|_0 Y(x)(f\circ \mathrm{Fl}^X_s) = Y(x)\tfrac{\partial}{\partial s}|_0 (f\circ \mathrm{Fl}^X_s) = Y(x)(Xf).
\end{aligned}$$

But on the other hand we have

$$\begin{aligned}
\tfrac{\partial}{\partial u}|_0 \alpha(u,-u) &= \tfrac{\partial}{\partial u}|_0 Y(\mathrm{Fl}^X(u,x))(f\circ \mathrm{Fl}^X_{-u})\\
&= \tfrac{\partial}{\partial u}|_0 \Big(T(\mathrm{Fl}^X_{-u})\circ Y\circ \mathrm{Fl}^X_u\Big)_x (f) = (\mathcal{L}_X Y)_x(f),
\end{aligned}$$

so the first assertion follows. For the second claim we compute as follows:

$$\begin{aligned}
\tfrac{\partial}{\partial t}(\mathrm{Fl}^X_t)^*Y &= \tfrac{\partial}{\partial s}|_0 \Big(T(\mathrm{Fl}^X_{-t})\circ T(\mathrm{Fl}^X_{-s})\circ Y\circ \mathrm{Fl}^X_s \circ \mathrm{Fl}^X_t\Big)\\
&= T(\mathrm{Fl}^X_{-t})\circ \tfrac{\partial}{\partial s}|_0 \Big(T(\mathrm{Fl}^X_{-s})\circ Y\circ \mathrm{Fl}^X_s\Big)\circ \mathrm{Fl}^X_t\\
&= T(\mathrm{Fl}^X_{-t})\circ [X,Y]\circ \mathrm{Fl}^X_t = (\mathrm{Fl}^X_t)^*[X,Y].\\
\tfrac{\partial}{\partial t}(\mathrm{Fl}^X_t)^*Y &= \tfrac{\partial}{\partial s}|_0 (\mathrm{Fl}^X_s)^*(\mathrm{Fl}^X_t)^*Y = \mathcal{L}_X(\mathrm{Fl}^X_t)^*Y. \quad \square
\end{aligned}$$

3.14. Lemma. *Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be f-related vector fields for a smooth mapping $f : M \to N$. Then we have $f\circ \mathrm{Fl}^X_t = \mathrm{Fl}^Y_t \circ f$, whenever both sides are defined. In particular, if f is a diffeomorphism, we have $\mathrm{Fl}^{f^*Y}_t = f^{-1}\circ \mathrm{Fl}^Y_t \circ f$.*

Proof. We have $\frac{d}{dt}(f\circ \mathrm{Fl}^X_t) = Tf\circ \frac{d}{dt}\mathrm{Fl}^X_t = Tf\circ X\circ \mathrm{Fl}^X_t = Y\circ f\circ Fl^X_t$ and $f(\mathrm{Fl}^X(0,x)) = f(x)$. So $t\mapsto f(\mathrm{Fl}^X(t,x))$ is an integral curve of the vector field Y on N with initial value $f(x)$, so we have $f(\mathrm{Fl}^X(t,x)) = \mathrm{Fl}^Y(t,f(x))$ or $f\circ \mathrm{Fl}^X_t = \mathrm{Fl}^Y_t\circ f$. $\square$

3.15. Corollary. *Let $X, Y \in \mathfrak{X}(M)$. Then the following assertions are equivalent*

(1) $\mathcal{L}_X Y = [X,Y] = 0$.
(2) $(\mathrm{Fl}^X_t)^*Y = Y$, *wherever defined.*
(3) $\mathrm{Fl}^X_t \circ \mathrm{Fl}^Y_s = \mathrm{Fl}^Y_s\circ \mathrm{Fl}^X_t$, *wherever defined.*

Proof. (1) $\Leftrightarrow$ (2) is immediate from lemma 3.13. To see (2) $\Leftrightarrow$ (3) we note that $\mathrm{Fl}^X_t \circ \mathrm{Fl}^Y_s = \mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t$ if and only if $\mathrm{Fl}^Y_s = \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t = \mathrm{Fl}^{(\mathrm{Fl}^X_t)^* Y}_s$ by lemma 3.14; and this in turn is equivalent to $Y = (\mathrm{Fl}^X_t)^* Y$. □

3.16. Theorem. *Let M be a manifold, let $\varphi^i : \mathbb{R} \times M \supset U_{\varphi^i} \to M$ be smooth mappings for $i = 1, \dots, k$ where each U_{φ^i} is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that each φ^i_t is a diffeomorphism on its domain, $\varphi^i_0 = Id_M$, and $\frac{\partial}{\partial t}\big|_0 \varphi^i_t = X_i \in \mathfrak{X}(M)$. We put $[\varphi^i, \varphi^j]_t = [\varphi^i_t, \varphi^j_t] := (\varphi^j_t)^{-1} \circ (\varphi^i_t)^{-1} \circ \varphi^j_t \circ \varphi^i_t$. Then for each formal bracket expression P of lenght k we have*

$$\begin{aligned} 0 &= \tfrac{\partial^\ell}{\partial t^\ell}|_0 P(\varphi^1_t, \dots, \varphi^k_t) \qquad \text{for } 1 \le \ell < k, \\ P(X_1, \dots, X_k) &= \tfrac{1}{k!} \tfrac{\partial^k}{\partial t^k}|_0 P(\varphi^1_t, \dots, \varphi^k_t) \in \mathfrak{X}(M) \end{aligned}$$

in the sense explained in step 2 of the proof. In particular we have for vector fields $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} 0 &= \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^Y_{-t} \circ \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_t), \\ [X, Y] &= \tfrac{1}{2} \tfrac{\partial^2}{\partial t^2}|_0 (\mathrm{Fl}^Y_{-t} \circ \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_t). \end{aligned}$$

Proof. **Step 1.** Let $c : \mathbb{R} \to M$ be a smooth curve. If $c(0) = x \in M$, $c'(0) = 0, \dots, c^{(k-1)}(0) = 0$, then $c^{(k)}(0)$ is a well defined tangent vector in $T_x M$ which is given by the derivation $f \mapsto (f \circ c)^{(k)}(0)$ at x.

For we have

$$\begin{aligned} ((f.g) \circ c)^{(k)}(0) &= ((f \circ c).(g \circ c))^{(k)}(0) = \sum_{j=0}^{k} \tbinom{k}{j} (f \circ c)^{(j)}(0) (g \circ c)^{(k-j)}(0) \\ &= (f \circ c)^{(k)}(0) g(x) + f(x) (g \circ c)^{(k)}(0), \end{aligned}$$

since all other summands vanish: $(f \circ c)^{(j)}(0) = 0$ for $1 \le j < k$.

Step 2. Let $\varphi : \mathbb{R} \times M \supset U_\varphi \to M$ be a smooth mapping where U_φ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that each φ_t is a diffeomorphism on its domain and $\varphi_0 = Id_M$. We say that φ_t is a *curve of local diffeomorphisms* though Id_M.

From step 1 we see that if $\frac{\partial^j}{\partial t^j}|_0 \varphi_t = 0$ for all $1 \le j < k$, then $X := \frac{1}{k!} \frac{\partial^k}{\partial t^k}|_0 \varphi_t$ is a well defined vector field on M. We say that X is the *first non-vanishing derivative* at 0 of the curve φ_t of local diffeomorphisms. We may paraphrase this as $(\partial^k_t|_0 \varphi^*_t) f = k! \mathcal{L}_X f$.

Claim 3. Let φ_t, ψ_t be curves of local diffeomorphisms through Id_M and let $f \in C^\infty(M,\mathbb{R})$. Then we have

$$\partial_t^k|_0(\varphi_t \circ \psi_t)^* f = \partial_t^k|_0(\psi_t^* \circ \varphi_t^*) f = \sum_{j=0}^k \binom{k}{j} (\partial_t^j|_0 \psi_t^*)(\partial_t^{k-j}|_0 \varphi_t^*) f.$$

Also the multinomial version of this formula holds:

$$\partial_t^k|_0(\varphi_t^1 \circ \ldots \circ \varphi_t^\ell)^* f = \sum_{j_1+\dots+j_\ell=k} \frac{k!}{j_1! \ldots j_\ell!} (\partial_t^{j_\ell}|_0 (\varphi_t^\ell)^*) \ldots (\partial_t^{j_1}|_0 (\varphi_t^1)^*) f.$$

We only show the binomial version. For a function $h(t,s)$ of two variables we have

$$\partial_t^k h(t,t) = \sum_{j=0}^k \binom{k}{j} \partial_t^j \partial_s^{k-j} h(t,s)|_{s=t},$$

since for $h(t,s) = f(t)g(s)$ this is just a consequence of the Leibnitz rule, and linear combinations of such decomposable tensors are dense in the space of all functions of two variables in the compact C^∞-topology, so that by continuity the formula holds for all functions. In the following form it implies the claim:

$$\partial_t^k|_0 f(\varphi(t,\psi(t,x))) = \sum_{j=0}^k \binom{k}{j} \partial_t^j \partial_s^{k-j} f(\varphi(t,\psi(s,x)))|_{t=s=0}.$$

Claim 4. Let φ_t be a curve of local diffeomorphisms through Id_M with first non-vanishing derivative $k!X = \partial_t^k|_0 \varphi_t$. Then the inverse curve of local diffeomorphisms φ_t^{-1} has first non-vanishing derivative $-k!X = \partial_t^k|_0 \varphi_t^{-1}$.

For we have $\varphi_t^{-1} \circ \varphi_t = Id$, so by claim 3 we get for $1 \le j \le k$

$$\begin{aligned} 0 = \partial_t^j|_0(\varphi_t^{-1} \circ \varphi_t)^* f &= \sum_{i=0}^j \binom{j}{i} (\partial_t^i|_0 \varphi_t^*)(\partial_t^{j-i} (\varphi_t^{-1})^*) f = \\ &= \partial_t^j|_0 \varphi_t^* (\varphi_0^{-1})^* f + \varphi_0^* \partial_t^j|_0 (\varphi_t^{-1})^* f, \end{aligned}$$

i.e. $\partial_t^j|_0 \varphi_t^* f = -\partial_t^j|_0 (\varphi_t^{-1})^* f$ as required.

Claim 5. Let φ_t be a curve of local diffeomorphisms through Id_M with first non-vanishing derivative $m!X = \partial_t^m|_0 \varphi_t$, and let ψ_t be a curve of local diffeomorphisms through Id_M with first non-vanishing derivative $n!Y = \partial_t^n|_0 \psi_t$.

Then the curve of local diffeomorphisms $[\varphi_t, \psi_t] = \psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t \circ \varphi_t$ has first non-vanishing derivative

$$(m+n)![X,Y] = \partial_t^{m+n}|_0[\varphi_t, \psi_t].$$

From this claim the theorem follows.

By the multinomial version of claim 3 we have

$$\begin{aligned}A_N f :&= \partial_t^N|_0(\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t \circ \varphi_t)^* f\\ &= \sum_{i+j+k+\ell=N} \frac{N!}{i!j!k!\ell!}(\partial_t^i|_0\varphi_t^*)(\partial_t^j|_0\psi_t^*)(\partial_t^k|_0(\varphi_t^{-1})^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)f.\end{aligned}$$

Let us suppose that $1 \le n \le m$, the case $m \le n$ is similar. If $N < n$ all summands are 0. If $N = n$ we have by claim 4

$$A_N f = (\partial_t^n|_0\varphi_t^*)f + (\partial_t^n|_0\psi_t^*)f + (\partial_t^n|_0(\varphi_t^{-1})^*)f + (\partial_t^n|_0(\psi_t^{-1})^*)f = 0.$$

If $n < N \le m$ we have, using again claim 4:

$$\begin{aligned}A_N f &= \sum_{j+\ell=N} \frac{N!}{j!\ell!}(\partial_t^j|_0\psi_t^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)f + \delta_N^m\left((\partial_t^m|_0\varphi_t^*)f + (\partial_t^m|_0(\varphi_t^{-1})^*)f\right)\\ &= (\partial_t^N|_0(\psi_t^{-1}\circ\psi_t)^*)f + 0 = 0.\end{aligned}$$

Now we come to the difficult case $m, n < N \le m+n$.

$$\begin{aligned}A_N f &= \partial_t^N|_0(\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t)^* f + \tbinom{N}{m}(\partial_t^m|_0\varphi_t^*)(\partial_t^{N-m}|_0(\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t)^*)f\\ &\quad + (\partial_t^N|_0\varphi_t^*)f, \end{aligned}\tag{1}$$

by claim 3, since all other terms vanish, see (3) below. By claim 3 again we get:

$$\begin{aligned}\partial_t^N|_0(\psi_t^{-1} \circ \varphi_t^{-1} \circ \psi_t)^* f &= \sum_{j+k+\ell=N} \frac{N!}{j!k!\ell!}(\partial_t^j|_0\psi_t^*)(\partial_t^k|_0(\varphi_t^{-1})^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)f\\ &= \sum_{j+\ell=N} \tbinom{N}{j}(\partial_t^j|_0\psi_t^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)f + \tbinom{N}{m}(\partial_t^{N-m}|_0\psi_t^*)(\partial_t^m|_0(\varphi_t^{-1})^*)f \qquad (2)\\ &\quad + \tbinom{N}{m}(\partial_t^m|_0(\varphi_t^{-1})^*)(\partial_t^{N-m}|_0(\psi_t^{-1})^*)f + \partial_t^N|_0(\varphi_t^{-1})^* f\\ &= 0 + \tbinom{N}{m}(\partial_t^{N-m}|_0\psi_t^*)m!\mathcal{L}_{-X}f + \tbinom{N}{m}m!\mathcal{L}_{-X}(\partial_t^{N-m}|_0(\psi_t^{-1})^*)f\\ &\quad + \partial_t^N|_0(\varphi_t^{-1})^* f\\ &= \delta_{m+n}^N(m+n)!(\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)f + \partial_t^N|_0(\varphi_t^{-1})^* f\\ &= \delta_{m+n}^N(m+n)!\mathcal{L}_{[X,Y]}f + \partial_t^N|_0(\varphi_t^{-1})^* f\end{aligned}$$

From the second expression in (2) one can also read off that

$$\partial_t^{N-m}|_0(\psi_t^{-1}\circ\varphi_t^{-1}\circ\psi_t)^*f = \partial_t^{N-m}|_0(\varphi_t^{-1})^*f. \tag{3}$$

If we put (2) and (3) into (1) we get, using claims 3 and 4 again, the final result which proves claim 3 and the theorem:

$$\begin{aligned}
A_N f &= \delta^N_{m+n}(m+n)!\mathcal{L}_{[X,Y]}f + \partial_t^N|_0(\varphi_t^{-1})^*f \\
&\quad + \tbinom{N}{m}(\partial_t^m|_0\varphi_t^*)(\partial_t^{N-m}|_0(\varphi_t^{-1})^*)f + (\partial_t^N|_0\varphi_t^*)f \\
&= \delta^N_{m+n}(m+n)!\mathcal{L}_{[X,Y]}f + \partial_t^N|_0(\varphi_t^{-1}\circ\varphi_t)^*f \\
&= \delta^N_{m+n}(m+n)!\mathcal{L}_{[X,Y]}f + 0. \quad \square
\end{aligned}$$

3.17. Theorem. *Let $X_1,\dots,X_m$ be vector fields on M defined in a neighborhood of a point $x\in M$ such that $X_1(x),\dots,X_m(x)$ are a basis for T_xM and $[X_i,X_j]=0$ for all i,j.*

Then there is a chart (U,u) of M centered at x such that $X_i\restriction U = \frac{\partial}{\partial u^i}$.

Proof. For small $t=(t^1,\dots,t^m)\in\mathbb{R}^m$ we put

$$f(t^1,\dots,t^m) = (\mathrm{Fl}^{X_1}_{t^1}\circ\dots\circ\mathrm{Fl}^{X_m}_{t^m})(x).$$

By 3.15 we may interchange the order of the flows arbitrarily. Therefore

$$\tfrac{\partial}{\partial t^i}f(t^1,\dots,t^m) = \tfrac{\partial}{\partial t^i}(\mathrm{Fl}^{X_i}_{t^i}\circ\mathrm{Fl}^{X_1}_{t^1}\circ\cdots)(x) = X_i((\mathrm{Fl}^{x_1}_{t^1}\circ\cdots)(x)).$$

So T_0f is invertible, f is a local diffeomorphism, and its inverse gives a chart with the desired properties. $\square$

3.27. The theorem of Frobenius. The next three subsections will be devoted to the theorem of Frobenius for distributions of constant rank. We will give a powerfull generalization for distributions of nonconstant rank below (3.18 — 3.25).

Let M be a manifold. By a *vector subbundle* E of TM of fiber dimension k we mean a subset $E\subset TM$ such that each $E_x := E\cap T_xM$ is a linear subspace of dimension k, and such that for each $x \operatorname{im} M$ there are k vector fields defined on an open neighborhood of M with values in E and spanning E, called a *local frame* for E. Such an E is also called a smooth *distribution* of constant rank k. See section 6 for a thorough discussion of the notion of vector bundles. The space of all vector fields with values in E will be called $C^\infty(E)$.

The vector subbundle E of TM is called *integrable* or *involutive*, if for all $X,Y\in C^\infty(E)$ we have $[X,Y]\in C^\infty(E)$.

Local version of Frobenius' theorem. *Let $E \subset TM$ be an integrable vector subbundle of fiber dimension k of TM.*

Then for each $x \in M$ there exists a chart (U, u) of M centered at x with $u(U) = V \times W \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$, such that $T(u^{-1}(V \times \{y\})) = E|(u^{-1}(V \times \{y\}))$ for each $y \in W$.

Proof. Let $x \in M$. We choose a chart (U, u) of M centered at x such that there exist k vector fields $X_1, \dots, X_k \in C^\infty(E)$ which form a frame of $E|U$. Then we have $X_i = \sum_{j=1}^m f_i^j \frac{\partial}{\partial u^j}$ for $f_i^j \in C^\infty(U, \mathbb{R})$. Then $f = (f_i^j)$ is a $(k \times m)$-matrix valued smooth function on U which has rank k on U. So some $(k \times k)$-submatrix, say the top one, is invertible at x and thus we may take U so small that this top $(k \times k)$-submatrix is invertible everywhere on U. Let $g = (g_i^j)$ be the inverse of this submatrix, so that $f.g = \binom{\mathrm{Id}}{*}$. We put

$$(1) \qquad Y_i := \sum_{j=1}^k g_i^j X_j = \sum_{j=1}^k \sum_{l=1}^m g_i^j f_j^l \frac{\partial}{\partial u^l} = \frac{\partial}{\partial u^i} + \sum_{p \ge k+1} h_i^p \frac{\partial}{\partial u^p}.$$

We claim that $[Y_i, Y_j] = 0$ for all $1 \le i, j \le k$. Since E is integrable we have $[Y_i, Y_j] = \sum_{l=1}^k c_{ij}^l Y_l$. But from (1) we conclude (using the coordinate formula in 3.4) that $[Y_i, Y_j] = \sum_{p \ge k+1} a^p \frac{\partial}{\partial u^p}$. Again by (1) this implies that $c_{ij}^l = 0$ for all l, and the claim follows.

Now we consider an $(m-k)$-dimensional linear subspace W_1 in $\mathbb{R}^m$ which is transversal to the k vectors $T_x u.Y_i(x) \in T_0\mathbb{R}^m$ spanning $\mathbb{R}^k$, and we define $f : V \times W \to U$ by

$$f(t^1, \dots, t^k, y) := \left(\mathrm{Fl}^{Y_1}_{t^1} \circ \mathrm{Fl}^{Y_2}_{t^2} \circ \dots \circ \mathrm{Fl}^{Y_k}_{t^k}\right)(u^{-1}(y)),$$

where $t = (t^1, \dots, t^k) \in V$, a small neighborhood of 0 in $\mathbb{R}^k$, and where $y \in W$, a small neighborhood of 0 in W_1. By 3.16 we may interchange the order of the flows in the definition of f arbitrarily. Thus

$$\frac{\partial}{\partial t^i} f(t, y) = \frac{\partial}{\partial t^i}\left(\mathrm{Fl}^{Y_i}_{t^i} \circ \mathrm{Fl}^{Y_1}_{t^1} \circ \dots\right)(u^{-1}(y)) = Y_i(f(t, y)),$$

$T_0 f$ is invertible and the inverse of f on a suitable neighborhood of x gives us the required chart. $\square$

3.28. Remark. Charts $(U, u : U \to V \times W \subset \mathbb{R}^k \times \mathbb{R}^{m-k})$ as constructed in theorem 3.27 with V and W open balls are called *distinguished charts* for E. The submanifolds $u^{-1}(V \times \{y\})$ are called *plaques.* Two plaques of different distinguished charts intersect in open subsets in both plaques or not at all: this follows immediately by flowing a point in the intersection into both plaques with the same construction as in in the proof of 3.27. Thus an atlas of distinguished charts on M has chart change mappings which respect the submersion $\mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^{m-k}$ (the plaque structure on M). Such an atlas (or the equivalence class of such atlases) is called the *foliation corresponding to the integrable vector subbundle* $E \subset TM$.

3.29. Global Version of Frobenius' theorem. *Let $E \subsetneq TM$ be an integrable vector subbundle of TM. Then, using the restrictions of distinguished charts to plaques as charts we get a new structure of a smooth manifold on M, which we denote by M_E. If $E \neq TM$ the topology of M_E is finer than that of M, M_E has uncountably many connected components called the leaves of the foliation, and the identity induces a bijective immersion $M_E \to M$. Each leaf L is a second countable initial submanifold of M, and it is a maximal integrable submanifold of M for E in the sense that $T_xL = E_x$ for each $x \in L$.*

Proof. Let $(U_\alpha, u_\alpha : U_\alpha \to V_\alpha \times W_\alpha \subseteq \mathbb{R}^k \times \mathbb{R}^{m-k})$ be an atlas of distuished charts corresponding to the integrable vector subbundle $E \subset TM$, as given by theorem 3.27. Let us now use for each plaque the homeomorphisms $\mathrm{pr}_1 \circ u_\alpha|(u_\alpha^{-1}(V_\alpha \times \{y\})) : u_\alpha^{-1}(V_\alpha \times \{y\}) \to V_\alpha \subset \mathbb{R}^{m-k}$ as charts, then we describe on M a new smooth manifold structure M_E with finer topology which however has uncountably many connected components, and the identity on M induces a bijective immersion $M_E \to M$. The connected components of M_E are called the *leaves of the foliation*.

In order to check the rest of the assertions made in the theorem let us construct the unique leaf L through an arbitrary point $x \in M$: choose a plaque containing x and take the union with any plaque meeting the first one, and keep going. Now choose $y \in L$ and a curve $c : [0,1] \to L$ with $c(0) = x$ and $c(1) = y$. Then there are finitely many distinguished charts $(U_1, u_1), \dots, (U_n, u_n)$ and $a_1, \dots, a_n \in \mathbb{R}^{m-k}$ such that $x \in u_1^{-1}(V_1 \times \{a_1\})$, $y \in u_n^{-1}(V_n \times \{a_n\})$ and such that for each i

$$(*) \qquad u_i^{-1}(V_i \times \{a_i\}) \cap u_{i+1}^{-1}(V_{i+1} \times \{a_{i+1}\}) \neq \emptyset.$$

Given u_i, u_{i+1} and a_i there are only countably many points a_{i+1} such that (*) holds: if not then we get a cover of the the separable submanifold $u_i^{-1}(V_i \times \{a_i\}) \cap U_{i+1}$ by uncountably many pairwise disjoint open sets of the form given in (*), which contradicts separability.

Finally, since (each component of) M is a Lindelöf space, any distinguished atlas contains a countable subatlas. So each leaf is the union of at most countably many plaques. The rest is clear. □

3.18. Distributions. Let M be a manifold. Suppose that for each $x \in M$ we are given a sub vector space E_x of T_xM. The disjoint union $E = \bigsqcup_{x\in M} E_x$ is called a *distribution* on M. We do not suppose, that the dimension of E_x is locally constant in x.

Let $\mathfrak{X}_{loc}(M)$ denote the set of all locally defined smooth vector fields on M, i.e. $\mathfrak{X}_{loc}(M) = \bigcup \mathfrak{X}(U)$, where U runs through all open sets in M. Furthermore let $\mathfrak{X}_E$ denote the set of all local vector fields $X \in \mathfrak{X}_{loc}(M)$ with $X(x) \in E_x$ whenever defined. We say that a subset $\mathcal{V} \subset \mathfrak{X}_E$ *spans* E, if for each $x \in M$ the vector space E_x is the linear hull of the set $\{X(x) : X \in \mathcal{V}\}$. We say that E is a *smooth distribution* if $\mathfrak{X}_E$ spans E. Note that every subset $\mathcal{W} \subset \mathfrak{X}_{loc}(M)$ spans a distribution denoted by $E(\mathcal{W})$, which is obviously smooth (the linear span of the empty set is the vector space 0). From now on we will consider only smooth distributions.

An *integral manifold* of a smooth distribution E is a connected immersed submanifold (N, i) (see 2.8) such that $T_xi(T_xN) = E_{i(x)}$ for all $x \in N$. We will see in theorem 3.22 below that any integral manifold is in fact an initial submanifold of M (see 2.14), so that we need not specify the injective immersion i. An integral manifold of E is called *maximal*, if it is not contained in any strictly larger integral manifold of E.

3.19. Lemma. *Let E be a smooth distribution on M. Then we have:*

1. If (N, i) is an integral manifold of E and $X \in \mathfrak{X}_E$, then i^*X makes sense and is an element of $\mathfrak{X}_{loc}(N)$, which is $i \restriction i^{-1}(U_X)$-related to X, where $U_X \subset M$ is the open domain of X.

2. If (N_j, i_j) are integral manifolds of E for $j = 1, 2$, then $i_1^{-1}(i_1(N_1)\cap i_2(N_2))$ and $i_2^{-1}(i_1(N_1)\cap i_2(N_2))$ are open subsets in N_1 and N_2, respectively; furthermore $i_2^{-1} \circ i_1$ is a diffeomorphism between them.

3. If $x \in M$ is contained in some integral submanifold of E, then it is contained in a unique maximal one.

Proof. 1. Let U_X be the open domain of $X \in \mathfrak{X}_E$. If $i(x) \in U_X$ for $x \in N$, we have $X(i(x)) \in E_{i(x)} = T_xi(T_xN)$, so $i^*X(x) := ((T_xi)^{-1} \circ X \circ i)(x)$ makes sense. It is clearly defined on an open subset of N and is smooth in x.

2. Let $X \in \mathfrak{X}_E$. Then $i_j^*X \in \mathfrak{X}_{loc}(N_j)$ and is i_j-related to X. So by lemma 3.14 for $j = 1, 2$ we have

$$i_j \circ \mathrm{Fl}_t^{i_j^*X} = Fl_t^X \circ i_j.$$

Now choose $x_j \in N_j$ such that $i_1(x_1) = i_2(x_2) = x_0 \in M$ and choose vector fields $X_1, \dots , X_n \in \mathfrak{X}_E$ such that $(X_1(x_0), \dots, X_n(x_0))$ is a basis of E_{x_0}. Then

$$f_j(t^1, \dots , t^n) := (\mathrm{Fl}_{t^1}^{i_j^*X_1} \circ \dots \circ \mathrm{Fl}_{t^n}^{i_j^*X_n})(x_j)$$

is a smooth mapping defined near zero $\mathbb{R}^n \to N_j$. Since obviously $\frac{\partial}{\partial t^k}|_0 f_j = i_j^* X_k(x_j)$ for $j = 1, 2$, we see that f_j is a diffeomorphism near 0. Finally we have

$$\begin{aligned}(i_2^{-1} \circ i_1 \circ f_1)(t^1, \dots, t^n) &= (i_2^{-1} \circ i_1 \circ \mathrm{Fl}_{t^1}^{i_1^* X_1} \circ \dots \circ \mathrm{Fl}_{t^n}^{i_1^* X_n})(x_1) \\ &= (i_2^{-1} \circ \mathrm{Fl}_{t^1}^{X_1} \circ \dots \circ \mathrm{Fl}_{t^n}^{X_n} \circ i_1)(x_1) \\ &= (\mathrm{Fl}_{t^1}^{i_2^* X_1} \circ \dots \circ \mathrm{Fl}_{t^n}^{i_2^* X_n} \circ i_2^{-1} \circ i_1)(x_1) \\ &= f_2(t^1, \dots, t^n).\end{aligned}$$

So $i_2^{-1} \circ i_1$ is a diffeomorphism, as required.

3. Let N be the union of all integral manifolds containing x. Choose the union of all the atlases of these integral manifolds as atlas for N, which is a smooth atlas for N by 2. Note that a connected immersed submanifold of a separable manifold is automatically separable (since it carries a Riemannian metric). □

3.20. Integrable distributions and foliations.

A smooth distribution E on a manifold M is called *integrable*, if each point of M is contained in some integral manifold of E. By 3.19.3 each point is then contained in a unique maximal integral manifold, so the maximal integral manifolds form a partition of M. This partition is called the *foliation* of M induced by the integrable distribution E, and each maximal integral manifold is called a *leaf* of this foliation. If $X \in \mathfrak{X}_E$ then by 3.19.1 the integral curve $t \mapsto \mathrm{Fl}^X(t, x)$ of X through $x \in M$ stays in the leaf through x.

Note, however, that usually a foliation is supposed to have constant dimensions of the leafs, so our notion here is sometimes called a *singular foliation*.

Let us now consider an arbitrary subset $\mathcal{V} \subset \mathfrak{X}_{loc}(M)$. We say that $\mathcal{V}$ is *stable* if for all $X, Y \in \mathcal{V}$ and for all t for which it is defined the local vector field $(\mathrm{Fl}_t^X)^* Y$ is again an element of $\mathcal{V}$.

If $\mathcal{W} \subset \mathfrak{X}_{loc}(M)$ is an arbitrary subset, we call $\mathcal{S}(\mathcal{W})$ the set of all local vector fields of the form $(\mathrm{Fl}_{t_1}^{X_1} \circ \dots \circ \mathrm{Fl}_{t_k}^{X_k})^* Y$ for $X_i, Y \in \mathcal{W}$. By lemma 3.14 the flow of this vector field is

$$\mathrm{Fl}((\mathrm{Fl}_{t_1}^{X_1} \circ \dots \circ \mathrm{Fl}_{t_k}^{X_k})^* Y, t) = \mathrm{Fl}_{-t_k}^{X_k} \circ \dots \circ \mathrm{Fl}_{-t_1}^{X_1} \circ \mathrm{Fl}_t^Y \circ \mathrm{Fl}_{t_1}^{X_1} \circ \dots \circ \mathrm{Fl}_{t_k}^{X_k},$$

so $\mathcal{S}(\mathcal{W})$ is the minimal stable set of local vector fields which contains $\mathcal{W}$.

Now let F be an arbitrary distribution. A local vector field $X \in \mathfrak{X}_{loc}(M)$ is called an *infinitesimal automorphism* of F, if $T_x(\mathrm{Fl}_t^X)(F_x) \subset F_{\mathrm{Fl}^X(t,x)}$ whenever defined. We denote by $aut(F)$ the set of all infinitesimal automorphisms of F. By arguments given just above, $aut(F)$ is stable.

3.21. Lemma. *Let E be a smooth distribution on a manifold M. Then the following conditions are equivalent:*

(1) *E is integrable.*
(2) *$\mathfrak{X}_E$ is stable.*
(3) *There exists a subset $\mathcal{W} \subset \mathfrak{X}_{loc}(M)$ such that $\mathcal{S}(\mathcal{W})$ spans E.*
(4) *$aut(E) \cap \mathfrak{X}_E$ spans E.*

Proof. (1) $\implies$ (2). Let $X \in \mathfrak{X}_E$ and let L be the leaf through $x \in M$, with $i : L \to M$ the inclusion. Then $\mathrm{Fl}^X_{-t} \circ i = i \circ \mathrm{Fl}^{i^*X}_{-t}$ by lemma 3.14, so we have

$$\begin{aligned} T_x(\mathrm{Fl}^X_{-t})(E_x) &= T(\mathrm{Fl}^X_{-t}).T_x i.T_x L = T(\mathrm{Fl}^X_{-t} \circ i).T_x L \\ &= Ti.T_x(\mathrm{Fl}^{i^*X}_{-t}).T_x L \\ &= Ti.T_{Fl^{i^*X}(-t,x)}L = E_{Fl^X(-t,x)}. \end{aligned}$$

This implies that $(\mathrm{Fl}^X_t)^* Y \in \mathfrak{X}_E$ for any $Y \in \mathfrak{X}_E$.

(2) $\implies$ (4). In fact (2) says that $\mathfrak{X}_E \subset aut(E)$.

(4) $\implies$ (3). We can choose $\mathcal{W} = aut(E) \cap \mathfrak{X}_E$: for $X, Y \in \mathcal{W}$ we have $(\mathrm{Fl}^X_t)^* Y \in \mathfrak{X}_E$; so $\mathcal{W} \subset \mathcal{S}(\mathcal{W}) \subset \mathfrak{X}_E$ and E is spanned by $\mathcal{W}$.

(3) $\implies$ (1). We have to show that each point $x \in M$ is contained in some integral submanifold for the distribution E. Since $\mathcal{S}(\mathcal{W})$ spans E and is stable we have

$$T(\mathrm{Fl}^X_t).E_x = E_{\mathrm{Fl}^X(t,x)} \tag{5}$$

for each $X \in \mathcal{S}(\mathcal{W})$. Let $\dim E_x = n$. There are $X_1, \dots, X_n \in \mathcal{S}(\mathcal{W})$ such that $X_1(x), \dots, X_n(x)$ is a basis of E_x, since E is smooth. As in the proof of 3.19.2 we consider the mapping

$$f(t^1, \dots, t^n) := (\mathrm{Fl}^{X_1}_{t^1} \circ \dots \circ \mathrm{Fl}^{X_n}_{t^n})(x),$$

defined and smooth near 0 in $\mathbb{R}^n$. Since the rank of f at 0 is n, the image under f of a small open neighborhood of 0 is a submanifold N of M. We claim that N is an integral manifold of E. The tangent space $T_{f(t^1,\dots,t^n)}N$ is linearly generated by

$$\begin{aligned} \tfrac{\partial}{\partial t^k}(\mathrm{Fl}^{X_1}_{t^1} \circ \dots \circ \mathrm{Fl}^{X_n}_{t^n})(x) &= T(\mathrm{Fl}^{X_1}_{t^1} \circ \dots \circ \mathrm{Fl}^{X_{k-1}}_{t^{k-1}})X_k((\mathrm{Fl}^{X_k}_{t^k} \circ \dots \circ \mathrm{Fl}^{X_n}_{t^n})(x)) \\ &= ((\mathrm{Fl}^{X_1}_{-t^1})^* \cdots (\mathrm{Fl}^{X_{k-1}}_{-t^{k-1}})^* X_k)(f(t^1, \dots, t^n)). \end{aligned}$$

Since $\mathcal{S}(\mathcal{W})$ is stable, these vectors lie in $E_{f(t)}$. From the form of f and from (5) we see that $\dim E_{f(t)} = \dim E_x$, so these vectors even span $E_{f(t)}$ and we have $T_{f(t)}N = E_{f(t)}$ as required. $\square$

3.22. Theorem (local structure of foliations). *Let E be an integrable distribution of a manifold M. Then for each $x \in M$ there exists a chart (U,u) with $u(U) = \{y \in \mathbb{R}^m : |y^i| < \varepsilon \text{ for all } i\}$ for some $\varepsilon > 0$, and a countable subset $A \subset \mathbb{R}^{m-n}$, such that for the leaf L through x we have*

$$u(U \cap L) = \{y \in u(U) : (y^{n+1}, \dots, y^m) \in A\}.$$

Each leaf is an initial submanifold.

If furthermore the distribution E has locally constant rank, this property holds for each leaf meeting U with the same n.

This chart (U,u) is called a *distinguished chart* for the distribution or the foliation. A connected component of $U \cap L$ is called a *plaque*.

Proof. Let L be the leaf through x, $\dim L = n$. Let $X_1, \dots, X_n \in \mathfrak{X}_E$ be local vector fields such that $X_1(x), \dots, X_n(x)$ is a basis of E_x. We choose a chart (V,v) centered at x on M such that the vectors

$$X_1(x), \dots, X_n(x), \tfrac{\partial}{\partial v^{n+1}}|_x, \dots, \tfrac{\partial}{\partial v^m}|_x$$

form a basis of T_xM. Then

$$f(t^1, \dots, t^m) = (\mathrm{Fl}^{X_1}_{t^1} \circ \dots \circ \mathrm{Fl}^{X_n}_{t^n})(v^{-1}(0, \dots, 0, t^{n+1}, \dots, t^m))$$

is a diffeomorphism from a neighborhood of 0 in $\mathbb{R}^m$ onto a neighborhood of x in M. Let (U,u) be the chart given by f^{-1}, suitably restricted. We have

$$y \in L \Longleftrightarrow (\mathrm{Fl}^{X_1}_{t^1} \circ \dots \circ \mathrm{Fl}^{X_n}_{t^n})(y) \in L$$

for all y and all $t^1, \dots, t^n$ for which both expressions make sense. So we have

$$f(t^1, \dots, t^m) \in L \Longleftrightarrow f(0, \dots, 0, t^{n+1}, \dots, t^m) \in L,$$

and consequently $L \cap U$ is the disjoint union of connected sets of the form $\{y \in U : (u^{n+1}(y), \dots, u^m(y)) = \text{ constant}\}$. Since L is a connected immersive submanifold of M, it is second countable and only a countable set of constants can appear in the description of $u(L \cap U)$ given above. From this description it is clear that L is an initial submanifold (2.14) since $u(C_x(L \cap U)) = u(U) \cap (\mathbb{R}^n \times 0)$.

The argument given above is valid for any leaf of dimension n meeting U, so also the assertion for an integrable distribution of constant rank follows. $\square$

3.23. Involutive distributions. A subset $\mathcal{V} \subset \mathfrak{X}_{loc}(M)$ is called *involutive* if $[X, Y] \in \mathcal{V}$ for all $X, Y \in \mathcal{V}$. Here $[X, Y]$ is defined on the intersection of the domains of X and Y.

A smooth distribution E on M is called *involutive* if there exists an involutive subset $\mathcal{V} \subset \mathfrak{X}_{loc}(M)$ spanning E.

For an arbitrary subset $\mathcal{W} \subset \mathfrak{X}_{loc}(M)$ let $\mathcal{L}(\mathcal{W})$ be the set consisting of all local vector fields on M which can be written as finite expressions using Lie brackets and starting from elements of $\mathcal{W}$. Clearly $\mathcal{L}(\mathcal{W})$ is the smallest involutive subset of $\mathfrak{X}_{loc}(M)$ which contains $\mathcal{W}$.

3.24. Lemma. *For each subset $\mathcal{W} \subset \mathfrak{X}_{loc}(M)$ we have*

$$E(\mathcal{W}) \subset E(\mathcal{L}(\mathcal{W})) \subset E(\mathcal{S}(\mathcal{W})).$$

In particular we have $E(\mathcal{S}(\mathcal{W})) = E(\mathcal{L}(\mathcal{S}(\mathcal{W})))$.

Proof. We will show that for $X, Y \in \mathcal{W}$ we have $[X, Y] \in \mathfrak{X}_{E(\mathcal{S}(\mathcal{W}))}$, for then by induction we get $\mathcal{L}(\mathcal{W}) \subset \mathfrak{X}_{E(\mathcal{S}(\mathcal{W}))}$ and $E(\mathcal{L}(\mathcal{W})) \subset E(\mathcal{S}(\mathcal{W}))$.

Let $x \in M$; since by 3.21 $E(\mathcal{S}(\mathcal{W}))$ is integrable, we can choose the leaf L through x, with the inclusion i. Then i^*X is i-related to X, i^*Y is i-related to Y, thus by 3.10 the local vector field $[i^*X, i^*Y] \in \mathfrak{X}_{loc}(L)$ is i-related to $[X, Y]$, and $[X, Y](x) \in E(\mathcal{S}(\mathcal{W}))_x$, as required. $\square$

3.25. Theorem. *Let $\mathcal{V} \subset \mathfrak{X}_{loc}(M)$ be an involutive subset. Then the distribution $E(\mathcal{V})$ spanned by $\mathcal{V}$ is integrable under each of the following conditions.*

(1) *M is real analytic and $\mathcal{V}$ consists of real analytic vector fields.*
(2) *The dimension of $E(\mathcal{V})$ is constant along all flow lines of vector fields in $\mathcal{V}$.*

Proof. (1). For $X, Y \in \mathcal{V}$ we have $\frac{d}{dt}(\mathrm{Fl}^X_t)^*Y = (\mathrm{Fl}^X_t)^*\mathcal{L}_X Y$, consequently $\frac{d^k}{dt^k}(\mathrm{Fl}^X_t)^*Y = (\mathrm{Fl}^X_t)^*(\mathcal{L}_X)^k Y$, and since everything is real analytic we get for $x \in M$ and small t

$$(\mathrm{Fl}^X_t)^*Y(x) = \sum_{k\geq 0} \frac{t^k}{k!}\frac{d^k}{dt^k}|_0 (\mathrm{Fl}^X_t)^*Y(x) = \sum_{k\geq 0}\frac{t^k}{k!}(\mathcal{L}_X)^k Y(x).$$

Since $\mathcal{V}$ is involutive, all $(\mathcal{L}_X)^k Y \in \mathcal{V}$. Therefore we get $(\mathrm{Fl}^X_t)^*Y(x) \in E(\mathcal{V})_x$ for small t. By the flow property of Fl^X the set of all t satisfying $(\mathrm{Fl}^X_t)^*Y(x) \in E(\mathcal{V})_x$ is open and closed, so it follows that 3.21.2 is satisfied and thus $E(\mathcal{V})$ is integrable.

(2). We choose $X_1,\dots,X_n\in\mathcal{V}$ such that $X_1(x),\dots,X_n(x)$ is a basis of $E(\mathcal{V})_x$. For $X\in\mathcal{V}$, by hypothesis, $E(\mathcal{V})_{\mathrm{Fl}^X(t,x)}$ has also dimension n and admits $X_1(\mathrm{Fl}^X(t,x)),\dots,X_n(\mathrm{Fl}^X(t,x))$ as basis for small t. So there are smooth functions $f_{ij}(t)$ such that

$$[X,X_i](\mathrm{Fl}^X(t,x))=\sum_{j=1}^n f_{ij}(t)X_j(\mathrm{Fl}^X(t,x)).$$

$$\begin{aligned}\tfrac{d}{dt}T(\mathrm{Fl}^X_{-t}).X_i(\mathrm{Fl}^X(t,x))&=T(\mathrm{Fl}^X_{-t}).[X,X_i](\mathrm{Fl}^X(t,x))=\\&=\sum_{j=1}^n f_{ij}(t)T(\mathrm{Fl}^X_{-t}).X_j(\mathrm{Fl}^X(t,x)).\end{aligned}$$

So the T_xM-valued functions $g_i(t)=T(\mathrm{Fl}^X_{-t}).X_i(\mathrm{Fl}^X(t,x))$ satisfy the linear ordinary differential equation $\frac{d}{dt}g_i(t)=\sum_{j=1}^n f_{ij}(t)g_j(t)$ and have initial values in the linear subspace $E(\mathcal{V})_x$, so they have values in it for all small t. Therefore $T(\mathrm{Fl}^X_{-t})E(\mathcal{V})_{\mathrm{Fl}^X(t,x)}\subset E(\mathcal{V})_x$ for small t. Using compact time intervals and the flow property one sees that condition 3.21.2 is satisfied and $E(\mathcal{V})$ is integrable. □

Example. The distribution spanned by $\mathcal{W}\subset\mathfrak{X}_{loc}(\mathbb{R}^2)$ is involutive, but not integrable, where $\mathcal{W}$ consists of all global vector fields with support in $\mathbb{R}^2\setminus\{0\}$ and the field $\frac{\partial}{\partial x^1}$; the leaf through 0 should have dimension 1 at 0 and dimension 2 elsewhere.

3.26. By a *time dependent vector field* on a manifold M we mean a smooth mapping $X:J\times M\to TM$ with $\pi_M\circ X=pr_2$, where J is an open interval. An integral curve of X is a smooth curve $c:I\to M$ with $\dot c(t)=X(t,c(t))$ for all $t\in I$, where I is a subinterval of J.

There is an associated vector field $\bar X\in X(J\times M)$, given by $\bar X(t,x)=(\frac{\partial}{\partial t},X(t,x))\in T_t\mathbb{R}\times T_xM$.

By the *evolution operator* of X we mean the mapping $\Phi^X:J\times J\times M\to M$, defined in a maximal open neighborhood of the diagonal $\times M$ and satisfying the differential equation

$$\begin{cases}\frac{d}{dt}\Phi^X(t,s,x)=X(t,\Phi^X(t,s,x))\\ \Phi^X(s,s,x)=x.\end{cases}$$

It is easily seen that $(t,\Phi^X(t,s,x))=\mathrm{Fl}^{\bar X}(t-s,(s,x))$, so the maximally defined evolution operator exists and is unique, and it satisfies

$$\Phi^X_{t,s}=\Phi^X_{t,r}\circ\Phi^X_{r,s}$$

whenever one side makes sense (with the restrictions of 3.7), where $\Phi^X_{t,s}(x) = \Phi(t,s,x)$.

Examples and Exercises

3.27. Compute the flow of the vector field $\xi_0(x,y) := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ in $\mathbb{R}^2$. Draw the flow lines. Is this a global flow?

3.28. Compute the flow of the vector field $\xi_1(x,y) := y\frac{\partial}{\partial x}$ in $\mathbb{R}^2$. Is it a global flow?

Answer the same questions for $\xi_2(x,y) := \frac{x^2}{2}\frac{\partial}{\partial y}$.

Now compute $[\xi_1,\xi_2]$ and investigate its flow. This time it is not global! In fact, $Fl_t^{[\xi_1,\xi_2]}(x,y) = \left(\frac{2x}{2+xt}, ye^{\int_0^t 2x/(2+xz)dz}\right)$, $xt+y>0$. Compute the integral.

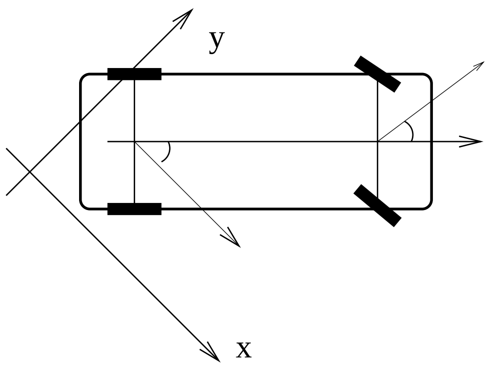
Investigate the flow of $\xi_1+\xi_2$. It is not global either!

3.29. Driving a car. The phase space consists of all $(x,y,\theta,\varphi) \in \mathbb{R}^2 \times S^1 \times (-\pi/4,\pi/4)$, where

$(x,y)\dots$ position of the midpoint of the rear axle,

$\theta\dots$ direction of the car axle,

$\varphi\dots$ steering angle of the front wheels.



There are two 'control' vector fields:

$$\begin{aligned}\text{steer} &= \frac{\partial}{\partial\varphi}\\ \text{drive} &= \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} + \tan(\varphi)\frac{1}{l}\frac{\partial}{\partial\theta} \text{ (why?)}\end{aligned}$$

Compute $[\text{steer},\text{drive}] =:$ park (why?) and $[\text{drive},\text{park}]$, and interpret the results. Is it not convenient that the two control vector fields do not span an integrable distribution?

3.30. Describe the Lie algebra of all vectorfields on S^1 in terms of Fourier expansion. This is nearly (up to a central extension) the Virasoro algebra of theoretical physics.

4. Lie Groups I

4.1. Definition. A *Lie group* G is a smooth manifold and a group such that the multiplication $\mu : G \times G \to G$ is smooth. We shall see in a moment, that then also the inversion $\nu : G \to G$ turns out to be smooth.

We shall use the following notation:
$\mu : G \times G \to G$, multiplication, $\mu(x, y) = x.y$.
$\mu_a : G \to G$, left translation, $\mu_a(x) = a.x$.
$\mu^a : G \to G$, right translation, $\mu^a(x) = x.a$.
$\nu : G \to G$, inversion, $\nu(x) = x^{-1}$.
$e \in G$, the unit element.
Then we have $\mu_a \circ \mu_b = \mu_{a.b}$, $\mu^a \circ \mu^b = \mu^{b.a}$, $\mu_a^{-1} = \mu_{a^{-1}}$, $(\mu^a)^{-1} = \mu^{a^{-1}}$, $\mu^a \circ \mu_b = \mu_b \circ \mu^a$. If $\varphi : G \to H$ is a smooth homomorphism between Lie groups, then we also have $\varphi \circ \mu_a = \mu_{\varphi(a)} \circ \varphi$, $\varphi \circ \mu^a = \mu^{\varphi(a)} \circ \varphi$, thus also $T\varphi.T\mu_a = T\mu_{\varphi(a)}.T\varphi$, etc. So $T_e\varphi$ is injective (surjective) if and only if $T_a\varphi$ is injective (surjective) for all $a \in G$.

4.2. Lemma. *$T_{(a,b)}\mu : T_aG \times T_bG \to T_{ab}G$ is given by*

$$T_{(a,b)}\mu.(X_a, Y_b) = T_a(\mu^b).X_a + T_b(\mu_a).Y_b.$$

Proof. Let $ri_a : G \to G \times G$, $ri_a(x) = (a, x)$ be the right insertion and let $li_b : G \to G \times G$, $li_b(x) = (x, b)$ be the left insertion. Then we have

$$\begin{aligned} T_{(a,b)}\mu.(X_a, Y_b) &= T_{(a,b)}\mu.(T_a(li_b).X_a + T_b(ri_a).Y_b) = \\ &= T_a(\mu \circ li_b).X_a + T_b(\mu \circ ri_a).Y_b = T_a(\mu^b).X_a + T_b(\mu_a).Y_b. \quad \square \end{aligned}$$

4.3. Corollary. *The inversion $\nu : G \to G$ is smooth and*

$$T_a\nu = -T_e(\mu^{a^{-1}}).T_a(\mu_{a^{-1}}) = -T_e(\mu_{a^{-1}}).T_a(\mu^{a^{-1}}).$$

Proof. The equation $\mu(x, \nu(x)) = e$ determines ν implicitly. Since we have $T_e(\mu(e,\quad)) = T_e(\mu_e) = Id$, the mapping ν is smooth in a neighborhood of e by the implicit function theorem. From $(\nu \circ \mu_a)(x) = x^{-1}.a^{-1} = (\mu^{a^{-1}} \circ \nu)(x)$ we may conclude that ν is everywhere smooth. Now we differentiate the equation $\mu(a, \nu(a)) = e$; this gives in turn

$$0_e = T_{(a,a^{-1})}\mu.(X_a, T_a\nu.X_a) = T_a(\mu^{a^{-1}}).X_a + T_{a^{-1}}(\mu_a).T_a\nu.X_a,$$
$$T_a\nu.X_a = -T_e(\mu_a)^{-1}.T_a(\mu^{a^{-1}}).X_a = -T_e(\mu_{a^{-1}}).T_a(\mu^{a^{-1}}).X_a. \quad \square$$

4.4. Example. The *general linear group* $GL(n,\mathbb{R})$ is the group of all invertible real $n\times n$-matrices. It is an open subset of $L(\mathbb{R}^n,\mathbb{R}^n)$, given by $\det\neq 0$ and a Lie group.

Similarly $GL(n,\mathbb{C})$, the group of invertible complex $n\times n$-matrices, is a Lie group; also $GL(n,\mathbb{H})$, the group of all invertible quaternionic $n\times n$-matrices, is a Lie group, since it is open in the real Banach algebra $L_{\mathbb{H}}(\mathbb{H}^n,\mathbb{H}^n)$ as a glance at the von Neumann series shows; but the quaternionic determinant is a more subtle instrument here.

4.5. Example. The *orthogonal group* $O(n,\mathbb{R})$ is the group of all linear isometries of $(\mathbb{R}^n,\langle\quad,\quad\rangle)$, where $\langle\quad,\quad\rangle$ is the standard positive definite inner product on $\mathbb{R}^n$. The *special orthogonal group* $SO(n,\mathbb{R}):=\{A\in O(n,\mathbb{R}):\det A=1\}$ is open in $O(n,\mathbb{R})$, since

$$O(n,\mathbb{R})=SO(n,\mathbb{R})\sqcup\begin{pmatrix}-1&0\\0&\mathbb{I}_{n-1}\end{pmatrix}SO(n,\mathbb{R}),$$

where $\mathbb{I}_k$ is short for the identity matrix $Id_{\mathbb{R}^k}$. We claim that $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ are submanifolds of $L(\mathbb{R}^n,\mathbb{R}^n)$. For that we consider the mapping $f:L(\mathbb{R}^n,\mathbb{R}^n)\to L(\mathbb{R}^n,\mathbb{R}^n)$, given by $f(A)=A.A^t$. Then $O(n,\mathbb{R})=f^{-1}(\mathbb{I}_n)$; so $O(n,\mathbb{R})$ is closed. Since it is also bounded, $O(n,\mathbb{R})$ is compact. We have $df(A).X=X.A^t+A.X^t$, so $\ker df(\mathbb{I}_n)=\{X:X+X^t=0\}$ is the space $\mathfrak{o}(n,\mathbb{R})$ of all skew symmetric $n\times n$-matrices. Note that $\dim\mathfrak{o}(n,\mathbb{R})=\frac{1}{2}(n-1)n$. If A is invertible, we get $\ker df(A)=\{Y:Y.A^t+A.Y^t=0\}=\{Y:Y.A^t\in\mathfrak{o}(n,\mathbb{R})\}=\mathfrak{o}(n,\mathbb{R}).(A^{-1})^t$. The mapping f takes values in $L_{sym}(\mathbb{R}^n,\mathbb{R}^n)$, the space of all symmetric $n\times n$-matrices, and $\dim\ker df(A)+\dim L_{sym}(\mathbb{R}^n,\mathbb{R}^n)=\frac{1}{2}(n-1)n+\frac{1}{2}n(n+1)=n^2=\dim L(\mathbb{R}^n,\mathbb{R}^n)$, so $f:GL(n,\mathbb{R})\to L_{sym}(\mathbb{R}^n,\mathbb{R}^n)$ is a submersion. Since obviously $f^{-1}(\mathbb{I}_n)\subset GL(n,\mathbb{R})$, we conclude from 1.12 that $O(n,\mathbb{R})$ is a submanifold of $GL(n,\mathbb{R})$. It is also a Lie group, since the group operations are smooth as the restrictions of the ones from $GL(n,\mathbb{R})$.

4.6. Example. The *special linear group* $SL(n,\mathbb{R})$ is the group of all $n\times n$-matrices of determinant 1. The function $\det:L(\mathbb{R}^n,\mathbb{R}^n)\to\mathbb{R}$ is smooth and $d\det(A)X=\operatorname{trace}(C(A).X)$, where $C(A)^i_j$, the cofactor of A^j_i, is the determinant of the matrix, which results from putting 1 instead of A^j_i into A and 0 in the rest of the j-th row and the i-th column of A. We recall Cramers rule $C(A).A=A.C(A)=\det(A).\mathbb{I}_n$. So if $C(A)\neq 0$ (i.e. $\operatorname{rank}(A)\geq n-1$) then the linear functional $df(A)$ is non zero. So $\det:GL(n,\mathbb{R})\to\mathbb{R}$ is a submersion and $SL(n,\mathbb{R})=(\det)^{-1}(1)$ is a manifold and a Lie group of dimension n^2-1. Note finally that $T_{\mathbb{I}_n}SL(n,\mathbb{R})=\ker d\det(\mathbb{I}_n)=\{X:\operatorname{trace}(X)=0\}$. This space of traceless matrices is usually called $\mathfrak{sl}(n,\mathbb{R})$.

4.7. Example. The *symplectic group* $Sp(n,\mathbb{R})$ is the group of all $2n \times 2n$-matrices A such that $\omega(Ax,Ay) = \omega(x,y)$ for all $x,y \in \mathbb{R}^{2n}$, where ω is a (the standard) non degenerate skew symmetric bilinear form on $\mathbb{R}^{2n}$.

Such a form exists on a vector space if and only if the dimension is even, and on $\mathbb{R}^n \times (\mathbb{R}^n)^*$ the form $\omega((x,x^*),(y,y^*)) = \langle x, y^*\rangle - \langle y, x^*\rangle$, in coordinates $\omega((x^i)_{i=1}^{2n},(y^j)_{j=1}^{2n}) = \sum_{i=1}^n (x^i y^{n+i} - x^{n+i}y^i)$, is such a form. Any symplectic form on $\mathbb{R}^{2n}$ looks like that after choosing a suitable basis. Let $(e_i)_{i=1}^{2n}$ be the standard basis in $\mathbb{R}^{2n}$. Then we have

$$(\omega(e_i,e_j)_j^i) = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix} =: J,$$

and the matrix J satisfies $J^t = -J$, $J^2 = -\mathbb{I}_{2n}$, $J\binom{x}{y} = \binom{y}{-x}$ in $\mathbb{R}^n \times \mathbb{R}^n$, and $\omega(x,y) = \langle x, Jy\rangle$ in terms of the standard inner product on $\mathbb{R}^{2n}$.

For $A \in L(\mathbb{R}^{2n},\mathbb{R}^{2n})$ we have $\omega(Ax,Ay) = \langle Ax, JAy\rangle = \langle x, A^tJAy\rangle$. Thus $A \in Sp(n,\mathbb{R})$ if and only if $A^tJA = J$.

We consider now the mapping $f : L(\mathbb{R}^{2n},\mathbb{R}^{2n}) \to L(\mathbb{R}^{2n},\mathbb{R}^{2n})$ given by $f(A) = A^tJA$. Then $f(A)^t = (A^tJA)^t = -A^tJA = -f(A)$, so f takes values in the space $\mathfrak{o}(2n,\mathbb{R})$ of skew symmetric matrices. We have $df(A)X = X^tJA + A^tJX$, and therefore

$$\begin{aligned}\ker df(\mathbb{I}_{2n}) &= \{X \in L(\mathbb{R}^{2n},\mathbb{R}^{2n}) : X^tJ + JX = 0\} \\ &= \{X : JX \text{ is symmetric}\} =: \mathfrak{sp}(n,\mathbb{R}).\end{aligned}$$

We see that $\dim \mathfrak{sp}(n,\mathbb{R}) = \frac{2n(2n+1)}{2} = \binom{2n+1}{2}$. Furthermore $\ker df(A) = \{X : X^tJA + A^tJX = 0\}$ and the mapping $X \mapsto A^tJX$ is an isomorphism $\ker df(A) \to L_{sym}(\mathbb{R}^{2n},\mathbb{R}^{2n})$, if A is invertible. Thus $\dim\ker df(A) = \binom{2n+1}{2}$ for all $A \in GL(2n,\mathbb{R})$. If $f(A) = J$, then $A^tJA = J$, so A has rank $2n$ and is invertible, and we have $\dim\ker df(A) + \dim\mathfrak{o}(2n,\mathbb{R}) = \binom{2n+1}{2} + \frac{2n(2n-1)}{2} = 4n^2 = \dim L(\mathbb{R}^{2n},\mathbb{R}^{2n})$. So $f : GL(2n,\mathbb{R}) \to \mathfrak{o}(2n,\mathbb{R})$ is a submersion and $f^{-1}(J) = Sp(n,\mathbb{R})$ is a manifold and a Lie group. It is the symmetry group of 'classical mechanics'.

4.8. Example. The complex general linear group $GL(n,\mathbb{C})$ of all invertible complex $n \times n$-matrices is open in $L_{\mathbb{C}}(\mathbb{C}^n,\mathbb{C}^n)$, so it is a real Lie group of real dimension $2n^2$; it is also a complex Lie group of complex dimension n^2. The complex special linear group $SL(n,\mathbb{C})$ of all matrices of determinant 1 is a submanifold of $GL(n,\mathbb{C})$ of complex codimension 1 (or real codimension 2).

The complex orthogonal group $O(n,\mathbb{C})$ is the set

$$\{A \in L(\mathbb{C}^n,\mathbb{C}^n) : g(Az,Aw) = g(z,w) \text{ for all } z,w\},$$

where $g(z,w)=\sum_{i=1}^n z^i w^i$. This is a complex Lie group of complex dimension $\frac{(n-1)n}{2}$, and it is *not* compact. Since $O(n,\mathbb{C})=\{A : A^t A=\mathbb{I}_n\}$, we have $1=\det_{\mathbb{C}}(\mathbb{I}_n)=\det_{\mathbb{C}}(A^tA)=\det_{\mathbb{C}}(A)^2$, so $\det_{\mathbb{C}}(A)=\pm 1$. Thus $SO(n,\mathbb{C}):=\{A\in O(n,\mathbb{C}) : \det_{\mathbb{C}}(A)=1\}$ is an open subgroup of index 2 in $O(n,\mathbb{C})$.

The group $Sp(n,\mathbb{C})=\{A\in L_{\mathbb{C}}(\mathbb{C}^{2n},\mathbb{C}^{2n}) : A^tJA=J\}$ is also a complex Lie group of complex dimension $n(2n+1)$.

These groups treated here are the *classical complex Lie groups.* The groups $SL(n,\mathbb{C})$ for $n\ge 2$, $SO(n,\mathbb{C})$ for $n\ge 3$, $Sp(n,\mathbb{C})$ for $n\ge 4$, and five more exceptional groups exhaust all simple complex Lie groups up to coverings.

4.9. Example. Let $\mathbb{C}^n$ be equipped with the standard hermitian inner product $(z,w)=\sum_{i=1}^n \overline{z}^i w^i$. The *unitary* group $U(n)$ consists of all complex $n\times n$-matrices A such that $(Az,Aw)=(z,w)$ for all z, w holds, or equivalently $U(n)=\{A : A^*A=\mathbb{I}_n\}$, where $A^*=\overline{A}^t$.

We consider the mapping $f : L_{\mathbb{C}}(\mathbb{C}^n,\mathbb{C}^n)\to L_{\mathbb{C}}(\mathbb{C}^n,\mathbb{C}^n)$, given by $f(A)=A^*A$. Then f is smooth but not holomorphic. Its derivative is $df(A)X=X^*A+A^*X$, so $\ker df(\mathbb{I}_n)=\{X : X^*+X=0\}=:\mathfrak{u}(n)$, the space of all skew hermitian matrices. We have $\dim_{\mathbb{R}}\mathfrak{u}(n)=n^2$. As above we may check that $f : GL(n,\mathbb{C})\to L_{herm}(\mathbb{C}^n,\mathbb{C}^n)$ is a submersion, so $U(n)=f^{-1}(\mathbb{I}_n)$ is a compact real Lie group of dimension n^2.

The *special unitary* group is $SU(n)=U(n)\cap SL(n,\mathbb{C})$. For $A\in U(n)$ we have $|\det_{\mathbb{C}}(A)|=1$, thus $\dim_{\mathbb{R}} SU(n)=n^2-1$.

4.10. Example. The group $Sp(n)$. Let $\mathbb{H}$ be the division algebra of quaternions. We will use the following description of quaternions: Let $(\mathbb{R}^3,\langle\quad,\quad\rangle,\Delta)$ be the oriented Euclidean space of dimension 3, where Δ is a determinant function with value 1 on a positive oriented orthonormal basis. The *vector product* on R^3 is then given by $\langle X\times Y,Z\rangle=\Delta(X,Y,Z)$. Now we let $\mathbb{H}:=\mathbb{R}^3\times\mathbb{R}$, equipped with the following product:

$$(X,s)(Y,t):=(X\times Y+sY+tX, st-\langle X,Y\rangle).$$

Now we take a positively oriented orthonormal basis of $\mathbb{R}^3$, call it (i,j,k), and indentify $(0,1)$ with 1. Then the last formula implies visibly the usual product rules for the basis $(1,i,j,k)$ of the quaternions.

The group $Sp(1):=S^3\subset\mathbb{H}\cong\mathbb{R}^4$ is then the group of unit quaternions, obviously a Lie group.

Now let V be a right vector space over $\mathbb{H}$. Since $\mathbb{H}$ is not commutative, we have to distinguish between left and right vector spaces and we choose right ones as basic, so that matrices can multiply from the left. By choosing a basis we get

$V = \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{H} = \mathbb{H}^n$. For $u = (u^i)$, $v = (v^i) \in \mathbb{H}^n$ we put $\langle u, v\rangle := \sum_{i=1}^n \overline{u}^i v^i$. Then $\langle\quad,\quad\rangle$ is $\mathbb{R}$-bilinear and $\langle ua, vb\rangle = \overline{a}\langle u, v\rangle b$ for $a, b \in \mathbb{H}$.

An $\mathbb{R}$ linear mapping $A : V \to V$ is called $\mathbb{H}$-*linear* or *quaternionically linear* if $A(ua) = A(u)a$ holds. The space of all such mappings shall be denoted by $L_{\mathbb{H}}(V, V)$. It is real isomorphic to the space of all quaternionic $n \times n$-matrices with the usual multiplication, since for the standard basis $(e_i)_{i=1}^n$ in $V = \mathbb{H}^n$ we have $A(u) = A(\sum_i e_i u^i) = \sum_i A(e_i)u^i = \sum_{i,j} e_j A^j_i u^i$. Note that $L_{\mathbb{H}}(V, V)$ is only a real vector space, if V is a right quaternionic vector space - any further structure must come from a second (left) quaternionic vector space structure on V.

$GL(n, \mathbb{H})$, the group of invertible $\mathbb{H}$-linear mappings of $\mathbb{H}^n$, is a Lie group, because it is $GL(4n, \mathbb{R}) \cap L_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^n)$, open in $L_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^n)$.

A quaternionically linear mapping A is called isometric or *quaternionically unitary*, if $\langle A(u), A(v)\rangle = \langle u, v\rangle$ for all $u, v \in \mathbb{H}^n$. We denote by $Sp(n)$ the group of all quaternionic isometries of $\mathbb{H}^n$, the *quaternionic unitary group*. The reason for its name is that $Sp(n) = Sp(n, \mathbb{C}) \cap U(2n)$, since we can decompose the quaternionic hermitian form $\langle\quad,\quad\rangle$ into a complex hermitian one and a complex symplectic one. Also we have $Sp(n) \subset O(4n, \mathbb{R})$, since the real part of $\langle\quad,\quad\rangle$ is a positive definite real inner product. For $A \in L_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^n)$ we put $A^* := \overline{A}^t$. Then we have $\langle u, A(v)\rangle = \langle A^*(u), v\rangle$, so $\langle A(u), A(v)\rangle = \langle A^*A(u), v\rangle$. Thus $A \in Sp(n)$ if and only if $A^*A = Id$.

Again $f : L_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^n) \to L_{\mathbb{H},herm}(\mathbb{H}^n, \mathbb{H}^n) = \{A : A^* = A\}$, given by $f(A) = A^*A$, is a smooth mapping with $df(A)X = X^*A + A^*X$. So we have $\ker df(Id) = \{X : X^* = -X\} =: \mathfrak{sp}(n)$, the space of quaternionic skew hermitian matrices. The usual proof shows that f has maximal rank on $GL(n, \mathbb{H})$, so $Sp(n) = f^{-1}(Id)$ is a compact real Lie group of dimension $2n(n-1) + 3n$.

The groups $SO(n, \mathbb{R})$ for $n \geq 3$, $SU(n)$ for $n \geq 2$, $Sp(n)$ for $n \geq 2$ and real forms of the exceptional complex Lie groups exhaust all simple compact Lie groups up to coverings.

4.11. Invariant vector fields and Lie algebras. Let G be a (real) Lie group. A vector field ξ on G is called *left invariant*, if $\mu_a^*\xi = \xi$ for all $a \in G$, where $\mu_a^*\xi = T(\mu_{a^{-1}}) \circ \xi \circ \mu_a$ as in section 3. Since by 3.11 we have $\mu_a^*[\xi, \eta] = [\mu_a^*\xi, \mu_a^*\eta]$, the space $\mathfrak{X}_L(G)$ of all left invariant vector fields on G is closed under the Lie bracket, so it is a sub Lie algebra of $\mathfrak{X}(G)$. Any left invariant vector field ξ is uniquely determined by $\xi(e) \in T_eG$, since $\xi(a) = T_e(\mu_a).\xi(e)$. Thus the Lie algebra $\mathfrak{X}_L(G)$ of left invariant vector fields is linearly isomorphic to T_eG, and on T_eG the Lie bracket on $\mathfrak{X}_L(G)$ induces a Lie algebra structure, whose bracket is again denoted by $[\quad,\quad]$. This Lie algebra will be denoted as usual by $\mathfrak{g}$, sometimes by $Lie(G)$.

We will also give a name to the isomorphism with the space of left invariant vector fields: $L : \mathfrak{g} \to \mathfrak{X}_L(G)$, $X \mapsto L_X$, where $L_X(a) = T_e\mu_a.X$. Thus $[X,Y] = [L_X, L_Y](e)$.

A vector field η on G is called *right invariant*, if $(\mu^a)^*\eta = \eta$ for all $a \in G$. If ξ is left invariant, then $\nu^*\xi$ is right invariant, since $\nu \circ \mu^a = \mu_{a^{-1}} \circ \nu$ implies that $(\mu^a)^*\nu^*\xi = (\nu \circ \mu^a)^*\xi = (\mu_{a^{-1}} \circ \nu)^*\xi = \nu^*(\mu_{a^{-1}})^*\xi = \nu^*\xi$. The right invariant vector fields form a sub Lie algebra $\mathfrak{X}_R(G)$ of $\mathfrak{X}(G)$, which is again linearly isomorphic to T_eG and induces also a Lie algebra structure on T_eG. Since $\nu^* : \mathfrak{X}_L(G) \to \mathfrak{X}_R(G)$ is an isomorphism of Lie algebras by 3.11, $T_e\nu = -Id : T_eG \to T_eG$ is an isomorphism between the two Lie algebra structures. We will denote by $R : \mathfrak{g} = T_eG \to \mathfrak{X}_R(G)$ the isomorphism discussed, which is given by $R_X(a) = T_e(\mu^a).X$.

4.12. Lemma. *If L_X is a left invariant vector field and R_Y is a right invariant one, then $[L_X, R_Y] = 0$. Thus the flows of L_X and R_Y commute.*

Proof. We consider the vector field $0 \times L_X \in \mathfrak{X}(G \times G)$, given by $(0 \times L_X)(a,b) = (0_a, L_X(b))$. Then $T_{(a,b)}\mu.(0_a, L_X(b)) = T_a\mu^b.0_a + T_b\mu_a.L_X(b) = L_X(ab)$, so $0 \times L_X$ is μ-related to L_X. Likewise $R_Y \times 0$ is μ-related to R_Y. But then $0 = [0 \times L_X, R_Y \times 0]$ is μ-related to $[L_X, R_Y]$ by 3.10. Since μ is surjective, $[L_X, R_Y] = 0$ follows. $\square$

4.13. Let $\varphi : G \to H$ be a homomorphism of Lie groups, so for the time being we require φ to be smooth.

Lemma. *Then $\varphi' := T_e\varphi : \mathfrak{g} = T_eG \to \mathfrak{h} = T_eH$ is a Lie algebra homomorphism.*

Proof. For $X \in \mathfrak{g}$ and $x \in G$ we have

$$\begin{aligned} T_x\varphi.L_X(x) &= T_x\varphi.T_e\mu_x.X = T_e(\varphi \circ \mu_x).X \\ &= T_e(\mu_{\varphi(x)} \circ \varphi).X = T_e(\mu_{\varphi(x)}).T_e\varphi.X = L_{\varphi'(X)}(\varphi(x)). \end{aligned}$$

So L_X is φ-related to $L_{\varphi'(X)}$. By 3.10 the field $[L_X, L_Y] = L_{[X,Y]}$ is φ-related to $[L_{\varphi'(X)}, L_{\varphi'(Y)}] = L_{[\varphi'(X),\varphi'(Y)]}$. So we have $T\varphi \circ L_{[X,Y]} = L_{[\varphi'(X),\varphi'(Y)]} \circ \varphi$. If we evaluate this at e the result follows. $\square$

Now we will determine the Lie algebras of all the examples given above.

4.14. For the Lie group $GL(n,\mathbb{R})$ we have $T_eGL(n,\mathbb{R}) = L(\mathbb{R}^n, \mathbb{R}^n) =: \mathfrak{gl}(n,\mathbb{R})$ and $TGL(n,\mathbb{R}) = GL(n,\mathbb{R}) \times L(\mathbb{R}^n, \mathbb{R}^n)$ by the affine structure of the surrounding vector space. For $A \in GL(n,\mathbb{R})$ we have $\mu_A(B) = A.B$, so μ_A extends to a linear isomorphism of $L(\mathbb{R}^n, \mathbb{R}^n)$, and for $(B,X) \in TGL(n,\mathbb{R})$

we get $T_B(\mu_A).(B,X) = (A.B, A.X)$. So the left invariant vector field $L_X \in \mathfrak{X}_L(GL(n,\mathbb{R}))$ is given by $L_X(A) = T_e(\mu_A).X = (A, A.X)$.

Let $f : GL(n,\mathbb{R}) \to \mathbb{R}$ be the restriction of a linear functional on $L(\mathbb{R}^n,\mathbb{R}^n)$. Then we have $L_X(f)(A) = df(A)(L_X(A)) = df(A)(A.X) = f(A.X)$, which we may write as $L_X(f) = f(\quad .X)$. Therefore

$$\begin{aligned} L_{[X,Y]}(f) &= [L_X, L_Y](f) = L_X(L_Y(f)) - L_Y(L_X(f)) \\ &= L_X(f(\quad .Y)) - L_Y(f(\quad .X)) = f(\quad .X.Y) - f(\quad .Y.X) \\ &= L_{XY-YX}(f). \end{aligned}$$

So the Lie bracket on $\mathfrak{gl}(n,\mathbb{R}) = L(\mathbb{R}^n,\mathbb{R}^n)$ is given by $[X,Y] = XY - YX$, the usual commutator.

4.15. Example. Let V be a vector space. Then $(V,+)$ is a Lie group, $T_0V = V$ is its Lie algebra, $TV = V\times V$, left translation is $\mu_v(w) = v+w$, $T_w(\mu_v).(w,X) = (v+w, X)$. So $L_X(v) = (v,X)$, a constant vector field. Thus the Lie bracket is 0.

4.16. Example. The special linear group is $SL(n,\mathbb{R}) = \det^{-1}(1)$ and its Lie algebra is given by $T_eSL(n,\mathbb{R}) = \ker d\det(\mathbb{I}) = \{X \in L(\mathbb{R}^n,\mathbb{R}^n) : \operatorname{trace} X = 0\} = \mathfrak{sl}(n,\mathbb{R})$ by 4.6. The injection $i : SL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is a smooth homomorphism of Lie groups, so $T_ei = i' : \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ is an injective homomorphism of Lie algebras. Thus the Lie bracket is given by $[X,Y] = XY - YX$.

The same argument gives the commutator as the Lie bracket in all other examples we have treated. We have already determined the Lie algebras as T_eG.

4.17. One parameter subgroups. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A *one parameter subgroup* of G is a Lie group homomorphism $\alpha : (\mathbb{R},+) \to G$, i.e. a smooth curve α in G with $\alpha(s+t) = \alpha(s).\alpha(t)$, and hence $\alpha(0) = e$.

Lemma. *Let $\alpha : \mathbb{R} \to G$ be a smooth curve with $\alpha(0) = e$. Let $X \in \mathfrak{g}$. Then the following assertions are equivalent.*

(1) *α is a one parameter subgroup with $X = \frac{\partial}{\partial t}\big|_0 \alpha(t)$.*
(2) *$\alpha(t) = \mathrm{Fl}^{L_X}(t,e)$ for all t.*
(3) *$\alpha(t) = \mathrm{Fl}^{R_X}(t,e)$ for all t.*
(4) *$x.\alpha(t) = \mathrm{Fl}^{L_X}(t,x)$, or $\mathrm{Fl}^{L_X}_t = \mu^{\alpha(t)}$, for all t .*
(5) *$\alpha(t).x = \mathrm{Fl}^{R_X}(t,x)$, or $\mathrm{Fl}^{R_X}_t = \mu_{\alpha(t)}$, for all t.*

Proof. (1) $\implies$ (4). We have $\frac{d}{dt}x.\alpha(t) = \frac{d}{ds}|_0 x.\alpha(t+s) = \frac{d}{ds}|_0 x.\alpha(t).\alpha(s) = \frac{d}{ds}|_0 \mu_{x.\alpha(t)}\alpha(s) = T_e(\mu_{x.\alpha(t)}).\frac{d}{ds}|_0\alpha(s) = L_X(x.\alpha(t))$. By uniqueness of solutions we get $x.\alpha(t) = \mathrm{Fl}^{L_X}(t,x)$.

(4) $\Longrightarrow$ (2). This is clear.

(2) $\Longrightarrow$ (1). We have

$$\begin{aligned}\tfrac{d}{ds}\alpha(t)\alpha(s) &= \tfrac{d}{ds}(\mu_{\alpha(t)}\alpha(s)) = T(\mu_{\alpha(t)})\tfrac{d}{ds}\alpha(s)\\ &= T(\mu_{\alpha(t)})L_X(\alpha(s)) = L_X(\alpha(t)\alpha(s))\end{aligned}$$

and $\alpha(t)\alpha(0) = \alpha(t)$. So we get $\alpha(t)\alpha(s) = \mathrm{Fl}^{L_X}(s,\alpha(t)) = \mathrm{Fl}^{L_X}_s \mathrm{Fl}^{L_X}_t(e) = \mathrm{Fl}^{L_X}(t+s,e) = \alpha(t+s)$.

(4) $\Longleftrightarrow$ (5). We have $\mathrm{Fl}^{\nu^*\xi}_t = \nu^{-1}\circ \mathrm{Fl}^{\xi}_t \circ\nu$ by 3.14. Therefore we have by 4.11

$$\begin{aligned}(\mathrm{Fl}^{R_X}_t(x^{-1}))^{-1} &= (\nu\circ \mathrm{Fl}^{R_X}_t \circ\nu)(x) = \mathrm{Fl}^{\nu^* R_X}_t(x)\\ &= \mathrm{Fl}^{L_X}_{-t}(x) = x.\alpha(-t).\end{aligned}$$

So $\mathrm{Fl}^{R_X}_t(x^{-1}) = \alpha(t).x^{-1}$, and $\mathrm{Fl}^{R_X}_t(y) = \alpha(t).y$.

(5) $\Longrightarrow$ (3) $\Longrightarrow$ (1) can be shown in a similar way. $\square$

An immediate consequence of the foregoing lemma is that left invariant and right invariant vector fields on a Lie group are always complete, so they have global flows, because a locally defined one parameter group can always be extended to a globally defined one by multiplying it up.

4.18. Definition. The *exponential mapping* $\exp : \mathfrak{g}\to G$ of a Lie group is defined by

$$\exp X = \mathrm{Fl}^{L_X}(1,e) = \mathrm{Fl}^{R_X}(1,e) = \alpha_X(1),$$

where α_X is the one parameter subgroup of G with $\dot\alpha_X(0) = X$.

Theorem.

(1) $\exp : \mathfrak{g}\to G$ *is smooth.*
(2) $\exp(tX) = \mathrm{Fl}^{L_X}(t,e)$.
(3) $\mathrm{Fl}^{L_X}(t,x) = x.\exp(tX)$.
(4) $\mathrm{Fl}^{R_X}(t,x) = \exp(tX).x$.
(5) $\exp(0) = e$ *and* $T_0\exp = Id : T_0\mathfrak{g} = \mathfrak{g}\to T_eG = \mathfrak{g}$*, thus* $\exp$ *is a diffeomorphism from a neighborhood of 0 in* $\mathfrak{g}$ *onto a neighborhood of* e *in* G.

Proof. (1) Let $0\times L\in\mathfrak{X}(\mathfrak{g}\times G)$ be given by $(0\times L)(X,x) = (0_X, L_X(x))$. Then $pr_2\,\mathrm{Fl}^{0\times L}(t,(X,e)) = \alpha_X(t)$ is smooth in (t,X).

(2) $\exp(tX) = \mathrm{Fl}^{t.L_X}(1,e) = \mathrm{Fl}^{L_X}(t,e) = \alpha_X(t)$.

(3) and (4) follow from lemma 4.17.

(5) $T_0\exp.X = \frac{d}{dt}|_0\exp(0+t.X) = \frac{d}{dt}|_0\,\mathrm{Fl}^{L_X}(t,e) = X$. $\square$

4.19. Remark. If G is connected and $U \subset \mathfrak{g}$ is open with $0 \in U$, then the group generated by $\exp(U)$ equals G.

For this group is a subgroup of G containing some open neighborhood of e, so it is open. The complement in G is also open (as union of the other cosets), so this subgroup is open and closed. Since G is connected, it coincides with G.

If G is not connected, then the subgroup generated by $\exp(U)$ is the connected component of e in G.

4.20. Remark. Let $\varphi : G \to H$ be a smooth homomorphism of Lie groups. Then the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi'"] \arrow[d, "\exp^G"'] & \mathfrak{h} \arrow[d, "\exp^H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes, since $t \mapsto \varphi(\exp^G(tX))$ is a one parameter subgroup of H and $\frac{d}{dt}|_0 \varphi(\exp^G tX) = \varphi'(X)$, so $\varphi(\exp^G tX) = \exp^H(t\varphi'(X))$.

If G is connected and $\varphi, \psi : G \to H$ are homomorphisms of Lie groups with $\varphi' = \psi' : \mathfrak{g} \to \mathfrak{h}$, then $\varphi = \psi$. For $\varphi = \psi$ on the subgroup generated by $\exp^G \mathfrak{g}$ which equals G by 4.19.

4.21. Theorem. *A continuous homomorphism $\varphi : G \to H$ between Lie groups is smooth. In particular a topological group can carry at most one compatible Lie group structure.*

Proof. Let first $\varphi = \alpha : (\mathbb{R}, +) \to G$ be a continuous one parameter subgroup. Then $\alpha(-\varepsilon, \varepsilon) \subset \exp(U)$, where U is an absolutely convex open neighborhood of 0 in $\mathfrak{g}$ such that $\exp \restriction 2U$ is a diffeomorphism, for some $\varepsilon > 0$. Put $\beta := (\exp \restriction 2U)^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to \mathfrak{g}$. Then for $|t| < \frac{\varepsilon}{2}$ we have $\exp(2\beta(t)) = \exp(\beta(t))^2 = \alpha(t)^2 = \alpha(2t) = \exp(\beta(2t))$, so $2\beta(t) = \beta(2t)$; thus $\beta(\frac{s}{2}) = \frac{1}{2}\beta(s)$ for $|s| < \varepsilon$. So we have $\alpha(\frac{s}{2}) = \exp(\beta(\frac{s}{2})) = \exp(\frac{1}{2}\beta(s))$ for all $|s| < \varepsilon$ and by recursion we get $\alpha(\frac{s}{2^n}) = \exp(\frac{1}{2^n}\beta(s))$ for $n \in \mathbb{N}$ and in turn $\alpha(\frac{k}{2^n}s) = \alpha(\frac{s}{2^n})^k = \exp(\frac{1}{2^n}\beta(s))^k = \exp(\frac{k}{2^n}\beta(s))$ for $k \in \mathbb{Z}$. Since the $\frac{k}{2^n}$ for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ are dense in R and since α is continuous we get $\alpha(ts) = \exp(t\beta(s))$ for all $t \in \mathbb{R}$. So α is smooth.

Now let $\varphi : G \to H$ be a continuous homomorphism. Let $X_1, \dots, X_n$ be a linear basis of $\mathfrak{g}$. We define a mapping $\psi : \mathbb{R}^n \to G$ as $\psi(t^1, \dots, t^n) = \exp(t^1 X_1) \cdots \exp(t^n X_n)$. Then $T_0\psi$ is invertible, so ψ is a diffeomorphism near 0. Sometimes ψ^{-1} is called a coordinate system of the second kind. $t \mapsto \varphi(\exp^G tX_i)$ is a continuous one parameter subgroup of H, so it is smooth by the first part of the proof.

We have $(\varphi \circ \psi)(t^1, \dots, t^n) = (\varphi \exp(t^1 X_1)) \cdots (\varphi \exp(t^n X_n))$, so $\varphi \circ \psi$ is smooth. Thus φ is smooth near $e \in G$ and consequently everywhere on G. $\square$

4.22. Theorem. *Let G and H be Lie groups (G separable is essential here), and let $\varphi : G \to H$ be a continuous bijective homomorphism. Then φ is a diffeomorphism.*

Proof. Our first aim is to show that φ is a homeomorphism. Let V be an open e-neighborhood in G, and let K be a compact e-neighborhood in G such that $K.K^{-1} \subset V$. Since G is separable there is a sequence $(a_i)_{i\in\mathbb{N}}$ in G such that $G = \bigcup_{i=1}^{\infty} a_i.K$. Since H is locally compact, it is a Baire space (V_i, for $i \in \mathbb{N}$ open and dense implies $\bigcap V_i$ dense). The set $\varphi(a_i)\varphi(K)$ is compact, thus closed. Since $H = \bigcup_i \varphi(a_i).\varphi(K)$, there is some i such that $\varphi(a_i)\varphi(K)$ has non empty interior, so $\varphi(K)$ has non empty interior. Choose $b \in G$ such that $\varphi(b)$ is an interior point of $\varphi(K)$ in H. Then $e_H = \varphi(b)\varphi(b^{-1})$ is an interior point of $\varphi(K)\varphi(K^{-1}) \subset \varphi(V)$. So if U is open in G and $a \in U$, then e_H is an interior point of $\varphi(a^{-1}U)$, so $\varphi(a)$ is in the interior of $\varphi(U)$. Thus $\varphi(U)$ is open in H, and φ is a homeomorphism.

Now by 4.21 φ and φ^{-1} are smooth. $\square$

4.23. Examples. We first describe the exponential mapping of the general linear group $GL(n,\mathbb{R})$. Let $X \in \mathfrak{gl}(n,\mathbb{R}) = L(\mathbb{R}^n,\mathbb{R}^n)$, then the left invariant vector field is given by $L_X(A) = (A, A.X) \in GL(n,\mathbb{R}) \times \mathfrak{gl}(n,\mathbb{R})$ and the one parameter group $\alpha_X(t) = \operatorname{Fl}^{L_X}(t,\mathbb{I})$ is given by the differential equation $\frac{d}{dt}\alpha_X(t) = L_X(\alpha_X(t)) = \alpha_X(t).X$, with initial condition $\alpha_X(0) = \mathbb{I}$. But the unique solution of this equation is $\alpha_X(t) = e^{tX} = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$. So

$$\exp^{GL(n,\mathbb{R})}(X) = e^X = \textstyle\sum_{k=0}^{\infty} \frac{1}{k!}\, X^k.$$

If $n = 1$ we get the usual exponential mapping of one real variable. For all Lie subgroups of $GL(n,\mathbb{R})$, the exponential mapping is given by the same formula $\exp(X) = e^X$; this follows from 4.20.

4.24. The adjoint representation. A *representation* of a Lie group G on a finite dimensional vector space V (real or complex) is a homomorphism $\rho : G \to GL(V)$ of Lie groups. Then by 4.13 $\rho' : \mathfrak{g} \to \mathfrak{gl}(V) = L(V,V)$ is a Lie algebra homomorphism.

For $a \in G$ we define $\operatorname{conj}_a : G \to G$ by $\operatorname{conj}_a(x) = axa^{-1}$. It is called the *conjugation* or the *inner automorphism* by $a \in G$. We have $\operatorname{conj}_a(xy) = \operatorname{conj}_a(x)\operatorname{conj}_a(y)$, $\operatorname{conj}_{ab} = \operatorname{conj}_a \circ \operatorname{conj}_b$, and conj is smooth in all variables.

Next we define for $a \in G$ the mapping $\operatorname{Ad}(a) = (\operatorname{conj}_a)' = T_e(\operatorname{conj}_a) : \mathfrak{g} \to \mathfrak{g}$. By 4.13 $\operatorname{Ad}(a)$ is a Lie algebra homomorphism, so we have $\operatorname{Ad}(a)[X,Y] =$

$[\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y]$. Furthermore $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is a representation, called the *adjoint representation* of G, since

$$\begin{aligned}\mathrm{Ad}(ab) &= T_e(\mathrm{conj}_{ab}) = T_e(\mathrm{conj}_a \circ \mathrm{conj}_b)\\ &= T_e(\mathrm{conj}_a) \circ T_e(\mathrm{conj}_b) = \mathrm{Ad}(a) \circ \mathrm{Ad}(b).\end{aligned}$$

The relations $\mathrm{Ad}(a) = T_e(\mathrm{conj}_a) = T_a(\mu^{a^{-1}}).T_e(\mu_a) = T_{a^{-1}}(\mu_a).T_e(\mu^{a^{-1}})$ will be used later.

Finally we define the (lower case) *adjoint representation* of the Lie algebra $\mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = L(\mathfrak{g}, \mathfrak{g})$, by $\mathrm{ad} := \mathrm{Ad}' = T_e \mathrm{Ad}$.

Lemma.

(1) $L_X(a) = R_{\mathrm{Ad}(a)X}(a)$ *for* $X \in \mathfrak{g}$ *and* $a \in G$.
(2) $\mathrm{ad}(X)Y = [X, Y]$ *for* $X, Y \in \mathfrak{g}$.

Proof. (1). $L_X(a) = T_e(\mu_a).X = T_e(\mu^a).T_e(\mu^{a^{-1}} \circ \mu_a).X = R_{\mathrm{Ad}(a)X}(a)$.

(2). Let $X_1, \dots, X_n$ be a linear basis of $\mathfrak{g}$ and fix $X \in \mathfrak{g}$. Then $\mathrm{Ad}(x)X = \sum_{i=1}^n f_i(x).X_i$ for $f_i \in C^\infty(G, \mathbb{R})$ and we have in turn

$$\begin{aligned}\mathrm{Ad}'(Y)X &= T_e(\mathrm{Ad}(\quad)X)Y = d(\mathrm{Ad}(\quad)X)|_e Y = d(\textstyle\sum f_i X_i)|_e Y\\ &= \textstyle\sum df_i|_e(Y) X_i = \sum L_Y(f_i)(e).X_i.\\ L_X(x) &= R_{\mathrm{Ad}(x)X}(x) = R(\textstyle\sum f_i(x) X_i)(x) = \sum f_i(x).R_{X_i}(x) \text{ by (1).}\\ [L_Y, L_X] &= [L_Y, \textstyle\sum f_i.R_{X_i}] = 0 + \sum L_Y(f_i).R_{X_i} \text{ by 3.4 and 4.12.}\\ [Y, X] &= [L_Y, L_X](e) = \textstyle\sum L_Y(f_i)(e).R_{X_i}(e) = \mathrm{Ad}'(Y)X = \mathrm{ad}(Y)X. \quad \square\end{aligned}$$

4.25. Corollary. *From 4.20 and 4.23 we have*

$$\begin{aligned}\mathrm{Ad} \circ exp^G &= exp^{GL(\mathfrak{g})} \circ \mathrm{ad}\\ \mathrm{Ad}(exp^G X)Y &= \sum_{k=0}^{\infty} \tfrac{1}{k!}\,(\mathrm{ad}\ X)^k Y = e^{\mathrm{ad}\ X} Y\\ &= Y + [X, Y] + \tfrac{1}{2!}[X, [X, Y]] + \tfrac{1}{3!}[X, [X, [X, Y]]] + \cdots\end{aligned}$$

so that also $\mathrm{ad}(X) = \frac{\partial}{\partial t}\big|_0 \mathrm{Ad}(\exp(tX))$.

4.26. The right logarithmic derivative. Let M be a manifold and let $f : M \to G$ be a smooth mapping into a Lie group G with Lie algebra $\mathfrak{g}$. We define the mapping $\delta f : TM \to \mathfrak{g}$ by the formula $\delta f(\xi_x) := T_{f(x)}(\mu^{f(x)^{-1}}).T_x f.\xi_x$. Then δf is a $\mathfrak{g}$-valued 1-form on M, $\delta f \in \Omega^1(M, \mathfrak{g})$, as we will write later. We call δf the *right logarithmic derivative* of f, since for $f : \mathbb{R} \to (\mathbb{R}^+, \cdot)$ we have $\delta f(x).1 = \frac{f'(x)}{f(x)} = (\log \circ f)'(x)$.

Lemma. *Let $f, g : M \to G$ be smooth. Then we have*

$$\delta(f.g)(x) = \delta f(x) + \mathrm{Ad}(f(x)).\delta g(x).$$

Proof.

$$\begin{aligned}
\delta(f.g)(x) &= T(\mu^{g(x)^{-1}.f(x)^{-1}}).T_x(f.g) \\
&= T(\mu^{f(x)^{-1}}).T(\mu^{g(x)^{-1}}).T_{(f(x),g(x))}\mu.(T_x f, T_x g) \\
&= T(\mu^{f(x)^{-1}}).T(\mu^{g(x)^{-1}}).\Big(T(\mu^{g(x)}).T_x f + T(\mu_{f(x)}).T_x g\Big) \\
&= \delta f(x) + \mathrm{Ad}(f(x)).\delta g(x). \quad \square
\end{aligned}$$

Remark. The *left logarithmic derivative* $\delta^{\text{left}} f \in \Omega^1(M, \mathfrak{g})$ of a smooth mapping $f : M \to G$ is given by $\delta^{\text{left}} f.\xi_x = T_{f(x)}(\mu_{f(x)^{-1}}).T_x f.\xi_x$. The corresponding Leibnitz rule for it is uglier that that for the right logarithmic derivative:

$$\delta^{\text{left}}(fg)(x) = \delta^{\text{left}} g(x) + Ad(g(x)^{-1})\delta^{\text{left}} f(x).$$

The form $\delta^{\text{left}}(Id_G) \in \Omega^1(G; \mathfrak{g})$ is also called the *Maurer-Cartan form* of the Lie group G.

4.27. Lemma. *For* $\exp : \mathfrak{g} \to G$ *and for* $g(z) := \dfrac{e^z - 1}{z}$ *we have*

$$\delta(\exp)(X) = T(\mu^{\exp(-X)}).T_X \exp = \sum_{p=0}^{\infty} \tfrac{1}{(p+1)!} (\mathrm{ad}\ X)^p = g(\mathrm{ad}\ X).$$

Proof. We put $M(X) = \delta(\exp)(X) : \mathfrak{g} \to \mathfrak{g}$. Then

$$\begin{aligned}
(s+t)M((s+t)X) &= (s+t)\delta(\exp)((s+t)X) \\
&= \delta(\exp((s+t) \quad))X \qquad \text{by the chain rule,} \\
&= \delta(\exp(s \quad).\exp(t \quad)).X \\
&= \delta(\exp(s \quad)).X + Ad(\exp(sX)).\delta(\exp(t \quad)).X \qquad \text{by 4.26,} \\
&= s.\delta(\exp)(sX) + Ad(\exp(sX)).t.\delta(\exp)(tX) \\
&= s.M(sX) + Ad(\exp(sX)).t.M(tX).
\end{aligned}$$

Next we put $N(t) := t.M(tX) \in L(\mathfrak{g}, \mathfrak{g})$, then we obtain $N(s+t) = N(s) + \mathrm{Ad}(\exp(sX)).N(t)$. We fix t, apply $\frac{d}{ds}|_0$, and get $N'(t) = N'(0) + \mathrm{ad}(X).N(t)$, where $N'(0) = M(0) + 0 = \delta(\exp)(0) = Id_{\mathfrak g}$. So we have the differential equation $N'(t) = Id_{\mathfrak g} + \operatorname{ad}(X).N(t)$ in $L(\mathfrak g, \mathfrak g)$ with initial condition $N(0) = 0$. The unique solution is

$$N(s) = \sum_{p=0}^{\infty} \tfrac{1}{(p+1)!}\ \operatorname{ad}(X)^p.s^{p+1}, \quad \text{and so}$$

$$\delta(\exp)(X) = M(X) = N(1) = \sum_{p=0}^{\infty} \tfrac{1}{(p+1)!}\ \operatorname{ad}(X)^p. \quad \square$$

4.28. Corollary. *$T_X \exp$ is bijective if and only if no eigenvalue of $\operatorname{ad}(X) : \mathfrak g \to \mathfrak g$ is of the form $\sqrt{-1}\,2k\pi$ for $k \in \mathbb Z \setminus \{0\}$.*

Proof. The zeros of $g(z) = \frac{e^z - 1}{z}$ are exactly $z = \sqrt{-1}\,2k\pi$ for $k \in \mathbb Z \setminus \{0\}$. The linear mapping $T_X \exp$ is bijective if and only if no eigenvalue of $g(\operatorname{ad}(X)) = T(\mu^{\exp(-X)}).T_X \exp$ is 0. But the eigenvalues of $g(\operatorname{ad}(X))$ are the images under g of the eigenvalues of $\operatorname{ad}(X)$. $\square$

4.29. Theorem. The Baker-Campbell-Hausdorff formula.
Let G be a Lie group with Lie algebra $\mathfrak g$. For complex z near 1 we consider the function $f(z) := \frac{\log(z)}{z-1} = \sum_{n\ge 0} \frac{(-1)^n}{n+1}(z-1)^n$.

Then for X, Y near 0 in $\mathfrak g$ we have $\exp X.\exp Y = \exp C(X,Y)$, where

$$\begin{aligned}
C(X,Y) &= Y + \int_0^1 f(e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y}).X\,dt \\
&= X + Y + \sum_{n\ge 1} \frac{(-1)^n}{n+1} \int_0^1 \Bigg(\sum_{\substack{k,\ell\ge 0\\ k+\ell\ge 1}} \frac{t^k}{k!\,\ell!}\,(\operatorname{ad} X)^k (\operatorname{ad} Y)^\ell \Bigg)^n X\,dt \\
&= X + Y + \sum_{n\ge 1} \frac{(-1)^n}{n+1} \sum_{\substack{k_1,\dots,k_n\ge 0\\ \ell_1,\dots \ell_n \ge 0\\ k_i+\ell_i\ge 1}} \frac{(\operatorname{ad} X)^{k_1}(\operatorname{ad} Y)^{\ell_1}\dots(\operatorname{ad} X)^{k_n}(\operatorname{ad} Y)^{\ell_n}}{(k_1+\dots+k_n+1)k_1!\dots k_n!\ell_1!\dots\ell_n!} X \\
&= X + Y + \tfrac12 [X,Y] + \tfrac1{12}([X,[X,Y]] - [Y,[Y,X]]) + \cdots
\end{aligned}$$

Proof. Let $C(X,Y) := \exp^{-1}(\exp X.\exp Y)$ for X, Y near 0 in $\mathfrak g$, and let $C(t) := C(tX, Y)$. Then by 4.27 we have

$$\begin{aligned}
T(\mu^{\exp(-C(t))})\tfrac{d}{dt}(\exp C(t)) &= \delta(\exp\circ C)(t).1 = \delta\exp(C(t)).\dot C(t) \\
&= \textstyle\sum_{k\ge 0} \frac{1}{(k+1)!} (\operatorname{ad}\, C(t))^k \dot C(t) = g(\operatorname{ad}\, C(t)).\dot C(t),
\end{aligned}$$

where $g(z) := \frac{e^z-1}{z} = \sum_{k\geq 0}\frac{z^k}{(k+1)!}$. We have $\exp C(t) = \exp(tX)\exp Y$ and $\exp(-C(t)) = \exp(C(t))^{-1} = \exp(-Y)\exp(-tX)$, therefore

$$\begin{aligned}
T(\mu^{\exp(-C(t))})\tfrac{d}{dt}(\exp C(t)) &= T(\mu^{\exp(-Y)\exp(-tX)})\tfrac{d}{dt}(\exp(tX)\exp Y)\\
&= T(\mu^{\exp(-tX)})T(\mu^{\exp(-Y)})T(\mu^{\exp Y})\tfrac{d}{dt}\exp(tX)\\
&= T(\mu^{\exp(-tX)}).R_X(\exp(tX)) = X, \quad \text{by 4.18.4 and 4.11.}\\
X &= g(\operatorname{ad} C(t)).\dot{C}(t).\\
e^{\operatorname{ad} C(t)} &= \operatorname{Ad}(\exp C(t)) \qquad \text{by 4.25}\\
&= \operatorname{Ad}(\exp(tX)\exp Y) = \operatorname{Ad}(\exp(tX)).\operatorname{Ad}(\exp Y)\\
&= e^{\operatorname{ad}(tX)}.e^{\operatorname{ad} Y} = e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y}.
\end{aligned}$$

If X, Y, and t are small enough we get $\operatorname{ad} C(t) = \log(e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y})$, where $\log(z) = \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}(z-1)^n$, thus we have

$$X = g(\operatorname{ad} C(t)).\dot{C}(t) = g(\log(e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y})).\dot{C}(t).$$

For z near 1 we put $f(z) := \frac{\log(z)}{z-1} = \sum_{n\geq 0}\frac{(-1)^n}{n+1}(z-1)^n$, which satisfies $g(\log(z)).f(z) = 1$. So we have

$$X = g(\log(e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y})).\dot{C}(t) = f(e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y})^{-1}.\dot{C}(t),$$

$$\begin{cases} \dot{C}(t) = f(e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y}).X, \\ C(0) = Y \end{cases}$$

Passing to the definite integral we get the desired formula

$$\begin{aligned}
C(X,Y) &= C(1) = C(0) + \int_0^1 \dot{C}(t)\,dt\\
&= Y + \int_0^1 f(e^{t.\operatorname{ad} X}.e^{\operatorname{ad} Y}).X\,dt\\
&= X + Y + \sum_{n\geq 1}\frac{(-1)^n}{n+1}\int_0^1\Bigg(\sum_{\substack{k,\ell\geq 0\\ k+\ell\geq 1}}\frac{t^k}{k!\,\ell!}(\operatorname{ad} X)^k(\operatorname{ad} Y)^\ell\Bigg)^n X\,dt\\
&= X + Y + \sum_{n\geq 1}\frac{(-1)^n}{n+1}\sum_{\substack{k_1,\dots,k_n\geq 0\\ \ell_1,\dots\ell_n\geq 0\\ k_i+\ell_i\geq 1}}\frac{(\operatorname{ad} X)^{k_1}(\operatorname{ad} Y)^{\ell_1}\dots(\operatorname{ad} X)^{k_n}(\operatorname{ad} Y)^{\ell_n}}{(k_1+\dots+k_n+1)k_1!\dots k_n!\ell_1!\dots\ell_n!}X\\
&= X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}([X,[X,Y]] - [Y,[Y,X]]) + \cdots \qquad \square
\end{aligned}$$

Remark. If G is a Lie group of differentiability class C^2, then we may define TG and the Lie bracket of vector fields. The proof above then makes sense and the theorem shows, that in the chart given by $\exp^{-1}$ the multiplication $\mu : G \times G \to G$ is C^ω near e, hence everywhere. So in this case G is a real analytic Lie group. See also remark 5.6 below.

4.30. Example. The group $SO(3,\mathbb{R})$. From 4.5 and 4.16 we know that the Lie algebra $\mathfrak{o}(3,\mathbb{R})$ of $SO(3,\mathbb{R})$ is the space $L_{\text{skew}}(\mathbb{R}^3,\mathbb{R}^3)$ of all linear mappings which are skew symmetric with respect to the inner product, with the commutator as Lie bracket.

The group $Sp(1) = S^3$ of unit quaternions has as Lie algebra $T_1S^3 = 1^\perp$, the space of imaginary quaternions, with the commutator of the quaternion multiplications as bracket. From 4.10 we see that this is $[X,Y] = 2X \times Y$.

Then we observe that the mapping

$$\alpha : \mathfrak{sp}(1) \to \mathfrak{o}(3,\mathbb{R}) = L_{\text{skew}}(\mathbb{R}^3,\mathbb{R}^3)$$
$$\alpha(X)Y = 2X \times Y$$

is an isomorphism of Lie algebras. Since S^3 is simply connected we may conclude that $Sp(1)$ is the universal cover of $SO(3)$.

We can also see this directly as follows: Consider the mapping $\tau : S^3 \subset \mathbb{H} \to SO(3,\mathbb{R})$ which is given by $\tau(P)X = PX\bar{P}$, where $X \in \mathbb{R}^3 \times \{0\} \subset \mathbb{H}$ is an imaginary quaternion. It is clearly a homomorphism $\tau : S^3 \to GL(3,\mathbb{R})$, and since $|\tau(P)X| = |PX\bar{P}| = |X|$ and S^3 is connected it has values in $SO(3,\mathbb{R})$. The tangent mapping of τ is computed as $(T_1\tau.X)Y = XY1 + 1Y(-X) = 2(X \times Y) = \alpha(X)Y$, which we already identified as an isomorphism. Thus τ is a local diffeomorphism, the image of τ is an open and compact (since S^3 is compact) subgroup of $SO(3,\mathbb{R})$, so τ is surjective since $SO(3,\mathbb{R})$ is connected. The kernel of τ is the set of all $P \in S^3$ with $PX\bar{P} = X$ for all $X \in \mathbb{R}^3$, that is the intersection of the center of $\mathbb{H}$ with S^3, the set $\{1,-1\}$. So τ is a two sheeted covering mapping.

So the universal cover of $SO(3,\mathbb{R})$ is the group $S^3 = Sp(1) = SU(2) = Spin(3)$. Here $Spin(n)$ is just a name for the universal cover of $SO(n)$, and the isomorphism $Sp(1) = SU(2)$ is just given by the fact that the quaternions can also be described as the set of all complex matrices

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \sim a1 + bj.$$

The fundamental group $\pi_1(SO(3,\mathbb{R})) = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

4.31. Example. The group $SO(4,\mathbb{R})$. We consider the smooth homomorphism $\rho : S^3\times S^3 \to SO(4,\mathbb{R})$ given by $\rho(P,Q)Z := PZ\bar{Q}$ in terms of multiplications of quaternions. The derived mapping is $\rho'(X,Y)Z = (T_{(1,1)}\rho.(X,Y))Z = XZ1 + 1Z(-Y) = XZ-ZY$, and its kernel consists of all pairs of imaginary quaternions (X,Y) with $XZ = ZY$ for all $Z \in \mathbb{H}$. If we put $Z = 1$ we get $X = Y$, then X is in the center of $\mathbb{H}$ which intersects $\mathfrak{sp}(1)$ in 0 only. So ρ' is a Lie algebra isomorphism since the dimensions are equal, and ρ is a local diffeomorphism. Its image is open and closed in $SO(4,\mathbb{R})$, so ρ is surjective, a covering mapping. The kernel of ρ is easily seen to be $\{(1,1),(-1,-1)\} \subset S^3\times S^3$. So the universal cover of $SO(4,\mathbb{R})$ is $S^3 \times S^3 = Sp(1) \times Sp(1) = Spin(4)$, and the fundamental group $\pi_1(SO(4,\mathbb{R})) = \mathbb{Z}_2$ again.

5. Lie Groups II. Lie Subgroups and Homogeneous Spaces

5.1. Definition. Let G be a Lie group. A subgroup H of G is called a *Lie subgroup*, if H is itself a Lie group (so it is separable) and the inclusion $i : H \to G$ is smooth.

In this case the inclusion is even an immersion. For that it suffices to check that $T_e i$ is injective: If $X \in \mathfrak{h}$ is in the kernel of $T_e i$, then $i \circ \exp^H(tX) = \exp^G(t.T_e i.X) = e$. Since i is injective, $X = 0$.

From the next result it follows that $H \subset G$ is then an initial submanifold in the sense of 2.14: If H_0 is the connected component of H, then $i(H_0)$ is the Lie subgroup of G generated by $i'(\mathfrak{h}) \subset \mathfrak{g}$, which is an initial submanifold, and this is true for all components of H.

5.2. Theorem. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, then there is a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$. H is an initial submanifold.*

Proof. Put $E_x := \{T_e(\mu_x).X : X \in \mathfrak{h}\} \subset T_x G$. Then $E := \bigsqcup_{x \in G} E_x$ is a distribution of constant rank on G, in the sense of 3.18. The set $\{L_X : X \in \mathfrak{h}\}$ is an involutive set in the sense of 3.23 which spans E. So by theorem 3.25 the distribution E is integrable and by theorem 3.22 the leaf H through e is an initial submanifold. It is even a subgroup, since for $x \in H$ the initial submanifold $\mu_x H$ is again a leaf (since E is left invariant) and intersects H (in x), so $\mu_x(H) = H$. Thus $H.H = H$ and consequently $H^{-1} = H$. The multiplication $\mu : H \times H \to G$ is smooth by restriction, and smooth as a mapping $H \times H \to H$, since H is an initial submanifold, by lemma 2.17. $\square$

5.3. Theorem. *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Then there exists a connected Lie group G whose Lie algebra is $\mathfrak{g}$.*

Sketch of Proof. By the theorem of Ado (see [Jacobson, 1962, p??] or [Varadarajan, 1974, p 237]) $\mathfrak{g}$ has a faithful (i.e. injective) representation on a finite dimensional vector space V, i.e. $\mathfrak{g}$ can be viewed as a Lie subalgebra of $\mathfrak{gl}(V) = L(V, V)$. By theorem 5.2 above there is a Lie subgroup G of $GL(V)$ with $\mathfrak{g}$ as its Lie algebra. $\square$

This is a rather involved proof, since the theorem of Ado needs the structure theory of Lie algebras for its proof. There are simpler proofs available, starting from a neighborhood of e in G (a neighborhood of 0 in $\mathfrak{g}$ with the Baker-Campbell-Hausdorff formula 4.29 as multiplication) and extending it.

5.4. Theorem. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $f : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras. Then there is a Lie group homomorphism φ, locally defined near e, from G to H, such that $\varphi' = T_e\varphi = f$. If G is simply connected, then there is a globally defined homomorphism of Lie groups $\varphi : G \to H$ with this property.*

Proof. Let $\mathfrak{k} := \operatorname{graph}(f) \subset \mathfrak{g} \times \mathfrak{h}$. Then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$, since f is a homomorphism of Lie algebras. $\mathfrak{g} \times \mathfrak{h}$ is the Lie algebra of $G \times H$, so by theorem 5.2 there is a connected Lie subgroup $K \subset G \times H$ with algebra $\mathfrak{k}$. We consider the homomorphism $g := pr_1 \circ incl : K \to G \times H \to G$, whose tangent mapping satisfies $T_e g(X, f(X)) = T_{(e,e)}pr_1.T_e incl.(X, f(X)) = X$, so is invertible. Thus g is a local diffeomorphism, so $g : K \to G_0$ is a covering of the connected component G_0 of e in G. If G is simply connected, g is an isomorphism. Now we consider the homomorphism $\psi := pr_2 \circ incl : K \to G \times H \to H$, whose tangent mapping satisfies $T_e\psi.(X, f(X)) = f(X)$. We see that $\varphi := \psi \circ (g \restriction U)^{-1} : G \supset U \to H$ solves the problem, where U is an e-neighborhood in K such that $g \restriction U$ is a diffeomorphism. If G is simply connected, $\varphi = \psi \circ g^{-1}$ is the global solution. $\square$

5.5. Theorem. *Let H be a closed subgroup of a Lie group G. Then H is a Lie subgroup and a submanifold of G.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. We consider the subset $\mathfrak{h} := \{c'(0) : c \in C^\infty(\mathbb{R}, G), c(\mathbb{R}) \subset H, c(0) = e\}$.

Claim 1. $\mathfrak{h}$ is a linear subspace.

If $c_i'(0) \in \mathfrak{h}$ and $t_i \in \mathbb{R}$, we define $c(t) := c_1(t_1.t).c_2(t_2.t)$. Then $c'(0) = T_{(e,e)}\mu.(t_1.c_1'(0), t_2.c_2'(0)) = t_1.c_1'(0) + t_2.c_2'(0) \in \mathfrak{h}$.

Claim 2. $\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}$.

Clearly we have '$\supseteq$'. To check the other inclusion, let $X = c'(0) \in \mathfrak{h}$ and consider $v(t) := (\exp^G)^{-1}c(t)$ for small t. Then we have $X = c'(0) = \frac{d}{dt}|_0 \exp(v(t)) = v'(0) = \lim_{n\to\infty} n.v(\frac{1}{n})$. We put $t_n = \frac{1}{n}$ and $X_n = n.v(\frac{1}{n})$, so that $\exp(t_n.X_n) = \exp(v(\frac{1}{n})) = c(\frac{1}{n}) \in H$. By claim 3 below we then get $\exp(tX) \in H$ for all t.

Claim 3. Let $X_n \to X$ in $\mathfrak{g}$, $0 < t_n \to 0$ in $\mathbb{R}$ with $\exp(t_nX_n) \in H$. Then $\exp(tX) \in H$ for all $t \in R$.

Let $t \in \mathbb{R}$ and take $m_n \in (\frac{t}{t_n} - 1, \frac{t}{t_n}] \cap \mathbb{Z}$. Then $t_n.m_n \to t$ and $m_n.t_n.X_n \to tX$, and since H is closed we may conclude that

$$\exp(tX) = \lim_n \exp(m_n.t_n.X_n) = \lim_n \exp(t_n.X_n)^{m_n} \in H.$$

Claim 4. Let $\mathfrak{k}$ be a complementary linear subspace for $\mathfrak{h}$ in $\mathfrak{g}$. Then there is an open 0-neighborhood W in $\mathfrak{k}$ such that $\exp(W) \cap H = \{e\}$.

If not there are $0 \neq Y_k \in \mathfrak{k}$ with $Y_k \to 0$ such that $\exp(Y_k) \in H$. Choose a norm $|\ \ |$ on $\mathfrak{g}$ and let $X_n = Y_n/|Y_n|$. Passing to a subsequence we may assume that $X_n \to X$ in $\mathfrak{k}$, then $|X| = 1$. But $\exp(|Y_n|.X_n) = \exp(Y_n) \in H$ and $0 < |Y_n| \to 0$, so by claim 3 we have $\exp(tX) \in H$ for all $t \in \mathbb{R}$. So by claim 2 $X \in \mathfrak{h}$, a contradiction.

Claim 5. Put $\varphi : \mathfrak{h} \times \mathfrak{k} \to G$, $\varphi(X,Y) = \exp X.\exp Y$. Then there are 0-neighborhoods V in $\mathfrak{h}$, W in $\mathfrak{k}$, and an e-neighborhood U in G such that $\varphi : V \times W \to U$ is a diffeomorphism and $U \cap H = \exp(V)$.

Choose V, W, and U so small that φ becomes a diffeomorphism. By claim 4 W may be chosen so small that $\exp(W) \cap H = \{e\}$. By claim 2 we have $\exp(V) \subseteq H \cap U$. Let $x \in H \cap U$. Since $x \in U$ we have $x = \exp X.\exp Y$ for unique $(X,Y) \in V \times W$. Then x and $\exp X \in H$, so $\exp Y \in H \cap \exp(W)$, thus $Y = 0$. So $x = \exp X \in \exp(V)$.

Claim 6. H is a submanifold and a Lie subgroup.

$(U, (\varphi \restriction V \times W)^{-1} =: u)$ is a submanifold chart for H centered at e by claim 5. For $x \in H$ the pair $(\mu_x(U), u \circ \mu_{x^{-1}})$ is a submanifold chart for H centered at x. So H is a closed submanifold of G, and the multiplication is smooth since it is a restriction. $\square$

5.6. Remark. The following stronger results on subgroups and the relation between topological groups and Lie groups in general are available.

Any arc wise connected subgroup of a Lie group is a connected Lie subgroup, [Yamabe, 1950].

Let G be a separable locally compact topological group. If it has an e-neighborhood which does not contain a proper subgroup, then G is a Lie group. This is the solution of the 5-th problem of Hilbert, see the book [Montgomery-Zippin, 1955, p. 107].

Any subgroup H of a Lie group G has a coarsest Lie group structure, but it might be non separable. To indicate a proof of this statement, consider all continuous curves $c : \mathbb{R} \to G$ with $c(\mathbb{R}) \subset H$, and equip H with the final topology with respect to them. Then the component of the identity satisfies the conditions of the Gleason-Yamabe theorem cited above.

5.7. Let $\mathfrak{g}$ be a Lie algebra. An *ideal* $\mathfrak{k}$ in $\mathfrak{g}$ is a linear subspace $\mathfrak{k}$ such that $[\mathfrak{k}, \mathfrak{g}] \subset \mathfrak{k}$. Then the quotient space $\mathfrak{g}/\mathfrak{k}$ carries a unique Lie algebra structure such that $\mathfrak{g} \to \mathfrak{g}/\mathfrak{k}$ is a Lie algebra homomorphism.

Lemma. *A connected Lie subgroup H of a connected Lie group G is a normal subgroup if and only if its Lie algebra $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

Proof. H normal in G means $xHx^{-1} = conj_x(H) \subset H$ for all $x \in G$. By remark 4.20 this is equivalent to $T_e(conj_x)(\mathfrak{h}) \subset \mathfrak{h}$, i.e. $\mathrm{Ad}(x)\mathfrak{h} \subset \mathfrak{h}$, for all $x \in G$. But this in turn is equivalent to $\operatorname{ad}(X)\mathfrak{h} \subset \mathfrak{h}$ for all $X \in \mathfrak{g}$, so to the fact that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. □

5.8. Let G be a connected Lie group. If $A \subset G$ is an arbitrary subset, the *centralizer* of A in G is the closed subgroup $Z_A := \{x \in G : xa = ax \text{ for all } a \in A\}$.

The Lie algebra $\mathfrak{z}_A$ of Z_A consists of all $X \in \mathfrak{g}$ such that $a.\exp(tX).a^{-1} = \exp(tX)$ for all $a \in A$, i.e. $\mathfrak{z}_A = \{X \in \mathfrak{g} : \operatorname{Ad}(a)X = X \text{ for all } a \in A\}$.

If A is itself a connected Lie subgroup of G, then $\mathfrak{z}_A = \{X \in \mathfrak{g} : \operatorname{ad}(Y)X = 0 \text{ for all } Y \in \mathfrak{a}\}$. This set is also called the *centralizer* of $\mathfrak{a}$ in $\mathfrak{g}$. If $A = G$ then Z_G is called the *center* of G and $\mathfrak{z}_G = \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$ is then the *center* of the Lie algebra $\mathfrak{g}$.

5.9. The *normalizer* of a subset A of a connected Lie group G is the subgroup $N_A = \{x \in G : \mu_x(A) = \mu^x(A)\} = \{x \in G : conj_x(A) = A\}$. If A is closed then N_A is also closed.

If A is a connected Lie subgroup of G then $N_A = \{x \in G : \operatorname{Ad}(x)\mathfrak{a} \subset \mathfrak{a}\}$ and its Lie algebra is $\mathfrak{n}_A = \{X \in \mathfrak{g} : \operatorname{ad}(X)\mathfrak{a} \subset \mathfrak{a}\}$ is then the *idealizer* of $\mathfrak{a}$ in $\mathfrak{g}$.

5.10. Group actions. A *left action* of a Lie group G on a manifold M is a smooth mapping $\ell : G \times M \to M$ such that $\ell_x \circ \ell_y = \ell_{xy}$ and $\ell_e = Id_M$, where $\ell_x(z) = \ell(x, z)$.

A *right action* of a Lie group G on a manifold M is a smooth mapping $r : M \times G \to M$ such that $r^x \circ r^y = r^{yx}$ and $r^e = Id_M$, where $r^x(z) = r(z, x)$.

A G-space is a manifold M together with a right or left action of G on M.

We will describe the following notions only for a left action of G on M. They make sense also for right actions.

The *orbit* through $z \in M$ is the set $G.z = \ell(G, z) \subset M$. The action is called *transitive*, if M is one orbit, i.e. for all $z, w \in M$ there is some $g \in G$ with $g.z = w$. The action is called *free*, if $g_1.z = g_2.z$ for some $z \in M$ implies already $g_1 = g_2$. The action is called *effective*, if $\ell_x = \ell_y$ implies $x = y$, i.e. if $\ell : G \to \operatorname{Diff}(M)$ is injective, where $\operatorname{Diff}(M)$ denotes the group of all diffeomorphisms of M.

More generally, a continuous transformation group of a topological space M is a pair (G, M) where G is a topological group and where to each element $x \in G$ there is given a homeomorphism ℓ_x of M such that $\ell : G \times M \to M$ is continuous, and $\ell_x \circ \ell_y = \ell_{xy}$. The continuity is an obvious geometrical requirement, but in accordance with the general observation that group properties often force more regularity than explicitly postulated (cf. 5.6), differentiability follows in many situations. So, if G is locally compact, M is a smooth or real analytic manifold, all ℓ_x are smooth or real analytic homeomorphisms and the action is effective, then G is a Lie group and ℓ is smooth or real analytic, respectively, see [Montgomery, Zippin, 55, p. 212].

5.11. Homogeneous spaces. Let G be a Lie group and let $H \subset G$ be a closed subgroup. By theorem 5.5 H is a Lie subgroup of G. We denote by G/H the space of all right cosets of G, i.e. $G/H = \{xH : x \in G\}$. Let $p : G \to G/H$ be the projection. We equip G/H with the quotient topology, i.e. $U \subset G/H$ is open if and only if $p^{-1}(U)$ is open in G. Since H is closed, G/H is a Hausdorff space.

G/H is called a *homogeneous space* of G. We have a left action of G on G/H, which is induced by the left translation and is given by $\bar{\mu}_x(zH) = xzH$.

Theorem. *If H is a closed subgroup of G, then there exists a unique structure of a smooth manifold on G/H such that $p : G \to G/H$ is a submersion. So $\dim G/H = \dim G - \dim H$.*

Proof. Surjective submersions have the universal property 2.4, thus the manifold structure on G/H is unique, if it exists. Let $\mathfrak{h}$ be the Lie algebra of the Lie subgroup H. We choose a complementary linear subspace $\mathfrak{k}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$.
Claim 1. We consider the mapping $f : \mathfrak{k} \times H \to G$, given by $f(X, h) := \exp X.h$. Then there is an open 0-neighborhood W in $\mathfrak{k}$ and an open e-neighborhood U in G such that $f : W \times H \to U$ is a diffeomorphism.
By claim 5 in the proof of theorem 5.5 there are open 0-neighborhoods V in $\mathfrak{h}$, W' in $\mathfrak{k}$, and an open e-neighborhood U' in G such that $\varphi : W' \times V \to U'$ is a diffeomorphism, where $\varphi(X, Y) = \exp X. \exp Y$, and such that $U' \cap H = \exp V$. Now we choose W in $W' \subset \mathfrak{k}$ so small that $\exp(W)^{-1}. \exp(W) \subset U'$. We will check that this W satisfies claim 1.
Claim 2. $f \restriction W \times H$ is injective.
$f(X_1, h_1) = f(X_2, h_2)$ means $\exp X_1.h_1 = \exp X_2.h_2$, consequently we have $h_2 h_1^{-1} = (\exp X_2)^{-1} \exp X_1 \in \exp(W)^{-1} \exp(W) \cap H \subset U' \cap H = \exp V$. So there is a unique $Y \in V$ with $h_2 h_1^{-1} = \exp Y$. But then $\varphi(X_1, 0) = \exp X_1 = \exp X_2.h_2.h_1^{-1} = \exp X_2. \exp Y = \varphi(X_2, Y)$. Since φ is injective, $X_1 = X_2$ and $Y = 0$, so $h_1 = h_2$.
Claim 3. $f \restriction W \times H$ is a local diffeomorphism.
The diagram

$$\begin{tikzcd}
W \times V \arrow[r, "Id \times \exp"] \arrow[d, "\varphi"'] & W \times (U' \cap H) \arrow[d, "f"] \\
\varphi(W \times V) \arrow[r, "incl"] & U'
\end{tikzcd}$$

commutes, and $Id_W \times \exp$ and φ are diffeomorphisms. So $f \restriction W \times (U' \cap H)$ is a diffeomorphism. Since $f(X, h) = f(X, e).h$ we conclude that $f \restriction W \times H$

is everywhere a local diffeomorphism. So finally claim 1 follows, where $U = f(W \times H)$.

Now we put $g := p \circ (\exp \restriction W) : \mathfrak{k} \supset W \to G/H$. Then the following diagram commutes:

$$\begin{array}{ccc} W \times H & \xrightarrow{f} & U \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{g} & G/H. \end{array}$$

Claim 4. g is a homeomorphism onto $p(U) =: \bar{U} \subset G/H$.
Clearly g is continuous, and g is open, since p is open. If $g(X_1) = g(X_2)$ then $\exp X_1 = \exp X_2.h$ for some $h \in H$, so $f(X_1, e) = f(X_2, h)$. By claim 1 we get $X_1 = X_2$, so g is injective. Finally $g(W) = \bar{U}$, so claim 4 follows.

For $a \in G$ we consider $\bar{U}_a = \bar{\mu}_a(\bar{U}) = a.\bar{U}$ and the mapping $u_a := g^{-1} \circ \bar{\mu}_{a^{-1}} : \bar{U}_a \to W \subset \mathfrak{k}$.
Claim 5. $(\bar{U}_a, u_a = g^{-1} \circ \bar{\mu}_{a^{-1}} : \bar{U}_a \to W)_{a \in G}$ is a smooth atlas for G/H.
Let a, $b \in G$ such that $\bar{U}_a \cap \bar{U}_b \neq \emptyset$. Then

$$\begin{aligned} u_a \circ u_b^{-1} &= g^{-1} \circ \bar{\mu}_{a^{-1}} \circ \bar{\mu}_b \circ g : u_b(\bar{U}_a \cap \bar{U}_b) \to u_a(\bar{U}_a \cap \bar{U}_b) \\ &= g^{-1} \circ \bar{\mu}_{a^{-1}b} \circ p \circ (\exp \restriction W) \\ &= g^{-1} \circ p \circ \mu_{a^{-1}b} \circ (\exp \restriction W) \\ &= pr_1 \circ f^{-1} \circ \mu_{a^{-1}b} \circ (\exp \restriction W) \quad \text{is smooth.} \quad \square \end{aligned}$$

5.12. Let $\ell : G \times M \to M$ be a left action. Then we have partial mappings $\ell_a : M \to M$ and $\ell^x : G \to M$, given by $\ell_a(x) = \ell^x(a) = \ell(a, x) = a.x$.

For any $X \in \mathfrak{g}$ we define the *fundamental vector field* $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ by $\zeta_X(x) = T_e(\ell^x).X = T_{(e,x)}\ell.(X, 0_x)$.

Lemma. *In this situation the following assertions hold:*

(1) $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ *is a linear mapping.*
(2) $T_x(\ell_a).\zeta_X(x) = \zeta_{\mathrm{Ad}(a)X}(a.x)$.
(3) $R_X \times 0_M \in \mathfrak{X}(G \times M)$ *is ℓ-related to* $\zeta_X \in \mathfrak{X}(M)$.
(4) $[\zeta_X, \zeta_Y] = -\zeta_{[X,Y]}$.

Proof. (1) is clear.

(2) We have $\ell_a \ell^x(b) = abx = aba^{-1}ax = \ell^{ax} conj_a(b)$, so

$$\begin{aligned} T_x(\ell_a).\zeta_X(x) &= T_x(\ell_a).T_e(\ell^x).X = T_e(\ell_a \circ \ell^x).X \\ &= T_e(\ell^{ax}).\mathrm{Ad}(a).X = \zeta_{\mathrm{Ad}(a)X}(ax). \end{aligned}$$

(3) We have $\ell\circ(Id\times\ell_a)=\ell\circ(\mu^a\times Id):G\times M\to M$, so

$$\begin{aligned}\zeta_X(\ell(a,x))&=T_{(e,ax)}\ell.(X,0_{ax})=T\ell.(Id\times T(\ell_a)).(X,0_x)\\&=T\ell.(T(\mu^a)\times Id).(X,0_x)=T\ell.(R_X\times 0_M)(a,x).\end{aligned}$$

(4) $[R_X\times 0_M,R_Y\times 0_M]=[R_X,R_Y]\times 0_M=-R_{[X,Y]}\times 0_M$ is ℓ-related to $[\zeta_X,\zeta_Y]$ by (3) and by 3.10. On the other hand $-R_{[X,Y]}\times 0_M$ is ℓ-related to $-\zeta_{[X,Y]}$ by (3) again. Since ℓ is surjective we get $[\zeta_X,\zeta_Y]=-\zeta_{[X,Y]}$. □

5.13. Let $r:M\times G\to M$ be a right action, so $\check r:G\to \operatorname{Diff}(M)$ is a group anti homomorphism. We will use the following notation: $r^a:M\to M$ and $r_x:G\to M$, given by $r_x(a)=r^a(x)=r(x,a)=x.a$.

For any $X\in\mathfrak g$ we define the *fundamental vector field* $\zeta_X=\zeta^M_X\in\mathfrak X(M)$ by $\zeta_X(x)=T_e(r_x).X=T_{(x,e)}r.(0_x,X)$.

Lemma. *In this situation the following assertions hold:*

(1) $\zeta:\mathfrak g\to\mathfrak X(M)$ *is a linear mapping.*
(2) $T_x(r^a).\zeta_X(x)=\zeta_{\operatorname{Ad}(a^{-1})X}(x.a)$.
(3) $0_M\times L_X\in\mathfrak X(M\times G)$ *is r-related to* $\zeta_X\in\mathfrak X(M)$.
(4) $[\zeta_X,\zeta_Y]=\zeta_{[X,Y]}$. □

5.14. Theorem. *Let $\ell:G\times M\to M$ be a smooth left action. For $x\in M$ let $G_x=\{a\in G:ax=x\}$ be the isotropy subgroup of x in G, a closed subgroup of G. Then $\ell^x:G\to M$ factors over $p:G\to G/G_x$ to an injective immersion $i^x:G/G_x\to M$, which is G-equivariant, i.e. $\ell_a\circ i^x=i^x\circ\bar\mu_a$ for all $a\in G$. The image of i^x is the orbit through x.*

The fundamental vector fields span an integrable distribution on M in the sense of 3.20. Its leaves are the connected components of the orbits, and each orbit is an initial submanifold.

Proof. Clearly ℓ^x factors over p to an injective mapping $i^x:G/G_x\to M$; by the universal property of surjective submersions i^x is smooth, and obviously it is equivariant. Thus $T_{p(a)}(i^x).T_{p(e)}(\bar\mu_a)=T_{p(e)}(i^x\circ\bar\mu_a)=T_{p(e)}(\ell_a\circ i^x)=T_x(\ell_a).T_{p(e)}(i^x)$ for all $a\in G$ and it suffices to show that $T_{p(e)}(i^x)$ is injective.

Let $X\in\mathfrak g$ and consider its fundamental vector field $\zeta_X\in\mathfrak X(M)$. By 3.14 and 5.12.3 we have

$$\ell(\exp(tX),x)=\ell(\operatorname{Fl}^{R_X\times 0_M}_t(e,x))=\operatorname{Fl}^{\zeta_X}_t(\ell(e,x))=\operatorname{Fl}^{\zeta_X}_t(x).$$

So $\exp(tX)\in G_x$, i.e. $X\in\mathfrak g_x$, if and only if $\zeta_X(x)=0_x$. In other words, $0_x=\zeta_X(x)=T_e(\ell^x).X=T_{p(e)}(i^x).T_ep.X$ if and only if $T_ep.X=0_{p(e)}$. Thus i^x is an immersion.

Since the connected components of the orbits are integral manifolds, the fundamental vector fields span an integrable distribution in the sense of 3.20; but also the condition 3.25.2 is satisfied. So by theorem 3.22 each orbit is an initial submanifold in the sense of 2.14. □

5.15. Semidirect products of Lie groups. Let H and K be two Lie groups and let $\ell : H \times K \to K$ be a smooth left action of H in K such that each $\ell_h : K \to K$ is a group homomorphism. So the associated mapping $\check{\ell} : H \to \operatorname{Aut}(K)$ is a smooth homomorphism into the automorphism group of K. Then we can introduce the following multiplication on $K \times H$

$$(1) \qquad (k,h)(k',h') := (k\ell_h(k'), hh').$$

It is easy to see that this defines a Lie group $G = K \ltimes_\ell H$ called the *semidirect product* of H and K with respect to ℓ. If the action ℓ is clear from the context we write $G = K \ltimes H$ only. The second projection $pr_2 : K \ltimes H \to H$ is a surjective smooth homomorphism with kernel $K \times \{e\}$, and the insertion $\operatorname{ins}_e : H \to K \ltimes H$, $\operatorname{ins}_e(h) = (e,h)$ is a smooth group homomorphism with $pr_2 \circ \operatorname{ins}_e = Id_H$.

Conversely we consider an exact sequence of Lie groups and homomorphisms

$$(2) \qquad \{e\} \to K \xrightarrow{j} G \xrightarrow{p} H \to \{e\}.$$

So j is injective, p is surjective, and the kernel of p equals the image of j. We suppose furthermore that the sequence splits, so that there is a smooth homomorphism $i : H \to G$ with $p \circ i = Id_H$. Then the rule $\ell_h(k) = i(h)ki(h^{-1})$ (where we suppress j) defines a left action of H on K by automorphisms. It is easily seen that the mapping $K \ltimes_\ell H \to G$ given by $(k,h) \mapsto ki(h)$ is an isomorphism of Lie groups. So we see that semidirect products of Lie groups correspond exactly to splitting short exact sequences.

5.16. The tangent group of a Lie group. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We will use the notation from 4.1. First note that TG is also a Lie group with multiplication $T\mu$ and inversion $T\nu$, given by (see 4.2) $T_{(a,b)}\mu.(\xi_a, \eta_b) = T_a(\mu^b).\xi_a + T_b(\mu_a).\eta_b$ and $T_a\nu.\xi_a = -T_e(\mu_{a^{-1}}).T_a(\mu^{a^{-1}}).\xi_a$.

Lemma. *Via the isomorphism $T\rho : \mathfrak{g} \times G \to TG$, $T\rho.(X,g) = T_e(\mu^g).X$, the group structure on TG looks as follows: $(X,a).(Y,b) = (X + \operatorname{Ad}(a)Y, a.b)$ and $(X,a)^{-1} = (-\operatorname{Ad}(a^{-1})X, a^{-1})$. So TG is isomorphic to the semidirect product $\mathfrak{g} \ltimes G$.*

Proof. $T_{(a,b)}\mu.(T\mu^a.X, T\mu^b.Y) = T\mu^b.T\mu^a.X + T\mu_a.T\mu^b.Y =$
$= T\mu^{ab}.X + T\mu^b.T\mu^a.T\mu^{a^{-1}}.T\mu_a.Y = T\mu^{ab}(X + \operatorname{Ad}(a)Y)$.
$T_a\nu.T\mu^a.X = -T\mu^{a^{-1}}.T\mu_{a^{-1}}.T\mu^a.X = -T\mu^{a^{-1}}.\operatorname{Ad}(a^{-1})X$. □

Remark. In the left trivialisation $T\lambda : G \times \mathfrak{g} \to TG$, $T\lambda.(g, X) = T_e(\mu_g).X$, the semidirect product structure looks somewhat awkward: $(a, X).(b, Y) = (ab, \operatorname{Ad}(b^{-1})X + Y)$ and $(a, X)^{-1} = (a^{-1}, -\operatorname{Ad}(a)X)$.

6. Vector Bundles

6.1. Vector bundles. Let $p : E \to M$ be a smooth mapping between manifolds. By a *vector bundle chart* on (E, p, M) we mean a pair (U, ψ), where U is an open subset in M and where ψ is a fiber respecting diffeomorphism as in the following diagram:

$$\begin{tikzcd}
E \restriction U := p^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & U \times V \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}$$

Here V is a fixed finite dimensional vector space, called the *standard fiber* or the *typical fiber*, real for the moment.

Two vector bundle charts (U_1, ψ_1) and (U_2, ψ_2) are called *compatible*, if $\psi_1 \circ \psi_2^{-1}$ is a fiber linear isomorphism, i.e. $(\psi_1 \circ \psi_2^{-1})(x, v) = (x, \psi_{1,2}(x)v)$ for some mapping $\psi_{1,2} : U_{1,2} := U_1 \cap U_2 \to GL(V)$. The mapping $\psi_{1,2}$ is then unique and smooth, and it is called the *transition function* between the two vector bundle charts.

A *vector bundle atlas* $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ for (E, p, M) is a set of pairwise compatible vector bundle charts (U_α, ψ_α) such that $(U_\alpha)_{\alpha \in A}$ is an open cover of M. Two vector bundle atlases are called *equivalent*, if their union is again a vector bundle atlas.

A *vector bundle* (E, p, M) consists of manifolds E (the *total space*), M (the *base*), and a smooth mapping $p : E \to M$ (the *projection*) together with an equivalence class of vector bundle atlases: So we must know at least one vector bundle atlas. p turns out to be a surjective submersion.

6.2. Let us fix a vector bundle (E, p, M) for the moment. On each *fiber* $E_x := p^{-1}(x)$ (for $x \in M$) there is a unique structure of a real vector space, induced from any vector bundle chart (U_α, ψ_α) with $x \in U_\alpha$. So $0_x \in E_x$ is a special element and $0 : M \to E$, $0(x) = 0_x$, is a smooth mapping, the *zero section*.

A *section* u of (E, p, M) is a smooth mapping $u : M \to E$ with $p \circ u = Id_M$. The *support* of the section u is the closure of the set $\{x \in M : u(x) \neq 0_x\}$ in M. The space of all smooth sections of the bundle (E, p, M) will be denoted by either $C^\infty(E) = C^\infty(E, p, M) = C^\infty(E \to M)$. Clearly it is a vector space with fiber wise addition and scalar multiplication.

If $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ is a vector bundle atlas for (E, p, M), then any smooth mapping $f_\alpha : U_\alpha \to V$ (the standard fiber) defines a local section $x \mapsto \psi_\alpha^{-1}(x, f_\alpha(x))$ on U_α. If $(g_\alpha)_{\alpha \in A}$ is a partition of unity subordinated to (U_α), then a global section can be formed by $x \mapsto \sum_\alpha g_\alpha(x) \cdot \psi_\alpha^{-1}(x, f_\alpha(x))$. So a smooth vector bundle has 'many' smooth sections.

6.3. We will now give a formal description of the amount of vector bundles with fixed base M and fixed standard fiber V.

Let us first fix an open cover $(U_\alpha)_{\alpha\in A}$ of M. If (E,p,M) is a vector bundle which admits a vector bundle atlas (U_α, ψ_α) with the given open cover, then we have $\psi_\alpha \circ \psi_\beta^{-1}(x,v) = (x, \psi_{\alpha\beta}(x)v)$ for transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to GL(V)$, which are smooth. This family of transition functions satisfies

$$(1) \qquad \begin{cases} \psi_{\alpha\beta}(x) \cdot \psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x) & \text{for each } x \in U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma \\ \psi_{\alpha\alpha}(x) = e & \text{for all } x \in U_\alpha \end{cases}$$

Condition (1) is called a *cocycle condition* and thus we call the family $(\psi_{\alpha\beta})$ the *cocycle of transition functions* for the vector bundle atlas (U_α, ψ_α).

Let us suppose now that the same vector bundle (E,p,M) is described by an equivalent vector bundle atlas $(U_\alpha, \varphi_\alpha)$ with the same open cover (U_α). Then the vector bundle charts (U_α, ψ_α) and $(U_\alpha, \varphi_\alpha)$ are compatible for each α, so $\varphi_\alpha \circ \psi_\alpha^{-1}(x,v) = (x, \tau_\alpha(x)v)$ for some $\tau_\alpha : U_\alpha \to GL(V)$. But then we have

$$\begin{aligned} (x, \tau_\alpha(x)\psi_{\alpha\beta}(x)v) &= (\varphi_\alpha \circ \psi_\alpha^{-1})(x, \psi_{\alpha\beta}(x)v) \\ &= (\varphi_\alpha \circ \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_\beta^{-1})(x,v) = (\varphi_\alpha \circ \psi_\beta^{-1})(x,v) \\ &= (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \psi_\beta^{-1})(x,v) = (x, \varphi_{\alpha\beta}(x)\tau_\beta(x)v). \end{aligned}$$

So we get

$$(2) \qquad \tau_\alpha(x)\psi_{\alpha\beta}(x) = \varphi_{\alpha\beta}(x)\tau_\beta(x) \quad \text{for all } x \in U_{\alpha\beta}.$$

We say that the two cocycles $(\psi_{\alpha\beta})$ and $(\varphi_{\alpha\beta})$ of transition functions over the cover (U_α) are *cohomologous.* The *cohomology classes* of cocycles $(\psi_{\alpha\beta})$ over the open cover (U_α) (where we identify cohomologous ones) form a set $\check{H}^1((U_\alpha), \underline{GL}(V))$ the first *Čech cohomology set* of the open cover (U_α) with values in the sheaf $C^\infty(\quad, GL(V)) =: \underline{GL}(V)$.

Now let $(W_i)_{i\in I}$ be an open cover of M that refines (U_α) with $W_i \subset U_{\varepsilon(i)}$, where $\varepsilon : I \to A$ is some refinement mapping, then for any cocycle $(\psi_{\alpha\beta})$ over (U_α) we define the cocycle $\varepsilon^*(\psi_{\alpha\beta}) =: (\varphi_{ij})$ by the prescription $\varphi_{ij} := \psi_{\varepsilon(i),\varepsilon(j)} \restriction W_{ij}$. The mapping ε^* respects the cohomology relations and induces therefore a mapping $\varepsilon^\sharp : \check{H}^1((U_\alpha), \underline{GL}(V)) \to \check{H}^1((W_i), \underline{GL}(V))$. One can show that the mapping ε^* depends on the choice of the refinement mapping ε only up to cohomology (use $\tau_i = \psi_{\varepsilon(i),\eta(i)} \restriction W_i$ if ε and η are two refinement mappings), so we may form the inductive limit $\varinjlim \check{H}^1(\mathcal{U}, \underline{GL}(V)) =: \check{H}^1(M, \underline{GL}(V))$ over all open covers of M directed by refinement.

Theorem. *There is a bijective correspondence between $\check{H}^1(M, \underline{GL}(V))$ and the set of all isomorphism classes of vector bundles over M with typical fiber V.*

Proof. Let $(\psi_{\alpha\beta})$ be a cocycle of transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} \to GL(V)$ over some open cover (U_α) of M. We consider the disjoint union $\bigsqcup_{\alpha\in A}\{\alpha\} \times U_\alpha \times V$ and the following relation on it: $(\alpha, x, v) \sim (\beta, y, w)$ if and only if $x = y$ and $\psi_{\beta\alpha}(x)v = w$.

By the cocycle property (1) of $(\psi_{\alpha\beta})$ this is an equivalence relation. The space of all equivalence classes is denoted by $E = VB(\psi_{\alpha\beta})$ and it is equipped with the quotient topology. We put $p : E \to M$, $p[(\alpha, x, v)] = x$, and we define the vector bundle charts (U_α, ψ_α) by $\psi_\alpha[(\alpha, x, v)] = (x, v)$, $\psi_\alpha : p^{-1}(U_\alpha) =: E \restriction U_\alpha \to U_\alpha \times V$. Then the mapping $\psi_\alpha \circ \psi_\beta^{-1}(x, v) = \psi_\alpha[(\beta, x, v)] = \psi_\alpha[(\alpha, x, \psi_{\alpha\beta}(x)v)] = (x, \psi_{\alpha\beta}(x)v)$ is smooth, so E becomes a smooth manifold. E is Hausdorff: let $u \neq v$ in E; if $p(u) \neq p(v)$ we can separate them in M and take the inverse image under p; if $p(u) = p(v)$, we can separate them in one chart. So (E, p, M) is a vector bundle.

Now suppose that we have two cocycles $(\psi_{\alpha\beta})$ over (U_α), and (φ_{ij}) over (V_i). Then there is a common refinement (W_γ) for the two covers (U_α) and (V_i). The construction described a moment ago gives isomorphic vector bundles if we restrict the cocycle to a finer open cover. So we may assume that $(\psi_{\alpha\beta})$ and $(\varphi_{\alpha\beta})$ are cocycles over the same open cover (U_α). If the two cocycles are cohomologous, so $\tau_\alpha \cdot \psi_{\alpha\beta} = \varphi_{\alpha\beta} \cdot \tau_\beta$ on $U_{\alpha\beta}$, then a fiber linear diffeomorphism $\tau : VB(\psi_{\alpha\beta}) \to VB(\varphi_{\alpha\beta})$ is given by $\varphi_\alpha \tau[(\alpha, x, v)] = (x, \tau_\alpha(x)v)$. By relation (2) this is well defined, so the vector bundles $VB(\psi_{\alpha\beta})$ and $VB(\varphi_{\alpha\beta})$ are isomorphic.

Most of the converse direction was already shown in the discussion before the theorem, and the argument can be easily refined to show also that isomorphic bundles give cohomologous cocycles. □

Remark. If $GL(V)$ is an abelian group (only if V is of real or complex dimension 1), then $\check{H}^1(M, \underline{GL}(V))$ is a usual cohomology group with coefficients in the sheaf $\underline{GL}(V)$ and it can be computed with the methods of algebraic topology. If $GL(V)$ is not abelian, then the situation is rather mysterious: there is no clear definition for $\check{H}^2(M, \underline{GL}(V))$ for example. So $\check{H}^1(M, \underline{GL}(V))$ is more a notation than a mathematical concept.

A coarser relation on vector bundles (stable isomorphism) leads to the concept of topological K-theory, which can be handled much better, but is only a quotient of the real situation.

6.4. Let (U_α, ψ_α) be a vector bundle atlas on a vector bundle (E, p, M). Let $(e_j)_{j=1}^k$ be a basis of the standard fiber V. We consider the section $s_j(x) := \psi_\alpha^{-1}(x, e_j)$ for $x \in U_\alpha$. Then the $s_j : U_\alpha \to E$ are local sections of E such that

$(s_j(x))_{j=1}^k$ is a basis of E_x for each $x \in U_\alpha$: we say that $s = (s_1, \dots, s_k)$ is a local *frame field* for E over U_α.

Now let conversely $U \subset M$ be an open set and let $s_j : U \to E$ be local sections of E such that $s = (s_1, \dots, s_k)$ is a local frame field of E over U. Then s determines a unique vector bundle chart (U, ψ) of E such that $s_j(x) = \psi^{-1}(x, e_j)$, in the following way. We define $f : U \times \mathbb{R}^k \to E \restriction U$ by $f(x, v^1, \dots, v^k) := \sum_{j=1}^k v^j s_j(x)$. Then f is smooth, invertible, and a fiber linear isomorphism, so $(U, \psi = f^{-1})$ is the vector bundle chart promised above.

6.5. Let (E, p, M) and (F, q, N) be vector bundles. A *vector bundle homomorphism* $\varphi : E \to F$ is a fiber respecting, fiber linear smooth mapping

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ M \arrow[r, "\underline{\varphi}"] & N. \end{tikzcd}$$

So we require that $\varphi_x : E_x \to F_{\underline{\varphi}(x)}$ is linear. We say that φ covers $\underline{\varphi}$. If φ is invertible, it is called a *vector bundle isomorphism.*

6.6. A *vector sub bundle* (F, p, M) of a vector bundle (E, p, M) is a vector bundle and a vector bundle homomorphism $\tau : F \to E$, which covers Id_M, such that $\tau_x : E_x \to F_x$ is a linear embedding for each $x \in M$.

Lemma. *Let $\varphi : (E, p, M) \to (E', q, N)$ be a vector bundle homomorphism such that* $\operatorname{rank}(\varphi_x : E_x \to E'_{\underline{\varphi}(x)})$ *is constant in $x \in M$. Then* $\ker \varphi$*, given by* $(\ker \varphi)_x = \ker(\varphi_x)$*, is a vector sub bundle of (E, p, M).*

Proof. This is a local question, so we may assume that both bundles are trivial: let $E = M \times \mathbb{R}^p$ and let $F = N \times \mathbb{R}^q$, then $\varphi(x, v) = (\underline{\varphi}(x), \overline{\varphi}(x).v)$, where $\overline{\varphi} : M \to L(\mathbb{R}^p, \mathbb{R}^q)$. The matrix $\overline{\varphi}(x)$ has rank k, so by the elimination procedure we can find $p-k$ linearly independent solutions $v_i(x)$ of the equation $\overline{\varphi}(x).v = 0$. The elimination procedure (with the same lines) gives solutions $v_i(y)$ for y near x, so near x we get a local frame field $v = (v_1, \dots, v_{p-k})$ for $\ker \varphi$. By 6.4 $\ker \varphi$ is then a vector sub bundle. $\square$

6.7. Constructions with vector bundles. Let $\mathcal{F}$ be a covariant functor from the category of finite dimensional vector spaces and linear mappings into itself, such that $\mathcal{F} : L(V, W) \to L(\mathcal{F}(V), \mathcal{F}(W))$ is smooth. Then $\mathcal{F}$ will be called a *smooth functor* for shortness sake. Well known examples of smooth functors are $\mathcal{F}(V) = \Lambda^k(V)$ (the k-th exterior power), or $\mathcal{F}(V) = \bigotimes^k V$, and the like.

If (E,p,M) is a vector bundle, described by a vector bundle atlas with cocycle of transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL(V)$, where (U_α) is an open cover of M, then we may consider the smooth functions $\mathcal{F}(\varphi_{\alpha\beta}) : x \mapsto \mathcal{F}(\varphi_{\alpha\beta}(x))$, $U_{\alpha\beta} \to GL(\mathcal{F}(V))$. Since $\mathcal{F}$ is a covariant functor, $\mathcal{F}(\varphi_{\alpha\beta})$ satisfies again the cocycle condition 6.3.1, and cohomology of cocycles 6.3.2 is respected, so there exists a unique vector bundle $(\mathcal{F}(E) := VB(\mathcal{F}(\varphi_{\alpha\beta})), p, M)$, the value at the vector bundle (E,p,M) of the canonical extension of the functor $\mathcal{F}$ to the category of vector bundles and their homomorphisms.

If $\mathcal{F}$ is a contravariant smooth functor like duality functor $\mathcal{F}(V) = V^*$, then we have to consider the new cocycle $\mathcal{F}(\varphi_{\alpha\beta}^{-1})$ instead of $\mathcal{F}(\varphi_{\alpha\beta})$.

If $\mathcal{F}$ is a contra-covariant smooth bifunctor like $L(V,W)$, then the construction $\mathcal{F}(VB(\psi_{\alpha\beta}), VB(\varphi_{\alpha\beta})) := VB(\mathcal{F}(\psi_{\alpha\beta}^{-1}, \varphi_{\alpha\beta}))$ describes the induced canonical vector bundle construction, and similarly in other constructions.

So for vector bundles (E,p,M) and (F,q,M) we have the following vector bundles with base M: $\Lambda^k E$, $E \oplus F$, E^*, $\Lambda E = \bigoplus_{k \geq 0} \Lambda^k E$, $E \otimes F$, $L(E,F) \cong E^* \otimes F$, and so on.

6.8. Pullbacks of vector bundles. Let (E,p,M) be a vector bundle and let $f : N \to M$ be smooth. Then the *pullback vector bundle* (f^*E, f^*p, N) with the same typical fiber and a vector bundle homomorphism

$$\begin{tikzcd}
f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\
N \arrow[r, "f"] & M
\end{tikzcd}$$

is defined as follows. Let E be described by a cocycle $(\psi_{\alpha\beta})$ of transition functions over an open cover (U_α) of M, $E = VB(\psi_{\alpha\beta})$. Then $(\psi_{\alpha\beta} \circ f)$ is a cocycle of transition functions over the open cover $(f^{-1}(U_\alpha))$ of N and the bundle is given by $f^*E := VB(\psi_{\alpha\beta} \circ f)$. As a manifold we have $f^*E = N \underset{(f,M,p)}{\times} E$ in the sense of 2.19.

The vector bundle f^*E has the following universal property: For any vector bundle (F,q,P), vector bundle homomorphism $\varphi : F \to E$ and smooth $g : P \to N$ such that $f \circ g = \underline{\varphi}$, there is a unique vector bundle homomorphism

$\psi : F \to f^*E$ with $\underline{\psi} = g$ and $p^*f \circ \psi = \varphi$.

$$\begin{tikzcd}
F \arrow[drr, "\varphi"] \arrow[dr, "\psi"'] \arrow[dd, "q"'] & & \\
& f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\
P \arrow[r, "g"] & N \arrow[r, "f"] & M.
\end{tikzcd}$$

6.9. Theorem. *Any vector bundle admits a finite vector bundle atlas.*

Proof. Let (E, p, M) be the vector bundle in question, where $\dim M = m$. Let $(U_\alpha, \psi_\alpha)_{\alpha\in A}$ be a vector bundle atlas. Since M is separable, by topological dimension theory there is a refinement of the open cover $(U_\alpha)_{\alpha\in A}$ of the form $(V_{ij})_{i=1,\dots,m+1; j\in\mathbb{N}}$, such that $V_{ij} \cap V_{ik} = \emptyset$ for $j \neq k$, see the remarks at the end of 1.1. We define the set $W_i := \bigcup_{j\in\mathbb{N}} V_{ij}$ (a disjoint union) and $\psi_i \upharpoonright V_{ij} = \psi_{\alpha(i,j)}$, where $\alpha : \{1, \dots, m+1\} \times \mathbb{N} \to A$ is a refining map. Then $(W_i, \psi_i)_{i=1,\dots,m+1}$ is a finite vector bundle atlas of E. $\square$

6.10. Theorem. *For any vector bundle (E, p, M) there is a second vector bundle (F, p, M) such that $(E \oplus F, p, M)$ is a trivial vector bundle, i.e. isomorphic to $M \times \mathbb{R}^N$ for some $N \in \mathbb{N}$.*

Proof. Let $(U_i, \psi_i)_{i=1}^n$ be a finite vector bundle atlas for (E, p, M). Let (g_i) be a smooth partition of unity subordinated to the open cover (U_i). Let $\ell_i : \mathbb{R}^k \to (\mathbb{R}^k)^n = \mathbb{R}^k \times \dots \times \mathbb{R}^k$ be the embedding on the i-th factor, where $\mathbb{R}^k$ is the typical fiber of E. Let us define $\psi : E \to M \times \mathbb{R}^{nk}$ by $\psi(u) = (p(u), \sum_{i=1}^n g_i(p(u))\,(\ell_i \circ pr_2 \circ \psi_i)(u))$, then ψ is smooth, fiber linear, and an embedding on each fiber, so E is a vector sub bundle of $M \times \mathbb{R}^{nk}$ via ψ. Now we define $F_x = E_x^\perp$ in $\{x\} \times \mathbb{R}^{nk}$ with respect to the standard inner product on $\mathbb{R}^{nk}$. Then $F \to M$ is a vector bundle and $E \oplus F \cong M \times \mathbb{R}^{nk}$. $\square$

6.11. The tangent bundle of a vector bundle. Let (E, p, M) be a vector bundle with fiber addition $+_E : E \times_M E \to E$ and fiber scalar multiplication $m_t^E : E \to E$. Then (TE, π_E, E), the tangent bundle of the manifold E, is itself a vector bundle, with fiber addition denoted by $+_{TE}$ and scalar multiplication denoted by m_t^{TE}.

If $(U_\alpha, \psi_\alpha : E \upharpoonright U_\alpha \to U_\alpha \times V)_{\alpha\in A}$ is a vector bundle atlas for E, such that (U_α, u_α) is also a manifold atlas for M, then $(E \upharpoonright U_\alpha, \psi'_\alpha)_{\alpha\in A}$ is an atlas for the manifold E, where

$$\psi'_\alpha := (u_\alpha \times Id_V) \circ \psi_\alpha : E \upharpoonright U_\alpha \to U_\alpha \times V \to u_\alpha(U_\alpha) \times V \subset \mathbb{R}^m \times V.$$

Hence the family $(T(E \restriction U_\alpha), T\psi'_\alpha : T(E \restriction U_\alpha) \to T(u_\alpha(U_\alpha) \times V) = u_\alpha(U_\alpha) \times V \times \mathbb{R}^m \times V)_{\alpha\in A}$ is the atlas describing the canonical vector bundle structure of (TE, π_E, E). The transition functions are in turn:

$$\begin{aligned}
(\psi_\alpha \circ \psi_\beta^{-1})(x,v) &= (x, \psi_{\alpha\beta}(x)v) \quad \text{ for } x \in U_{\alpha\beta} \\
(u_\alpha \circ u_\beta^{-1})(y) &= u_{\alpha\beta}(y) \quad \text{ for } y \in u_\beta(U_{\alpha\beta}) \\
(\psi'_\alpha \circ (\psi'_\beta)^{-1})(y,v) &= (u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v) \\
(T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(y,v;\xi,w) &= \big(u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v; d(u_{\alpha\beta})(y)\xi, \\
&\qquad (d(\psi_{\alpha\beta} \circ u_\beta^{-1})(y))\xi)v + \psi_{\alpha\beta}(u_\beta^{-1}(y))w\big).
\end{aligned}$$

So we see that for fixed (y,v) the transition functions are linear in $(\xi, w) \in \mathbb{R}^m \times V$. This describes the vector bundle structure of the tangent bundle (TE, π_E, E).

For fixed (y,ξ) the transition functions of TE are also linear in $(v,w) \in V \times V$. This gives a vector bundle structure on (TE, Tp, TM). Its fiber addition will be denoted by $T(+_E) : T(E \times_M E) = TE \times_{TM} TE \to TE$, since it is the tangent mapping of $+_E$. Likewise its scalar multiplication will be denoted by $T(m_t^E)$. One may say that the second vector bundle structure on TE, that one over TM, is the derivative of the original one on E.

The space $\{\Xi \in TE : Tp.\Xi = 0 \text{ in } TM\} = (Tp)^{-1}(0)$ is denoted by VE and is called the *vertical bundle* over E. The local form of a vertical vector Ξ is $T\psi'_\alpha.\Xi = (y,v;0,w)$, so the transition function looks like $(T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(y,v;0,w) = (u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v; 0, \psi_{\alpha\beta}(u_\beta^{-1}(y))w)$. They are linear in $(v,w) \in V \times V$ for fixed y, so VE is a vector bundle over M. It coincides with $0_M^*(TE, Tp, TM)$, the pullback of the bundle $TE \to TM$ over the zero section. We have a canonical isomorphism $vl_E : E \times_M E \to VE$, called the *vertical lift*, given by $vl_E(u_x, v_x) := \frac{d}{dt}|_0(u_x + tv_x)$, which is fiber linear over M. The local representation of the vertical lift is $(T\psi'_\alpha \circ vl_E \circ (\psi'_\alpha \times \psi'_\alpha)^{-1})((y,u),(y,v)) = (y,u;0,v)$.

If (and only if) $\varphi : (E,p,M) \to (F,q,N)$ is a vector bundle homomorphism, then we have $vl_F \circ (\varphi \times_M \varphi) = T\varphi \circ vl_E : E \times_M E \to VF \subset TF$. So vl is a natural transformation between certain functors on the category of vector bundles and their homomorphisms.

The mapping $vpr_E := pr_2 \circ vl_E^{-1} : VE \to E$ is called the *vertical projection*. Note also the relation $pr_1 \circ vl_E^{-1} = \pi_E \restriction VE$.

6.12. The second tangent bundle of a manifold. All of 6.11 is valid for the second tangent bundle $T^2M = TTM$ of a manifold, but here we have one more natural structure at our disposal. The *canonical flip* or *involution* $\kappa_M : T^2M \to T^2M$ is defined locally by

$$(T^2u \circ \kappa_M \circ T^2u^{-1})(x,\xi;\eta,\zeta) = (x,\eta;\xi,\zeta),$$

where (U, u) is a chart on M. Clearly this definition is invariant under changes of charts.

The flip κ_M has the following properties:

(1) $\kappa_N \circ T^2 f = T^2 f \circ \kappa_M$ for each $f \in C^\infty(M, N)$.
(2) $T(\pi_M) \circ \kappa_M = \pi_{TM}$.
(3) $\pi_{TM} \circ \kappa_M = T(\pi_M)$.
(4) $\kappa_M^{-1} = \kappa_M$.
(5) κ_M is a linear isomorphism from the bundle $(TTM, T(\pi_M), TM)$ to the bundle (TTM, π_{TM}, TM), so it interchanges the two vector bundle structures on TTM.
(6) It is the unique smooth mapping $TTM \to TTM$ which satisfies the equation $\frac{\partial}{\partial t}\frac{\partial}{\partial s}c(t, s) = \kappa_M \frac{\partial}{\partial s}\frac{\partial}{\partial t}c(t, s)$ for each $c : \mathbb{R}^2 \to M$.

All this follows from the local formula given above.

6.13. Lemma. *For vector fields X, $Y \in \mathfrak{X}(M)$ we have*

$$[X, Y] = vpr_{TM} \circ (TY \circ X - \kappa_M \circ TX \circ Y).$$

We will give global proofs of this result later on: the first one is 6.19.

Proof. We prove this locally, so we may assume that M is open in $\mathbb{R}^m$, $X(x) = (x, \bar{X}(x))$, and $Y(x) = (x, \bar{Y}(x))$. Then by 3.4 we have

$$[X, Y](x) = (x, d\bar{Y}(x).\bar{X}(x) - d\bar{X}(x).\bar{Y}(x)),$$

and thus

$$\begin{aligned}
vpr_{TM} \circ (TY \circ X - \kappa_M \circ TX \circ Y)(x) &= \\
&= vpr_{TM} \circ (TY.(x, \bar{X}(x)) - \kappa_M \circ TX.(x, \bar{Y}(x))) = \\
&= vpr_{TM}\big((x, \bar{Y}(x); \bar{X}(x), d\bar{Y}(x).\bar{X}(x)) - \\
&\qquad - \kappa_M((x, \bar{X}(x); \bar{Y}(x), d\bar{X}(x).\bar{Y}(x))\big) = \\
&= vpr_{TM}(x, \bar{Y}(x); 0, d\bar{Y}(x).\bar{X}(x) - d\bar{X}(x).\bar{Y}(x)) = \\
&= (x, d\bar{Y}(x).\bar{X}(x) - d\bar{X}(x).\bar{Y}(x)). \quad \square
\end{aligned}$$

6.14. Natural vector bundles or **vector bundle functors.** Let $\mathcal{M}f_m$ denote the category of all m-dimensional smooth manifolds and local diffeomorphisms (i.e. immersions) between them. A *vector bundle functor* or *natural* *vector bundle* is a functor F which associates a vector bundle $(F(M), p_M, M)$ to each m-manifold M and a vector bundle homomorphism

$$\begin{CD} F(M) @>F(f)>> F(N) \\ @Vp_MVV @VVp_NV \\ M @>f>> N \end{CD}$$

to each $f : M \to N$ in $\mathcal{M}f_m$, which covers f and is fiberwise a linear isomorphism. We also require that for smooth $f : \mathbb{R} \times M \to N$ the mapping $(t, x) \mapsto F(f_t)(x)$ is also smooth $\mathbb{R} \times F(M) \to F(N)$. We will say that F maps smoothly parametrized families to smoothly parametrized families.

Examples. 1. TM, the tangent bundle. This is even a functor on the category $\mathcal{M}f$.

2. T^*M, the cotangent bundle, where by 6.7 the action on morphisms is given by $(T^*f)_x := ((T_xf)^{-1})^* : T_x^*M \to T_{f(x)}^*N$. This functor is defined on $\mathcal{M}f_m$ only.

3. $\Lambda^k T^*M$, $\Lambda T^*M = \bigoplus_{k\ge 0} \Lambda^k T^*M$.

4. $\bigotimes^k T^*M \otimes \bigotimes^\ell TM = T^*M \otimes \dots \otimes T^*M \otimes TM \otimes \dots \otimes TM$, where the action on morphisms involves Tf^{-1} in the T^*M-parts and Tf in the TM-parts.

5. $\mathcal{F}(TM)$, where $\mathcal{F}$ is any smooth functor on the category of finite dimensional vector spaces and linear mappings, as in 6.7.

6.15. Lie derivative. Let F be a vector bundle functor on $\mathcal{M}f_m$ as described in 6.14. Let M be a manifold and let $X \in \mathfrak{X}(M)$ be a vector field on M. Then the flow Fl_t^X, for fixed t, is a diffeomorphism defined on an open subset of M, which we do not specify. The mapping

$$\begin{CD} F(M) @>F(\mathrm{Fl}_t^X)>> F(M) \\ @Vp_MVV @VVp_MV \\ M @>\mathrm{Fl}_t^X>> M \end{CD}$$

is then a vector bundle isomorphism, defined over an open subset of M.

We consider a section $s \in C^\infty(F(M))$ of the vector bundle $(F(M), p_M, M)$ and we define for $t \in \mathbb{R}$

$$(\mathrm{Fl}_t^X)^* s := F(\mathrm{Fl}_{-t}^X) \circ s \circ \mathrm{Fl}_t^X,$$

a local section of the bundle $F(M)$. For each $x \in M$ the value $((\mathrm{Fl}^X_t)^* s)(x) \in F(M)_x$ is defined, if t is small enough. So in the vector space $F(M)_x$ the expression $\frac{d}{dt}|_0((\mathrm{Fl}^X_t)^* s)(x)$ makes sense and therefore the section

$$\mathcal{L}_X s := \tfrac{d}{dt}|_0 (\mathrm{Fl}^X_t)^* s$$

is globally defined and is an element of $C^\infty(F(M))$. It is called the *Lie derivative* of s along X.

Lemma. *In this situation we have*

(1) $(\mathrm{Fl}^X_t)^*(\mathrm{Fl}^X_r)^* s = (\mathrm{Fl}^X_{t+r})^* s$, *whenever defined.*

(2) $\frac{d}{dt}(\mathrm{Fl}^X_t)^* s = (\mathrm{Fl}^X_t)^* \mathcal{L}_X s = \mathcal{L}_X (\mathrm{Fl}^X_t)^* s$, *so* $[\mathcal{L}_X, (\mathrm{Fl}^X_t)^*] := \mathcal{L}_X \circ (\mathrm{Fl}^X_t)^* - (\mathrm{Fl}^X_t)^* \circ \mathcal{L}_X = 0$, *whenever defined.*

(3) $(\mathrm{Fl}^X_t)^* s = s$ *for all relevant t if and only if* $\mathcal{L}_X s = 0$.

Proof. (1) is clear. (2) is seen by the following computations.

$$\tfrac{d}{dt}(\mathrm{Fl}^X_t)^* s = \tfrac{d}{dr}|_0 (\mathrm{Fl}^X_r)^*(\mathrm{Fl}^X_t)^* s = \mathcal{L}_X (\mathrm{Fl}^X_t)^* s.$$

$$\begin{aligned}
\tfrac{d}{dt}((\mathrm{Fl}^X_t)^* s)(x) &= \tfrac{d}{dr}|_0 ((\mathrm{Fl}^X_t)^*(\mathrm{Fl}^X_r)^* s)(x) \\
&= \tfrac{d}{dr}|_0 F(\mathrm{Fl}^X_{-t})(F(\mathrm{Fl}^X_{-r}) \circ s \circ \mathrm{Fl}^X_r)(\mathrm{Fl}^X_t(x)) \\
&= F(\mathrm{Fl}^X_{-t}) \tfrac{d}{dr}|_0 (F(\mathrm{Fl}^X_{-r}) \circ s \circ \mathrm{Fl}^X_r)(\mathrm{Fl}^X_t(x)) \\
&= ((\mathrm{Fl}^X_t)^* \mathcal{L}_X s)(x),
\end{aligned}$$

since $F(\mathrm{Fl}^X_{-t}) : F(M)_{\mathrm{Fl}^X_t(x)} \to F(M)_x$ is linear.

(3) follows from (2). $\square$

6.16. Let F_1, F_2 be two vector bundle functors on $\mathcal{M}f_m$. Then the tensor product $(F_1 \otimes F_2)(M) := F_1(M) \otimes F_2(M)$ is again a vector bundle functor and for $s_i \in C^\infty(F_i(M))$ there is a section $s_1 \otimes s_2 \in C^\infty((F_1 \otimes F_2)(M))$, given by the pointwise tensor product.

Lemma. *In this situation, for $X \in \mathfrak{X}(M)$ we have*

$$\mathcal{L}_X(s_1 \otimes s_2) = \mathcal{L}_X s_1 \otimes s_2 + s_1 \otimes \mathcal{L}_X s_2.$$

In particular, for $f \in C^\infty(M, \mathbb{R})$ we have $\mathcal{L}_X(f s) = df(X)\, s + f\, \mathcal{L}_X s$.

Proof. Using the bilinearity of the tensor product we have

$$\begin{aligned}
\mathcal{L}_X(s_1 \otimes s_2) &= \tfrac{d}{dt}|_0 (\mathrm{Fl}^X_t)^*(s_1 \otimes s_2) \\
&= \tfrac{d}{dt}|_0 ((\mathrm{Fl}^X_t)^* s_1 \otimes (\mathrm{Fl}^X_t)^* s_2) \\
&= \tfrac{d}{dt}|_0 (\mathrm{Fl}^X_t)^* s_1 \otimes s_2 + s_1 \otimes \tfrac{d}{dt}|_0 (\mathrm{Fl}^X_t)^* s_2 \\
&= \mathcal{L}_X s_1 \otimes s_2 + s_1 \otimes \mathcal{L}_X s_2. \quad \square
\end{aligned}$$

6.17. Let $\varphi : F_1 \to F_2$ be a linear *natural transformation* between vector bundle functors on $\mathcal{M}f_m$. So for each $M \in \mathcal{M}f_m$ we have a vector bundle homomorphism $\varphi_M : F_1(M) \to F_2(M)$ covering the identity on M, such that $F_2(f) \circ \varphi_M = \varphi_N \circ F_1(f)$ holds for any $f : M \to N$ in $\mathcal{M}f_m$.

Lemma. *In this situation, for $s \in C^\infty(F_1(M))$ and $X \in \mathfrak{X}(M)$, we have $\mathcal{L}_X(\varphi_M\, s) = \varphi_M(\mathcal{L}_X s)$.*

Proof. Since φ_M is fiber linear and natural we can compute as follows.

$$\begin{aligned}\mathcal{L}_X(\varphi_M\, s)(x) &= \tfrac{d}{dt}|_0((\mathrm{Fl}^X_t)^*(\varphi_M\, s))(x)\\ &= \tfrac{d}{dt}|_0(F_2(\mathrm{Fl}^X_{-t}) \circ \varphi_M \circ s \circ \mathrm{Fl}^X_t)(x)\\ &= \varphi_M \circ \tfrac{d}{dt}|_0(F_1(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t)(x)\\ &= (\varphi_M\, \mathcal{L}_X s)(x). \quad \square\end{aligned}$$

6.18. A *tensor field* of type $\binom{p}{q}$ is a smooth section of the natural bundle $\bigotimes^q T^*M \otimes \bigotimes^p TM$. For such tensor fields, by 6.15 the Lie derivative along any vector field is defined, by 6.16 it is a derivation with respect to the tensor product, and by 6.17 it commutes with any kind of contraction or 'permutation of the indices'. For functions and vector fields the Lie derivative was already defined in section 3.

6.19. Let F be a vector bundle functor on $\mathcal{M}f_m$ and let $X \in \mathfrak{X}(M)$ be a vector field. We consider the local vector bundle homomorphism $F(\mathrm{Fl}^X_t)$ on $F(M)$. Since $F(\mathrm{Fl}^X_t) \circ F(\mathrm{Fl}^X_s) = F(\mathrm{Fl}^X_{t+s})$ and $F(\mathrm{Fl}^X_0) = Id_{F(M)}$ we have $\frac{d}{dt}F(\mathrm{Fl}^X_t) = \frac{d}{ds}|_0 F(\mathrm{Fl}^X_s) \circ F(\mathrm{Fl}^X_t) = X^F \circ F(\mathrm{Fl}^X_t)$, so we get $F(\mathrm{Fl}^X_t) = \mathrm{Fl}^{X^F}_t$, where $X^F = \frac{d}{ds}|_0 F(\mathrm{Fl}^X_s) \in \mathfrak{X}(F(M))$ is a vector field on $F(M)$, which is called the *flow prolongation* or the *natural lift* of X to $F(M)$.

Lemma.
(1) *$X^T = \kappa_M \circ TX$.*
(2) *$[X,Y]^F = [X^F, Y^F]$.*
(3) *$X^F : (F(M), p_M, M) \to (TF(M), T(p_M), TM)$ is a vector bundle homomorphism for the $T(+)$-structure.*
(4) *For $s \in C^\infty(F(M))$ and $X \in \mathfrak{X}(M)$ we have*
$\mathcal{L}_X s = vpr_{F(M)}(Ts \circ X - X^F \circ s)$.
(5) *$\mathcal{L}_X s$ is linear in X and s.*

Proof. (1) is an easy computation. $F(\mathrm{Fl}^X_t)$ is fiber linear and this implies (3). (4) is seen as follows:

$$\begin{aligned}
(\mathcal{L}_X s)(x) &= \tfrac{d}{dt}|_0 (F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t)(x) \quad \text{in } F(M)_x\\
&= vpr_{F(M)}(\tfrac{d}{dt}|_0 (F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t)(x) \text{ in } VF(M))\\
&= vpr_{F(M)}(-X^F \circ s \circ \mathrm{Fl}^X_0(x) + T(F(\mathrm{Fl}^X_0)) \circ Ts \circ X(x))\\
&= vpr_{F(M)}(Ts \circ X - X^F \circ s)(x).
\end{aligned}$$

(5). $\mathcal{L}_X s$ is homogeneous of degree 1 in X by formula (4), and it is smooth as a mapping $\mathfrak{X}(M) \to C^\infty(F(M))$, so it is linear. See [Frölicher, Kriegl, 88] for the convenient calculus in infinite dimensions.

(2). Note first that F induces a smooth mapping between appropriate spaces of local diffeomorphisms which are infinite dimensional manifolds (see [Kriegl, Michor, 91]). By 3.16 we have

$$\begin{aligned}
0 &= \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^Y_{-t} \circ \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_t),\\
[X,Y] &= \tfrac{1}{2}\tfrac{\partial^2}{\partial t^2}|_0 (\mathrm{Fl}^Y_{-t} \circ \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_t)\\
&= \tfrac{\partial}{\partial t}\big|_0 \mathrm{Fl}^{[X,Y]}_t.
\end{aligned}$$

Applying F to these curves (of local diffeomorphisms) we get

$$\begin{aligned}
0 &= \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^{Y^F}_{-t} \circ \mathrm{Fl}^{X^F}_{-t} \circ \mathrm{Fl}^{Y^F}_t \circ \mathrm{Fl}^{X^F}_t),\\
[X^F,Y^F] &= \tfrac{1}{2}\tfrac{\partial^2}{\partial t^2}|_0 (\mathrm{Fl}^{Y^F}_{-t} \circ \mathrm{Fl}^{X^F}_{-t} \circ \mathrm{Fl}^{Y^F}_t \circ \mathrm{Fl}^{X^F}_t)\\
&= \tfrac{1}{2}\tfrac{\partial^2}{\partial t^2}|_0 F(\mathrm{Fl}^Y_{-t} \circ \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_t)\\
&= \tfrac{\partial}{\partial t}\big|_0 F(\mathrm{Fl}^{[X,Y]}_t) = [X,Y]^F.
\end{aligned}$$

□

6.20. Theorem. *For any vector bundle functor F on $\mathcal{M}f_m$ and $X, Y \in \mathfrak{X}(M)$ we have*

$$[\mathcal{L}_X, \mathcal{L}_Y] := \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]} : C^\infty(F(M)) \to C^\infty(F(M)).$$

So $\mathcal{L} : \mathfrak{X}(M) \to \operatorname{End} C^\infty(F(M))$ is a Lie algebra homomorphism.

7. Differential Forms

7.1. The *cotangent bundle* of a manifold M is the vector bundle $T^*M := (TM)^*$, the (real) dual of the tangent bundle.

If (U,u) is a chart on M, then $(\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^m})$ is the associated frame field over U of TM. Since $\frac{\partial}{\partial u^i}|_x(u^j) = du^j(\frac{\partial}{\partial u^i}|_x) = \delta_i^j$ we see that $(du^1,\dots,du^m)$ is the dual frame field on T^*M over U. It is also called a *holonomous* frame field. A section of T^*M is also called a 1-*form*.

7.2. According to 6.18 a *tensor field* of type $\binom{p}{q}$ on a manifold M is a smooth section of the vector bundle

$$\bigotimes^p TM \otimes \bigotimes^q T^*M = TM \overbrace{\otimes \cdots \otimes}^{p \text{ times}} TM \otimes T^*M \overbrace{\otimes \cdots \otimes}^{q \text{ times}} T^*M.$$

The position of p (up) and q (down) can be explained as follows: If (U,u) is a chart on M, we have the holonomous frame field

$$\left(\tfrac{\partial}{\partial u^{i_1}} \otimes \tfrac{\partial}{\partial u^{i_2}} \otimes \cdots \otimes \tfrac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}\right)_{i\in\{1,\dots,m\}^p, j\in\{1,\dots,m\}^q}$$

over U of this tensor bundle, and for any $\binom{p}{q}$-tensor field A we have

$$A \mid U = \sum_{i,j} A^{i_1\dots i_p}_{j_1\dots j_q} \tfrac{\partial}{\partial u^{i_1}} \otimes \cdots \otimes \tfrac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}.$$

The coefficients have p indices up and q indices down, they are smooth functions on U. From a categorical point of view one should look, where the indices of the frame field are, but this convention here has a long tradition.

7.3. Lemma. *Let $\Phi : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) = \mathfrak{X}(M)^k \to C^\infty(\bigotimes^l TM)$ be a mapping which is k-linear over $C^\infty(M,\mathbb{R})$ then Φ is given by the action of a $\binom{l}{k}$-tensor field.*

Proof. For simplicity's sake we put $k=1$, $\ell=0$, so $\Phi : \mathfrak{X}(M) \to C^\infty(M,\mathbb{R})$ is a $C^\infty(M,\mathbb{R})$-linear mapping: $\Phi(f.X) = f.\Phi(X)$.

Claim 1. If $X \mid U = 0$ for some open subset $U \subset M$, then we have $\Phi(X) \mid U = 0$.
Let $x \in U$. We choose $f \in C^\infty(M,\mathbb{R})$ with $f(x) = 0$ and $f \mid M \setminus U = 1$. Then $f.X = X$, so $\Phi(X)(x) = \Phi(f.X)(x) = f(x).\Phi(X)(x) = 0$.

Claim 2. If $X(x) = 0$ then also $\Phi(X)(x) = 0$.
Let (U,u) be a chart centered at x, let V be open with $x \in V \subset \bar{V} \subset U$. Then

$X \mid U = \sum X^i \frac{\partial}{\partial u^i}$ and $X^i(x) = 0$. We choose $g \in C^\infty(M, \mathbb{R})$ with $g \mid V \equiv 1$ and $\operatorname{supp} g \subset U$. Then $(g^2.X) \mid V = X \mid V$ and by claim 1 $\Phi(X) \mid V$ depends only on $X \mid V$ and $g^2.X = \sum_i (g.X^i)(g.\frac{\partial}{\partial u^i})$ is a decomposition which is globally defined on M. Therefore we have $\Phi(X)(x) = \Phi(g^2.X)(x) = \Phi\left(\sum_i (g.X^i)(g.\frac{\partial}{\partial u^i})\right)(x) = \sum (g.X^i)(x).\Phi(g.\frac{\partial}{\partial u^i})(x) = 0$.

So we see that for a general vector field X the value $\Phi(X)(x)$ depends only on the value $X(x)$, for each $x \in M$. So there is a linear map $\varphi_x : T_xM \to \mathbb{R}$ for each $x \in M$ with $\Phi(X)(x) = \varphi_x(X(x))$. Then $\varphi : M \to T^*M$ is smooth since $\varphi \mid V = \sum_i \Phi(g.\frac{\partial}{\partial u^i})\, du^i$ in the setting of claim 2. $\square$

7.4. Definition. A *differential form* of degree k or a *k-form* for short is a section of the (natural) vector bundle $\Lambda^k T^*M$. The space of all k-forms will be denoted by $\Omega^k(M)$. It may also be viewed as the space of all skew symmetric $\binom{0}{k}$-tensor fields, i. e. (by 7.3) the space of all mappings

$$\Phi : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) = \mathfrak{X}(M)^k \to C^\infty(M, \mathbb{R}),$$

which are k-linear over $C^\infty(M, \mathbb{R})$ and are skew symmetric:

$$\Phi(X_{\sigma 1}, \dots, X_{\sigma k}) = \operatorname{sign} \sigma \cdot \Phi(X_1, \dots, X_k)$$

for each permutation $\sigma \in \mathcal{S}_k$.

We put $\Omega^0(M) := C^\infty(M, \mathbb{R})$. Then the space

$$\Omega(M) := \bigoplus_{k=0}^{\dim M} \Omega^k(M)$$

is an algebra with the following product. For $\varphi \in \Omega^k(M)$ and $\psi \in \Omega^\ell(M)$ and for X_i in $\mathfrak{X}(M)$ (or in T_xM) we put

$$\begin{aligned}(\varphi \wedge \psi)&(X_1, \dots, X_{k+\ell}) = \\ &= \tfrac{1}{k!\,\ell!} \sum_{\sigma \in \mathcal{S}_{k+\ell}} \operatorname{sign} \sigma \cdot \varphi(X_{\sigma 1}, \dots, X_{\sigma k}).\psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}).\end{aligned}$$

This product is defined fiber wise, i. e. $(\varphi \wedge \psi)_x = \varphi_x \wedge \psi_x$ for each $x \in M$. It is also associative, i.e $(\varphi \wedge \psi) \wedge \tau = \varphi \wedge (\psi \wedge \tau)$, and graded commutative, i. e. $\varphi \wedge \psi = (-1)^{k\ell} \psi \wedge \varphi$. These properties are proved in multilinear algebra.

7.5. If $f : N \to M$ is a smooth mapping and $\varphi \in \Omega^k(M)$, then the pullback $f^*\varphi \in \Omega^k(N)$ is defined for $X_i \in T_xN$ by

$$(f^*\varphi)_x(X_1, \dots, X_k) := \varphi_{f(x)}(T_xf.X_1, \dots, T_xf.X_k). \tag{1}$$

Then we have $f^*(\varphi \wedge \psi) = f^*\varphi \wedge f^*\psi$, so the linear mapping $f^* : \Omega(M) \to \Omega(N)$ is an algebra homomorphism. Moreover we have $(g \circ f)^* = f^* \circ g^* : \Omega(P) \to \Omega(N)$ if $g : M \to P$, and $(Id_M)^* = Id_{\Omega(M)}$.

So $M \mapsto \Omega(M) = C^\infty(\Lambda T^*M)$ is a contravariant functor from the category $\mathcal{M}f$ of all manifolds and all smooth mappings into the category of real graded commutative algebras, whereas $M \mapsto \Lambda T^*M$ is a covariant vector bundle functor defined only on $\mathcal{M}f_m$, the category of m-dimensional manifolds and local diffeomorphisms, for each m separately.

7.6. The Lie derivative of differential forms. Since $M \mapsto \Lambda^k T^*M$ is a vector bundle functor on $\mathcal{M}f_m$, by 6.15 for $X \in \mathfrak{X}(M)$ the *Lie derivative* of a k-form φ along X is defined by

$$\mathcal{L}_X\varphi = \tfrac{d}{dt}|_0(\mathrm{Fl}^X_t)^*\varphi.$$

Lemma. *The Lie derivative has the following properties.*

(1) $\mathcal{L}_X(\varphi \wedge \psi) = \mathcal{L}_X\varphi \wedge \psi + \varphi \wedge \mathcal{L}_X\psi$, *so* $\mathcal{L}_X$ *is a derivation.*

(2) *For* $Y_i \in \mathfrak{X}(M)$ *we have*

$$(\mathcal{L}_X\varphi)(Y_1, \dots, Y_k) = X(\varphi(Y_1, \dots, Y_k)) - \sum_{i=1}^k \varphi(Y_1, \dots, [X, Y_i], \dots, Y_k).$$

(3) $[\mathcal{L}_X, \mathcal{L}_Y]\varphi = \mathcal{L}_{[X,Y]}\varphi$.

Proof. The mapping $Alt : \bigotimes^k T^*M \to \Lambda^k T^*M$, given by

$$(AltA)(Y_1, \dots, Y_k) := \tfrac{1}{k!}\sum_\sigma \operatorname{sign}(\sigma)\ A(Y_{\sigma 1}, \dots, Y_{\sigma k}),$$

is a linear natural transformation in the sense of 6.17 and induces an algebra homomorphism from $\bigoplus_{k \ge 0} C^\infty(\bigotimes^k T^*M)$ onto $\Omega(M)$. So (1) follows from 6.16 and 6.17.

(2). Again by 6.16 and 6.17 we may compute as follows, where Trace is the full evaluation of the form on all vector fields:

$$\begin{aligned}
X(\varphi(Y_1, \dots, Y_k)) &= \mathcal{L}_X \circ \operatorname{Trace}(\varphi \otimes Y_1 \otimes \dots \otimes Y_k)\\
&= \operatorname{Trace} \circ \mathcal{L}_X(\varphi \otimes Y_1 \otimes \dots \otimes Y_k)\\
&= \operatorname{Trace}\big(\mathcal{L}_X\varphi \otimes (Y_1 \otimes \dots \otimes Y_k)\\
&\qquad + \varphi \otimes (\textstyle\sum_i Y_1 \otimes \dots \otimes \mathcal{L}_X Y_i \otimes \dots \otimes Y_k)\big).
\end{aligned}$$

Now we use $\mathcal{L}_X Y_i = [X, Y_i]$.

(3) is a special case of 6.20. $\square$

7.7. The insertion operator. For a vector field $X \in \mathfrak{X}(M)$ we define the *insertion operator* $i_X = i(X) : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$(i_X\varphi)(Y_1, \dots, Y_{k-1}) := \varphi(X, Y_1, \dots, Y_{k-1}).$$

Lemma.

(1) *i_X is a graded derivation of degree -1 of the graded algebra $\Omega(M)$, so we have $i_X(\varphi \wedge \psi) = i_X\varphi \wedge \psi + (-1)^{\deg \varphi}\varphi \wedge i_X\psi$.*

(2) *$[\mathcal{L}_X, i_Y] := \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X,Y]}$.*

Proof. (1). For $\varphi \in \Omega^k(M)$ and $\psi \in \Omega^\ell(M)$ we have

$$\begin{aligned}
(i_{X_1}(\varphi \wedge \psi))(X_2, \dots, X_{k+\ell}) &= (\varphi \wedge \psi)(X_1, \dots, X_{k+\ell}) \\
&= \tfrac{1}{k!\,\ell!} \sum_\sigma \operatorname{sign}(\sigma)\, \varphi(X_{\sigma 1}, \dots, X_{\sigma k})\psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}).
\end{aligned}$$

$$\begin{aligned}
&(i_{X_1}\varphi \wedge \psi + (-1)^k \varphi \wedge i_{X_1}\psi)(X_2, \dots, X_{k+\ell}) \\
&\quad= \tfrac{1}{(k-1)!\,\ell!} \sum_\sigma \operatorname{sign}(\sigma)\, \varphi(X_1, X_{\sigma 2}, \dots, X_{\sigma k})\psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}) \\
&\qquad+ \frac{(-1)^k}{k!\,(\ell-1)!} \sum_\sigma \operatorname{sign}(\sigma)\, \varphi(X_{\sigma 2}, \dots, X_{\sigma(k+1)})\psi(X_1, X_{\sigma(k+2)}, \dots).
\end{aligned}$$

Using the skew symmetry of φ and ψ we may distribute X_1 to each position by adding an appropriate sign. These are $k+\ell$ summands. Since $\frac{1}{(k-1)!\,\ell!} + \frac{1}{k!\,(\ell-1)!} = \frac{k+\ell}{k!\,\ell!}$, and since we can generate each permutation in $\mathcal{S}_{k+\ell}$ in this way, the result follows.

(2). By 6.16 and 6.17 we have:

$$\begin{aligned}
\mathcal{L}_X i_Y \varphi &= \mathcal{L}_X \operatorname{Trace}_1(Y \otimes \varphi) = \operatorname{Trace}_1 \mathcal{L}_X(Y \otimes \varphi) \\
&= \operatorname{Trace}_1(\mathcal{L}_X Y \otimes \varphi + Y \otimes \mathcal{L}_X\varphi) = i_{[X,Y]}\varphi + i_Y\mathcal{L}_X\varphi. \quad \square
\end{aligned}$$

7.8. The exterior differential. We want to construct a differential operator $\Omega^k(M) \to \Omega^{k+1}(M)$ which is natural. We will show that the simplest choice will work and (later) that it is essentially unique.

Let U be open in $\mathbb{R}^n$, let $\varphi \in \Omega^k(U) = C^\infty(U, L^k_{alt}(\mathbb{R}^n, \mathbb{R}))$. We consider the derivative $D\varphi \in C^\infty(U, L(\mathbb{R}^n, L^k_{alt}(\mathbb{R}^n, \mathbb{R})))$, and we take its canonical image in

$C^\infty(U, L^{k+1}_{alt}(\mathbb{R}^n,\mathbb{R}))$. Here we write D for the derivative in order to distinguish it from the exterior differential, which we define as $d\varphi := (k+1)\operatorname{Alt} D\varphi$, more explicitly as

$$(1)\qquad (d\varphi)_x(X_0,\dots,X_k) = \tfrac{1}{k!}\sum_\sigma \operatorname{sign}(\sigma)\, D\varphi(x)(X_{\sigma 0})(X_{\sigma 1},\dots,X_{\sigma k})$$
$$= \sum_{i=0}^k (-1)^i D\varphi(x)(X_i)(X_0,\dots,\widehat{X_i},\dots,X_k),$$

where the hat over a symbol means that this is to be omitted, and where $X_i\in\mathbb{R}^n$.

Now we pass to an arbitrary manifold M. For a k-form $\varphi\in\Omega^k(M)$ and vector fields $X_i\in\mathfrak{X}(M)$ we try to replace $D\varphi(x)(X_i)(X_0,\dots)$ in formula (1) by Lie derivatives. We differentiate $X_i(\varphi(x)(X_0, dotsc)) = D\varphi(x)(X_i)(X_0,\dots) + \sum_{0\le j\le k, j\ne i}\varphi(x)(X_0,\dots,DX_j(x)X_i,\dots)$, so inserting this expression into formula (1) we get (cf. 3.4) our working definition

$$(2)\qquad d\varphi(X_0,\dots,X_k) := \sum_{i=0}^k (-1)^i X_i(\varphi(X_0,\dots,\widehat{X_i},\dots,X_k))$$
$$+\sum_{i<j}(-1)^{i+j}\varphi([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k).$$

$d\varphi$, given by this formula, is $(k+1)$-linear over $C^\infty(M,\mathbb{R})$, as a short computation involving 3.4 shows. It is obviously skew symmetric, so by 7.3 $d\varphi$ is a $(k+1)$-form, and the operator $d:\Omega^k(M)\to\Omega^{k+1}(M)$ is called the *exterior derivative*.

If (U,u) is a chart on M, then we have

$$\varphi\restriction U = \sum_{i_1<\dots<i_k}\varphi_{i_1,\dots,i_k}\,du^{i_1}\wedge\dots\wedge du^{i_k},$$

where $\varphi_{i_1,\dots,i_k} = \varphi(\frac{\partial}{\partial u^{i_1}},\dots,\frac{\partial}{\partial u^{i_k}})$. An easy computation shows that (2) leads to

$$(3)\qquad d\varphi\restriction U = \sum_{i_1<\dots<i_k} d\varphi_{i_1,\dots,i_k}\wedge du^{i_1}\wedge\dots\wedge du^{i_k},$$

so that formulas (1) and (2) really define the same operator.

7.9. Theorem. *The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ has the following properties:*

(1) $d(\varphi\wedge\psi) = d\varphi\wedge\psi + (-1)^{\deg\varphi}\varphi\wedge d\psi$, *so d is a graded derivation of degree 1.*
(2) $\mathcal{L}_X = i_X\circ d + d\circ i_X$ *for any vector field X.*
(3) $d^2 = d\circ d = 0$.
(4) $f^*\circ d = d\circ f^*$ *for any smooth $f : N\to M$.*
(5) $\mathcal{L}_X\circ d = d\circ\mathcal{L}_X$ *for any vector field X.*
(6) $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ *for any two vector fields X, Y.*

Remark. In terms of the graded commutator

$$[D_1, D_2] := D_1\circ D_2 - (-1)^{\deg(D_1)\deg(D_2)} D_2\circ D_1$$

for graded homomorphisms and graded derivations (see 13.1) the assertions of this theorem take the following form:

(2) $\mathcal{L}_X = [i_X, d]$.
(3) $\frac{1}{2}[d, d] = 0$.
(4) $[f^*, d] = 0$.
(5) $[\mathcal{L}_X, d] = 0$.

This point of view will be developed in section 13 below. The equation (6) is a special case of 6.20.

Proof. (2) For $\varphi\in\Omega^k(M)$ and $X_i\in\mathfrak{X}(M)$ we have

$$\begin{aligned}(\mathcal{L}_{X_0}\varphi)(X_1,\dots,X_k) &= X_0(\varphi(X_1,\dots,X_k)) + \\ &\quad + \sum_{j=1}^k (-1)^{0+j}\varphi([X_0,X_j], X_1,\dots,\widehat{X_j},\dots,X_k) \text{ by 7.6.2,}\\ (i_{X_0}d\varphi)(X_1,\dots,X_k) &= d\varphi(X_0,\dots,X_k)\\ &= \sum_{i=0}^k(-1)^i X_i(\varphi(X_0,\dots,\widehat{X_i},\dots,X_k)) + \\ &\quad + \sum_{0\le i<j}(-1)^{i+j}\varphi([X_i,X_j], X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k).\\ (di_{X_0}\varphi)(X_1,\dots,X_k) &= \sum_{i=1}^k(-1)^{i-1}X_i((i_{X_0}\varphi)(X_1,\dots,\widehat{X_i},\dots,X_k)) + \\ &\quad + \sum_{1\le i<j}(-1)^{i+j-2}(i_{X_0}\varphi)([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k)\end{aligned}$$

$$\begin{aligned}
&= -\sum_{i=1}^{k}(-1)^i X_i(\varphi(X_0, X_1, \dots, \widehat{X_i}, \dots, X_k)) - \\
&\quad - \sum_{1\le i<j}(-1)^{i+j}\varphi([X_i, X_j], X_0, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k).
\end{aligned}$$

By summing up the result follows.

(1) Let $\varphi \in \Omega^p(M)$ and $\psi \in \Omega^q(M)$. We prove the result by induction on $p+q$.
$p+q=0$: $d(f\cdot g) = df\cdot g + f\cdot dg$.
Suppose that (1) is true for $p+q<k$. Then for $X \in \mathfrak{X}(M)$ we have by part (2) and 7.6, 7.7 and by induction

$$\begin{aligned}
i_X\, d(\varphi\wedge\psi) &= \mathcal{L}_X(\varphi\wedge\psi) - d\, i_X(\varphi\wedge\psi) \\
&= \mathcal{L}_X\varphi\wedge\psi + \varphi\wedge\mathcal{L}_X\psi - d(i_X\varphi\wedge\psi + (-1)^p\varphi\wedge i_X\psi) \\
&= i_Xd\varphi\wedge\psi + di_X\varphi\wedge\psi + \varphi\wedge i_Xd\psi + \varphi\wedge di_X\psi - di_X\varphi\wedge\psi \\
&\qquad - (-1)^{p-1}i_X\varphi\wedge d\psi - (-1)^p d\varphi\wedge i_X\psi - \varphi\wedge di_X\psi \\
&= i_X(d\varphi\wedge\psi + (-1)^p\varphi\wedge d\psi).
\end{aligned}$$

Since X is arbitrary, (1) follows.

(3) By (1) d is a graded derivation of degree 1, so $d^2 = \frac{1}{2}[d,d]$ is a graded derivation of degree 2 (see 13.1), and is obviously local: $d^2(\varphi\wedge\psi) = d^2(\varphi)\wedge\psi + \varphi\wedge d(\psi)$. Since $\Omega(M)$ is locally generated as an algebra by $C^\infty(M,\mathbb{R})$ and $\{df : f\in C^\infty(M,\mathbb{R})\}$, it suffices to show that $d^2f=0$ for each $f\in C^\infty(M,\mathbb{R})$ ($d^3f=0$ is a consequence). But this is easy: $d^2f(X,Y) = X df(Y) - Y df(X) - df([X,Y]) = XYf - YXf - [X,Y]f = 0$.

(4) $f^* : \Omega(M)\to\Omega(N)$ is an algebra homomorphism by 7.6, so $f^*\circ d$ and $d\circ f^*$ are both graded derivations over f^* of degree 1. So if $f^*\circ d$ and $d\circ f^*$ agree on φ and on ψ, then also on $\varphi\wedge\psi$. By the same argument as in the proof of (3) above it suffices to show that they agree on g and dg for all $g\in C^\infty(M,\mathbb{R})$. We have $(f^*dg)_y(Y) = (dg)_{f(y)}(T_yf.Y) = (T_yf.Y)(g) = Y(g\circ f)(y) = (df^*g)_y(Y)$, thus also $df^*dg = ddf^*g = 0$, and $f^*ddg = 0$.

(5) $d\mathcal{L}_X = d\, i_X\, d + ddi_X = di_Xd + i_Xdd = \mathcal{L}_Xd$.

(6) We use the graded commutator alluded to in the remarks. By the (graded) Jacobi identity and by lemma 7.7.2 we have

$$\mathcal{L}_X, \mathcal{L}_Y] = [\mathcal{L}_X, [i_Y, d]] = [[\mathcal{L}_X, i_Y], d] + [i_Y, [\mathcal{L}_X, d]] = [i_{[X,Y]}, d] + 0 = \mathcal{L}_{[X,Y]}. \quad \square$$

7.10. A differential form $\omega \in \Omega^k(M)$ is called *closed* if $d\omega = 0$, and it is called *exact* if $\omega = d\varphi$ for some $\varphi \in \Omega^{k-1}(M)$. Since $d^2 = 0$, any exact form is closed. The quotient space

$$H^k(M) := \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

is called the k-th De Rham cohomology space of M. As a preparation for our treatment of cohomology we finish with the

Lemma of Poincaré. *A closed differential form is locally exact. More precisely: let $\omega \in \Omega^k(M)$ with $d\omega = 0$. Then for any $x \in M$ there is an open neighborhood U of x in M and a $\varphi \in \Omega^{k-1}(U)$ with $d\varphi = \omega \restriction U$.*

Proof. Let (U, u) be chart on M centered at x such that $u(U) = \mathbb{R}^m$. So we may just assume that $M = \mathbb{R}^m$.

We consider $\alpha : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, given by $\alpha(t, x) = \alpha_t(x) = tx$. Let $I \in \mathfrak{X}(\mathbb{R}^m)$ be the vector field $I(x) = x$, then $\alpha(e^t, x) = \operatorname{Fl}^I_t(x)$. So for $t > 0$ we have

$$\begin{aligned}\tfrac{d}{dt}\alpha_t^*\omega &= \tfrac{d}{dt}(\operatorname{Fl}^I_{\log t})^*\omega = \tfrac{1}{t}(\operatorname{Fl}^I_{\log t})^*\mathcal{L}_I\omega \\ &= \tfrac{1}{t}\alpha_t^*(i_I d\omega + d i_I\omega) = \tfrac{1}{t} d\alpha_t^* i_I\omega.\end{aligned}$$

Note that $T_x(\alpha_t) = t.Id$. Therefore

$$\begin{aligned}(\tfrac{1}{t}\alpha_t^* i_I\omega)_x(X_2, \dots, X_k) &= \tfrac{1}{t}(i_I\omega)_{tx}(tX_2, \dots, tX_k) \\ &= \tfrac{1}{t}\omega_{tx}(tx, tX_2, \dots, tX_k) = \omega_{tx}(x, tX_2, \dots, tX_k).\end{aligned}$$

So if $k \geq 1$, the $(k-1)$-form $\frac{1}{t}\alpha_t^* i_I\omega$ is defined and smooth in (t, x) for all $t \in \mathbb{R}$. Clearly $\alpha_1^*\omega = \omega$ and $\alpha_0^*\omega = 0$, thus

$$\begin{aligned}\omega = \alpha_1^*\omega - \alpha_0^*\omega &= \int_0^1 \tfrac{d}{dt}\alpha_t^*\omega\, dt \\ &= \int_0^1 d(\tfrac{1}{t}\alpha_t^* i_I\omega)dt = d\left(\int_0^1 \tfrac{1}{t}\alpha_t^* i_I\omega\, dt\right) = d\varphi. \quad \square\end{aligned}$$

8. Integration on Manifolds

8.1. Let $U \subset \mathbb{R}^n$ be an open subset, let dx denote Lebesgue-measure on $\mathbb{R}^n$ (which depends on the Euclidean structure), let $g : U \to g(U)$ be a diffeomorphism onto some other open subset in $\mathbb{R}^n$, and let $f : g(U) \to \mathbb{R}$ be an integrable continuous function. Then the *transformation formula for multiple integrals* reads

$$\int_{g(U)} f(y)\, dy = \int_U f(g(x))|\det dg(x)|dx.$$

This suggests that the suitable objects for integration on a manifold are sections of 1-dimensional vector bundle whose cocycle of transition functions is given by the absolute value of the Jacobi matrix of the chart changes. They will be called *densities* below.

8.2. The volume bundle. Let M be a manifold and let (U_α, u_α) be a smooth atlas for it. The *volume bundle* $(\operatorname{Vol}(M), \pi_M, M)$ of M is the one dimensional vector bundle (line bundle) which is given by the following cocycle of transition functions, see 6.2:

$$\psi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to \mathbb{R} \setminus \{0\} = GL(1, \mathbb{R}),$$
$$\psi_{\alpha\beta}(x) = |\det d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x))| = \frac{1}{|\det d(u_\alpha \circ u_\beta^{-1})(u_\beta(x))|}.$$

Lemma. Vol(M) *is a trivial line bundle over M.*

But there is no natural trivialization.

Proof. We choose a positive local section over each U_α and we glue them with a partition of unity. Since positivity is invariant under the transitions, the resulting global section μ is nowhere 0. By 6.4 μ is a global frame field and trivializes $\operatorname{Vol}(M)$. □

Definition. Sections of the line bundle $\operatorname{Vol}(M)$ are called densities.

8.3. Integral of a density. Let $\mu \in C^\infty(\operatorname{Vol}(M))$ be a density with compact support on the manifold M. We define the *integral of the density* μ as follows:

Let (U_α, u_α) be an atlas on M, let f_α be a partition of unity with $\operatorname{supp}(f_\alpha) \subset U_\alpha$. Then we put

$$\begin{aligned}\int_M \mu &= \sum_\alpha \int_{U_\alpha} f_\alpha \mu = \\ &:= \sum_\alpha \int_{u_\alpha(U_\alpha)} f_\alpha(u_\alpha^{-1}(y)).\psi_\alpha(\mu(u_\alpha^{-1}(y)))\, dy.\end{aligned}$$

If μ does not have compact support we require that $\sum\int_{U_\alpha} f_\alpha\,|\mu| < \infty$. The series is then absolutely convergent.

Lemma. *$\int_M \mu$ is well defined.*

Proof. Let (V_β, v_β) be another atlas on M, let (g_β) be a partition of unity with $\operatorname{supp}(g_\beta)\subset V_\beta$. Let (U_α,ψ_α) be the vector bundle atlas of $\operatorname{Vol}(M)$ induced by the atlas (U_α, u_α), and let (V_β,φ_β) be the one induced by (V_β, v_β). Then we have by the transition formula for the diffeomorphisms $u_\alpha\circ v_\beta^{-1} : v_\beta(U_\alpha\cap V_\beta)\to u_\alpha(U_\alpha\cap V_\beta)$

$$\begin{aligned}
\sum_\alpha\int_{U_\alpha} f_\alpha\mu &= \sum_\alpha\int_{u_\alpha(U_\alpha)}(f_\alpha\circ u_\alpha^{-1})(y)\psi_\alpha(\mu(u_\alpha^{-1}(y)))\,dy\\
&= \sum_\alpha\int_{u_\alpha(U_\alpha)}\sum_\beta(g_\beta\circ u_\alpha^{-1})(y)(f_\alpha\circ u_\alpha^{-1})(y)\psi_\alpha(\mu(u_\alpha^{-1}(y)))\,dy\\
&= \sum_{\alpha\beta}\int_{u_\alpha(U_\alpha\cap V_\beta)}(g_\beta\circ u_\alpha^{-1})(y)(f_\alpha\circ u_\alpha^{-1})(y)\psi_\alpha(\mu(u_\alpha^{-1}(y)))\,dy\\
&= \sum_{\alpha\beta}\int_{v_\beta(U_\alpha\cap V_\beta)}(g_\beta\circ v_\beta^{-1})(x)(f_\alpha\circ v_\beta^{-1})(x)\\
&\qquad\qquad \psi_\alpha(\mu(v_\beta^{-1}(x)))|\det d(u_\alpha\circ v_\beta^{-1})(x)|\,dx\\
&= \sum_{\alpha\beta}\int_{v_\beta(U_\alpha\cap V_\beta)}(g_\beta\circ v_\beta^{-1})(x)(f_\alpha\circ v_\beta^{-1})(x)\varphi_\beta(\mu(v_\beta^{-1}(x)))\,dx\\
&= \sum_\beta\int_{V_\beta} g_\beta\mu. \quad\square
\end{aligned}$$

If $\mu\in C^\infty(\operatorname{Vol}(M))$ is an arbitrary section and $f\in C_c^\infty(M,\mathbb{R})$ is a function with compact support, then we may define the integral of f with respect to μ by $\int_M f\mu$, since $f\mu$ is a density with compact support. In this way μ defines a Radon measure on M.

For the converse we note first that (C^1 suffices) diffeomorphisms between open subsets on $\mathbb{R}^m$ map sets of Lebesque measure zero to sets of Lebesque measure zero. Thus on a manifold we have a well defined notion of sets of Lebesque measure zero — but no measure. If ν is a Radon measure on M which is absolutely continuous, i. e. the $|\nu|$-measure of a set of Lebesque measure zero is zero, then is given by a uniquely determined measurable section if the line bundle Vol.

8.4. Remark. For $0 \leq p \leq 1$ let $\operatorname{Vol}^p(M)$ be the line bundle defined by the cocycle of transition functions

$$\begin{aligned} &\psi^p_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R} \setminus \{0\} \\ &\psi^p_{\alpha\beta}(x) = |\det d(u_\alpha \circ u_\beta^{-1})(u_\beta(x))|^{-p}. \end{aligned}$$

This is also a trivial line bundle. Its sections are called p-densities. 1-densities are just densities, 0-densities are functions. If μ is a p-density and ν is a q-density with $p+q \leq 1$ then $\mu.\nu := \mu \otimes \nu$ is a $p+q$-density, i. e. $\operatorname{Vol}^p(M) \otimes \operatorname{Vol}^q(M) = \operatorname{Vol}^{p+q}(M)$. Thus the product of two $\frac12$-densities with compact support can be integrated, so $C^\infty_c(\operatorname{Vol}^{1/2}(M))$ is a pre Hilbert space in a natural way.

Distributions on M (in the sense of generalized functions) are elements of the dual space of the space $C^\infty_c(\operatorname{Vol}(M))$ of densities with compact support equipped with the inductive limit topology — so they contain functions.

8.5. Example. The density of a Riemannian metric. Let g be a Riemannian metric on a manifold M. So g is a symmetric $\binom{0}{2}$ tensor field such that g_x is a positive definite inner product on T_xM for each $x \in M$. If (U,u) is a chart on M then we have

$$g|U = \sum_{i,j=1}^m g^u_{ij}\, du^i \otimes du^j$$

where the functions $g^u_{ij} = g(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j})$ form a positive definite symmetric matrix. So $\det(g^u_{ij}) = \det((g(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}))^m_{i,j=1}) > 0$. We put

$$\operatorname{vol}(g)^u := \sqrt{\det((g(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j}))^m_{i,j=1})}.$$

If (V,v) is another chart we have

$$\begin{aligned} \operatorname{vol}(g)^u &= \sqrt{\det((g(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j}))^m_{i,j=1})} \\ &= \sqrt{\det((g(\sum_k \tfrac{\partial v^k}{\partial u^i}\tfrac{\partial}{\partial v^k}, \sum_\ell \tfrac{\partial v^\ell}{\partial u^j}\tfrac{\partial}{\partial v^\ell}))^m_{i,j=1})} \\ &= \sqrt{\det((\tfrac{\partial v^k}{\partial u^i})_{k,i})^2 \det((g(\tfrac{\partial}{\partial v^\ell}, \tfrac{\partial}{\partial v^j}))_{\ell,j})} \\ &= |\det d(v \circ u^{-1})|\, \operatorname{vol}(g)^v, \end{aligned}$$

so these local representatives determine a section $\operatorname{vol}(g) \in C^\infty(\operatorname{Vol}(M))$, which is called the *density or volume of the Riemannian metric g*. If M is compact then $\int_M \operatorname{vol}(g)$ is called the *volume* of the Riemannian manifold (M,g).

8.6. The orientation bundle. For a manifold M with $\dim M = m$ and an atlas (U_α, u_α) for M the line bundle $\Lambda^m T^* M$ is given by the cocycle of transition functions

$$\varphi_{\alpha\beta}(x) = \det d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x)).$$

We consider the line bundle $\operatorname{Or}(M)$ which is given by the cocycle of transition functions

$$\tau_{\alpha\beta}(x) = \operatorname{sign} \varphi_{\alpha\beta}(x) = \operatorname{sign} \det d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x)).$$

Since $\tau_{\alpha\beta}(x)\varphi_{\alpha\beta}(x) = \psi_{\alpha\beta}(x)$, the cocycle of the volume bundle of 8.2, we have

$$\begin{aligned} \operatorname{Vol}(M) &= \operatorname{Or}(M) \otimes \Lambda^m T^* M \\ \Lambda^m T^* M &= \operatorname{Or}(M) \otimes \operatorname{Vol}(M) \end{aligned}$$

8.7. Definition. A manifold M is called *orientable* if the orientation bundle $\operatorname{Or}(M)$ is trivial. Obviously this is the case if and only if there exists an atlas (U_α, u_α) for the smooth structure of M such that $\det d(u_\alpha \circ u_\beta^{-1})(u_\beta(x)) > 0$ for all $x \in U_{\alpha\beta}$.

If M is orientable there are two distinguished global frames for the orientation bundle $\operatorname{Or}(M)$, namely those with absolute value $|s(x)| = 1$. Since the transition functions of $\operatorname{Or}(M)$ take only the values $+1$ and -1 there is a well defined notion of a fiberwise absolute value on $\operatorname{Or}(M)$, given by $|s(x)| := pr_2\, \tau_\alpha(s(x))$, where (U_α, τ_α) is a vector bundle chart of $\operatorname{Or}(M)$ induced by an atlas for M.

The two normed frames s_1 and s_2 of $\operatorname{Or}(M)$ will be called the two possible *orientations* of the orientable manifold M. M is called an *oriented manifold* if one of these two normed frames of $\operatorname{Or}(M)$ is specified: it is denoted by $\mathfrak{o}_M$.

If M is oriented then $\operatorname{Or}(M) \cong M \times \mathbb{R}$ with the help of the orientation, so we have also

$$\Lambda^m T^* M = \operatorname{Or}(M) \otimes \operatorname{Vol}(M) = (M \times \mathbb{R}) \otimes \operatorname{Vol}(M) = \operatorname{Vol}(M).$$

So an orientation gives us a canonical identification of m-forms and densities. Thus for an m-form $\omega \in \Omega^m(M)$ the integral

$$\int_M \omega$$

is defined by the isomorphism above as the integral of the associated density, see 8.3. If (U_α, u_α) is an oriented atlas (i. e. in each induced vector bundle chart

(U_α, τ_α) for $\operatorname{Or}(M)$ we have $\tau_\alpha(\mathfrak{o}_M) = 1)$ then the integral of the m-form ω is given by

$$\begin{aligned}\int_M \omega &= \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \\ &:= \sum_\alpha \int_{U_\alpha} f_\alpha . \omega^\alpha \, du^1 \wedge \dots \wedge du^m \\ &:= \sum_\alpha \int_{u_\alpha(U_\alpha)} f_\alpha(u_\alpha^{-1}(y)).\omega^\alpha(u_\alpha^{-1}(y)) \, dy^1 \wedge \dots \wedge dy^m,\end{aligned}$$

where the last integral has to be interpreted as an oriented integral on an open subset in $\mathbb{R}^m$.

8.8. Manifolds with boundary. A *manifold with boundary M* is a second countable metrizable topological space together with an equivalence class of smooth atlases (U_α, u_α) which consist of *charts with boundary*: so $u_\alpha : U_\alpha \to u_\alpha(U_\alpha)$ is a homeomorphism from U_α onto an open subset of a *half space* $(-\infty, 0] \times \mathbb{R}^{m-1} = \{(x_1, \dots, x_m) : x_1 \le 0\}$, and all chart changes $u_{\alpha\beta} : u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta)$ are smooth in the sense that they are restrictions of smooth mappings defined on open (in $\mathbb{R}^m$) neighborhoods of the respective domains. There is a more intrinsic treatment of this notion of smoothness by means of Whitney jets, see [Tougeron, 1972].

We have $u_{\alpha\beta}(u_\beta(U_\alpha \cap U_\beta) \cap (0 \times \mathbb{R}^{m-1})) = u_\alpha(U_\alpha \cap U_\beta) \cap (0 \times \mathbb{R}^{m-1})$ since interiour points (with respect to $\mathbb{R}^m$) are mapped to interior points by the inverse function theorem.

Thus the boundary of M, denoted by ∂M, is uniquely given as the set of all points $x \in M$ such that $u_\alpha(x) \in 0 \times \mathbb{R}^{m-1}$ for one (equivalently any) chart (U_α, u_α) of M. Obviously the boundary ∂M is itself a smooth manifold of dimension $m - 1$.

A simple example: the closed unit ball $B^m = \{x \in \mathbb{R}^m : |x| \le 1\}$ is a manifold with boundary, its boundary is $\partial B^m = S^{m-1}$.

The notions of smooth functions, smooth mappings, tangent bundle (use the approach 1.9 without any change in notation) are analogous to the usual ones. If $x \in \partial M$ we may distinguish in $T_x M$ tangent vectors pointing into the interior, pointing into the exterior, and those in $T_x(\partial M)$.

8.9. Lemma. *Let M be a manifold with boundary of dimension M. Then M is a submanifold with boundary of an m-dimensional manifold $\tilde{M}$ without boundary.*

Proof. Using partitions of unity we construct a vector field X on M which points strictly into the interior of M. We may multiply X by a strictly positive function so that the flow Fl^X_t exists for all $0 \le t < 2\varepsilon$ for some $\varepsilon > 0$. Then $\mathrm{Fl}^X_\varepsilon : M \to M \setminus \partial M$ is a diffeomorphism onto its image which embeds M as a submanifold with boundary of $M \setminus \partial M$. $\square$

8.10. Lemma. *Let M be an oriented manifold with boundary. Then there is a canonical induced orientation on the boundary ∂M.*

Proof. Let (U_α, u_α) be an oriented atlas for M. Then $u_{\alpha\beta} : u_\beta(U_{\alpha\beta} \cap \partial M) \to u_\alpha(U_{\alpha\beta} \cap \partial M)$, thus for $x \in u_\beta(U_{\alpha\beta} \cap \partial M)$ we have $du_{\alpha\beta}(x) : 0 \times \mathbb{R}^{m-1} \to 0 \times \mathbb{R}^{m-1}$,

$$du_{\alpha\beta}(x) = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ * & & * & \end{pmatrix},$$

where $\lambda > 0$ since $du_{\alpha\beta}(x)(-e_1)$ is again downwards pointing. So

$$\det du_{\alpha\beta}(x) = \lambda \det(du_{\alpha\beta}(x)|0 \times \mathbb{R}^{m-1}) > 0,$$

consequently $\det(du_{\alpha\beta}(x)|0 \times \mathbb{R}^{m-1}) > 0$ and the restriction of the atlas (U_α, u_α) is an oriented atlas for ∂M. $\square$

8.11. Theorem of Stokes. *Let M be an m-dimensional oriented manifold with boundary ∂M. Then for any $(m-1)$-form $\omega \in \Omega^{m-1}_c(M)$ with compact support on M we have*

$$\int_M d\omega = \int_{\partial M} i^*\omega = \int_{\partial M} \omega,$$

where $i : \partial M \to M$ is the embedding.

Proof. Clearly $d\omega$ has again compact support. Let (U_α, u_α) be an oriented smooth atlas for M and let (f_α) be a smooth partition of unity with $\operatorname{supp}(f_\alpha) \subset U_\alpha$. Then we have $\sum_\alpha f_\alpha \omega = \omega$ and $\sum_\alpha d(f_\alpha\omega) = d\omega$. Consequently $\int_M d\omega = \sum_\alpha \int_{U_\alpha} d(f_\alpha\omega)$ and $\int_{\partial M} \omega = \sum_\alpha \int_{\partial U_\alpha} f_\alpha\omega$. It suffices to show that for each α we have $\int_{U_\alpha} d(f_\alpha\omega) = \int_{\partial U_\alpha} f_\alpha\omega$. For simplicity's sake we now omit the index α. The form $f\omega$ has compact support in U and we have in turn

$$\begin{aligned} f\omega &= \sum_{k=1}^m \omega_k du^1 \wedge \dots \wedge \widehat{du^k} \dots \wedge du^m \\ d(f\omega) &= \sum_{k=1}^m \frac{\partial \omega_k}{\partial u^k} du^k \wedge du^1 \wedge \dots \wedge \widehat{du^k} \dots \wedge du^m \\ &= \sum_{k=1}^m (-1)^{k-1} \frac{\partial \omega_k}{\partial u^k} du^1 \wedge \dots \wedge du^m. \end{aligned}$$

Since $i^* du^1 = 0$ we have $f\omega|\partial U = i^*(f\omega) = \omega_1 du^2 \wedge \dots \wedge du^m$, where $i : \partial U \to U$ is the embedding. Finally we get

$$\begin{aligned}
\int_U d(f\omega) &= \int_U \sum_{k=1}^m (-1)^{k-1} \frac{\partial \omega_k}{\partial u^k} du^1 \wedge \dots \wedge du^m \\
&= \sum_{k=1}^m (-1)^{k-1} \int_U \frac{\partial \omega_k}{\partial u^k} du^1 \wedge \dots \wedge du^m \\
&= \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^0 \frac{\partial \omega_1}{\partial x^1} dx^1 \right) dx^2 \dots dx^m \\
&\quad + \sum_{k=2}^m (-1)^{k-1} \int_{(-\infty,0]\times\mathbb{R}^{m-2}} \left(\int_{-\infty}^\infty \frac{\partial \omega_k}{\partial x^k} dx^k \right) dx^1 \dots \widehat{dx^k} dx^m \\
&= \int_{\mathbb{R}^{m-1}} (\omega_1(0, x^2, \dots, x^m) - 0) dx^2 \dots dx^m \\
&= \int_{\partial U} (\omega_1|\partial U) du^2 \dots du^m = \int_{\partial U} f\omega.
\end{aligned}$$

We used the fundamental theorem of calculus:

$$\begin{aligned}
&\int_{-\infty}^0 \frac{\partial \omega_1}{\partial x^1} dx^1 = \omega_1(0, x^2, \dots, x^m) - 0, \\
&\int_{-\infty}^\infty \frac{\partial \omega_k}{\partial x^k} dx^k = 0,
\end{aligned}$$

since $f\omega$ has compact support in U. $\square$

9. De Rham cohomology

9.1. De Rham cohomology. Let M be a smooth manifold which may have boundary. We consider the graded algebra $\Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M)$ of all differential forms on M. Then the space $Z(M) := \{\omega \in \Omega(M) : d\omega = 0\}$ of *closed forms* is a graded subalgebra of Ω (i. e. it is a subalgebra and $\Omega^K(M) \cap Z(M) = Z^k(M)$), and the space $B(M) := \{d\varphi : \varphi \in \Omega(M)\}$ is a graded ideal in $Z(M)$. This follows directly from the derivation property $d(\varphi\wedge\psi) = d\varphi\wedge\psi + (-1)^{\deg\varphi}\varphi\wedge d\psi$ of the exterior derivative.

Definition. The algebra

$$H^*(M) := \frac{Z(M)}{B(M)} = \frac{\{\omega \in \Omega(M) : d\omega = 0\}}{\{d\varphi : \varphi \in \Omega(M)\}}$$

is called the *De Rham cohomology algebra* of the manifold M. It is graded by

$$H^*(M) = \bigoplus_{k=0}^{\dim M} H^k(M) = \bigoplus_{k=0}^{\dim M} \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im} d : \Omega^{k-1}(M) \to \Omega^k(M)}.$$

If $f : M \to N$ is a smooth mapping between manifolds then $f^* : \Omega(N) \to \Omega(N)$ is a homomorphism of graded algebras by 7.5 which satisfies $d \circ f^* = f^* \circ d$ by 7.9. Thus f^* induces an algebra homomorphism which we call again $f^* : H^*(N) \to H^*(M)$.

9.2. Remark. Since $\Omega^k(M) = 0$ for $k > \dim M =: m$ we have

$$\begin{aligned}
H^m(M) &= \frac{\Omega^m(M)}{\{d\varphi : \varphi \in \Omega^{m-1}(M)\}}. \\
H^k(M) &= 0 \qquad \text{for } k > m. \\
H^0(M) &= \frac{\{f \in \Omega^0(M) = C^\infty(M,\mathbb{R}) : df = 0\}}{0} \\
&= \text{ the space of locally constant functions on } M \\
&= \mathbb{R}^{b_0(M)},
\end{aligned}$$

where $b_0(M)$ is the number of arcwise connected components of M. We put $b_k(M) := \dim_{\mathbb{R}} H^k(M)$ and call it the k-th *Betti number* of M. If $b_k(M) < \infty$ for all k we put

$$f_M(t) := \sum_{k=0}^{m} b_k(M) t^k$$

and call it the *Poincaré polynomial* of M. The number

$$\chi_M := \sum_{k=0}^{m} b_k(M)(-1)^k = f_M(-1)$$

is called the *Euler Poincaré characteristic* of M, see also 11.7 below.

9.3. Examples. We have $H^0(\mathbb{R}^m) = \mathbb{R}$ since it has only one connected component. We have $H^k(\mathbb{R}^m) = 0$ for $k > 0$ by the proof of the lemma of Poincaré 7.10.

For the one dimensional sphere we have $H^0(S^1) = \mathbb{R}$ since it is connected, and clearly $H^k(S^1) = 0$ for $k > 1$ by reasons of dimension. And we have

$$\begin{aligned} H^1(S^1) &= \frac{\{\omega \in \Omega^1(M) : d\omega = 0\}}{\{d\varphi : \varphi \in \Omega^0(M)\}} \\ &= \frac{\Omega^1(M)}{\{df : f \in C^\infty(S^1, \mathbb{R})\}}, \\ \Omega^1(S^1) &= \{f\, dt : f \in C^\infty(S^1, \mathbb{R})\} \\ &\cong \{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f \text{ is periodic with period } 2\pi\}, \end{aligned}$$

where dt denotes the global coframe of T^*S^1. If f is periodic with period 2π then $f\, dt$ is exact if and only if $\int f\, dt$ is also 2π periodic, i. e. $\int_0^{2\pi} f(t)dt = 0$. So we have

$$\begin{aligned} H^1(S^1) &= \frac{\{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f \text{ is periodic with period } 2\pi\}}{\{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f \text{ is periodic with period } 2\pi, \int_0^{2\pi} = 0\}} \\ &= \mathbb{R}, \end{aligned}$$

where $f \mapsto \int_0^{2\pi} f\, dt$ factors to the isomorphism.

9.4. Lemma. *Let f, $g : M \to N$ be smooth mappings between manifolds which are C^∞-homotopic: there exists $h \in C^\infty(\mathbb{R} \times M, N)$ with $h(0, x) = f(x)$ and $h(1, x) = g(x)$.*

Then f and g induce the same mapping in cohomology: $f^* = g^* : H(N) \to H(M)$.

Remark. f, $g \in C^\infty(M, N)$ are called homotopic if there exists a continuous mapping $h : [0, 1] \times M \to N$ with with $h(0, x) = f(x)$ and $h(1, x) = g(x)$. This seemingly looser relation in fact coincides with the relation of C^∞-homotopy. We sketch a proof of this statement: let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth function with

$\varphi((-\infty,1/4])=0$, $\varphi([3/4,\infty))=1$, and φ monotone in between. Then consider $\bar h:\mathbb{R}\times M\to N$, given by $\bar h(t,x)=h(\varphi(t),x)$. Now we may approximate $\bar h$ by smooth functions $\tilde h:\mathbb{R}\times M\to N$ whithout changing it on $(-\infty,1/8)\times M$ where it equals f, and on $(7/8,\infty)\times M$ where it equals g. This is done chartwise by convolution with a smooth function with small support on $\mathbb{R}^m$. See [Bröcker-Jänich, 1973] for a careful presentation of the approximation.

So we will use the equivalent concept of homotopic mappings below.

Proof. For $\omega\in\Omega^k(M)$ we have $h^*\omega\in\Omega^k(\mathbb{R}\times M)$. We consider the insertion operator $\operatorname{ins}_t:M\to\mathbb{R}\times M$, given by $\operatorname{ins}_t(x)=(t,x)$. For $\varphi\in\Omega^k(\mathbb{R}\times M)$ we then have a smooth curve $t\mapsto\operatorname{ins}_t^*\varphi$ in $\Omega^k(M)$ (this can be made precise with the help of the calculus in infinite dimensions of [Frölicher-Kriegl, 1988]). We define the integral operator $I_0^1:\Omega^k(\mathbb{R}\times M)\to\Omega^k(M)$ by $I_0^1(\varphi):=\int_0^1\operatorname{ins}_t^*\varphi\,dt$. Looking at this locally on M one sees that it is well defined, even without Frölicher-Kriegl calculus. Let $T:=\frac{\partial}{\partial t}\in\mathfrak{X}(\mathbb{R}\times M)$ be the unit vector field in direction $\mathbb{R}$.

We have $\operatorname{ins}_{t+s}=\operatorname{Fl}_t^T\circ\operatorname{ins}_s$ for $s,\,t\in\mathbb{R}$, so

$$\begin{aligned}\tfrac{\partial}{\partial s}\operatorname{ins}_s^*\varphi &= \tfrac{\partial}{\partial t}\big|_0(\operatorname{Fl}_t^T\circ\operatorname{ins}_s)^*\varphi=\tfrac{\partial}{\partial t}\big|_0\operatorname{ins}_s^*(\operatorname{Fl}_t^T)^*\varphi\\ &=\operatorname{ins}_s^*\tfrac{\partial}{\partial t}\big|_0(\operatorname{Fl}_t^T)^*\varphi=(\operatorname{ins}_s)^*\mathcal{L}_T\varphi \qquad\text{by 7.6.}\end{aligned}$$

We have used that $(ins_s)^*:\Omega^k(\mathbb{R}\times M)\to\Omega^k(M)$ is linear and continuous and so one may differentiate through it by the chain rule. This can also be checked by evaluating at $x\in M$. Then we have in turn

$$\begin{aligned}d\,I_0^1\,\varphi&=d\int_0^1\operatorname{ins}_t^*\varphi\,dt=\int_0^1 d\,\operatorname{ins}_t^*\varphi\,dt\\ &=\int_0^1\operatorname{ins}_t^*\,d\varphi\,dt=I_0^1\,d\,\varphi\qquad\text{by 7.9.(4).}\\ (\operatorname{ins}_1^*-\operatorname{ins}_0^*)\varphi&=\int_0^1\tfrac{\partial}{\partial t}\operatorname{ins}_t^*\varphi\,dt=\int_0^1\operatorname{ins}_t^*\mathcal{L}_T\varphi\,dt\\ &=I_0^1\,\mathcal{L}_T\varphi=I_0^1(d\,i_T+i_T\,d)\varphi\qquad\text{by 7.9.}\end{aligned}$$

Now we define the *homotopy operator* $\bar h:=I_0^1\circ i_T\circ h^*:\Omega^k(M)\to\Omega^{k-1}(M)$. Then we get

$$\begin{aligned}g^*-f^*&=(h\circ\operatorname{ins}_1)^*-(h\circ\operatorname{ins}_0)^*=(\operatorname{ins}_1^*-\operatorname{ins}_0^*)\circ h^*\\ &=(d\circ I_0^1\circ i_T+I_0^1\circ i_T\circ d)\circ h^*=d\circ\bar h-\bar h\circ d,\end{aligned}$$

which implies the desired result since for $\omega\in\Omega^k(M)$ with $d\omega=0$ we have $g^*\omega-f^*\omega=\bar hd\omega+d\bar h\omega=d\bar h\omega$. $\square$

9.5. Lemma. *If a manifold is decomposed into a disjoint union $M = \bigsqcup_\alpha M_\alpha$ of open submanifolds, then $H^k(M) = \prod_\alpha H^k(M_\alpha)$ for all k.*

Proof. $\Omega^k(M)$ is isomorphic to $\prod_\alpha \Omega^k(M_\alpha)$ via $\varphi \mapsto (\varphi|M_\alpha)_\alpha$. This isomorphism commutes with exterior differential d and induces the result. □

9.6. The setting for the Mayer-Vietoris Sequence. Let M be a smooth manifold, let U, $V \subset M$ be open subsets such that $M = U \cup V$. We consider the following embeddings:

$$\begin{tikzcd} & U \cap V \arrow[dl, hook', "j_U"'] \arrow[dr, hook, "j_V"] & \\ U \arrow[dr, hook, "i_U"'] & & V \arrow[dl, hook', "i_V"] \\ & M. & \end{tikzcd}$$

Lemma. *In this situation the sequence*

$$0 \to \Omega(M) \xrightarrow{\alpha} \Omega(U) \oplus \Omega(V) \xrightarrow{\beta} \Omega(U \cap V) \to 0$$

is exact, where $\alpha(\omega) := (i_U^*\omega, i_V^*\omega)$ and $\beta(\varphi, \psi) = j_U^*\varphi - j_V^*\psi$. We also have $(d \oplus d) \circ \alpha = \alpha \circ d$ and $d \circ \beta = \beta \circ (d \oplus d)$.

Proof. We have to show that α is injective, $\ker \beta = \operatorname{im} \alpha$, and that β is surjective. The first two assertions are obvious and for the last one we we let $\{f_U, f_V\}$ be a partition of unity with $\operatorname{supp} f_U \subset U$ and $\operatorname{supp} f_V \subset V$. For $\varphi \in \Omega(U \cap V)$ we consider $f_V\varphi \in \Omega(U \cap V)$, note that $\operatorname{supp}(f_V\varphi)$ is closed in the set $U \cap V$ which is open in U, so we may extend $f_V\varphi$ by 0 to $\varphi_U \in \Omega(U)$. Likewise we extend $-f_U\varphi$ by 0 to $\varphi_V \in \Omega(V)$. Then we have $\beta(\varphi_U, \varphi_V) = (f_U + f_V)\varphi = \varphi$. □

Now we are in the situation where we may apply the main theorem of homological algebra, 9.8. So we deviate now to develop the basics of homological algebra.

9.7. The essentials of homological algebra. A *graded differential space* (GDS) $K = (K, d)$ is a sequence

$$\cdots \to K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \to \cdots$$

of abelian groups K^n and group homomorphisms $d^n : K^n \to K^{n+1}$ such that $d^{n+1} \circ d^n = 0$. In our case these are the vector spaces $K^n = \Omega^n(M)$ and the exterior derivative. The group

$$H^n(K) := \frac{\ker(d^n : K^n \to K^{n+1})}{\operatorname{im}(d^{n-1} : K^{n-1} \to K^n)}$$

is called the n-th *cohomology group* of the GDS K. We consider also the direct sum

$$H^*(K) := \bigoplus_{n=-\infty}^{\infty} H^n(K)$$

as a graded group. A homomorphism $f : K \to L$ of graded differential spaces is a sequence of homomorphisms $f^n : K^n \to L^n$ such that $d^n \circ f^n = f^{n+1} \circ d^n$. It induces a homomorphism $f_* = H^*(f) : H^*(K) \to H^*(L)$ and H^* has clearly the properties of a functor from the category of graded differential spaces into the category of graded group: $H^*(Id_K) = Id_{H^*(K)}$ and $H^*(f \circ g) = H^*(f) \circ H^*(g)$.

A graded differential space (K, d) is called a graded differential algebra if $\bigoplus_n K^n$ is an associative algebra which is graded (so $K^n.K^m \subset K^{n+m}$), such that the differential d is a graded derivation: $d(x.y) = dx.y + (-1)^{\deg x} x.dy$. The cohomology group $H^*(K, d)$ of a graded differential algebra is a graded algebra, see 9.1.

By a *short exact sequence* of graded differential spaces we mean a sequence

$$0 \to K \xrightarrow{i} L \xrightarrow{p} M \to 0$$

of homomorphism of graded differential spaces which is degreewise exact: For each n the sequence $0 \to K^n \to L^n \to M^n \to 0$ is exact.

9.8. Theorem. *Let*

$$0 \to K \xrightarrow{i} L \xrightarrow{p} M \to 0$$

be an exact sequence of graded differential spaces. Then there exists a graded homomorphism $\delta = (\delta^n : H^n(M) \to H^{n+1}(K))_{n \in \mathbb{Z}}$ called the "connecting homomorphism" such that the following is an exact sequence of abelian groups:

$$\cdots \to H^{n-1}(M) \xrightarrow{\delta} H^n(K) \xrightarrow{i_*} H^n(L) \xrightarrow{p_*} H^n(M) \xrightarrow{\delta} H^{n+1}(K) \to \cdots$$

It is called the "long exact sequence in cohomology". δ is a natural transformation in the following sense: Let

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "k"'] & L \arrow[r, "p"] \arrow[d, "\ell"'] & M \arrow[r] \arrow[d, "m"'] & 0 \\ 0 \arrow[r] & K' \arrow[r, "i'"'] & L' \arrow[r, "p'"'] & M' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of homomorphisms of graded differential spaces with exact lines. Then also the following diagram is commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(M) \arrow[r, "\delta"] \arrow[d, "m_*"'] & H^n(K) \arrow[r, "i_*"] \arrow[d, "k_*"'] & H^n(L) \arrow[r, "p_*"] \arrow[d, "\ell_*"'] & H^n(M) \arrow[r] \arrow[d, "m_*"'] & \cdots \\
\cdots \arrow[r] & H^{n-1}(M') \arrow[r, "\delta'"'] & H^n(K') \arrow[r, "i'_*"'] & H^n(L') \arrow[r, "p'_*"'] & H^n(M) \arrow[r] & \cdots
\end{tikzcd}$$

The long exact sequence in cohomology is also written in the following way:

$$\begin{tikzcd}
H^*(K) \arrow[rr, "i_*"] & & H^*(L) \arrow[dl, "p_*"] \\
& H^*(M) \arrow[ul, "\delta"] &
\end{tikzcd}$$

Definition of δ. The connecting homomorphism is defined by '$\delta = i^{-1} \circ d \circ p^{-1}$' or $\delta[p\ell] = [i^{-1} d\ell]$. This is meant as follows.

$$\begin{tikzcd}
& & L^{n-1} \arrow[r, "p^{n-1}"] \arrow[d, "d^{n-1}"'] & M^{n-1} \arrow[r] \arrow[d, "d^{n-1}"'] & 0 \\
0 \arrow[r] & K^n \arrow[r, "i^n"] \arrow[d, "d^n"'] & L^n \arrow[r, "p^n"] \arrow[d, "d^n"'] & M^n \arrow[r] \arrow[d, "d^n"'] & 0 \\
0 \arrow[r] & K^{n+1} \arrow[r, "i^{n+1}"] \arrow[d, "d^{n+1}"'] & L^{n+1} \arrow[r, "p^{n+1}"] \arrow[d, "d^{n+1}"'] & M^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & K^{n+2} \arrow[r, "i^{n+2}"] & L^{n+2} & &
\end{tikzcd}$$

The following argument is called a diagram chase. Let $[m] \in H^n(M)$. Then $m \in M^n$ with $dm = 0$. Since p is surjective there is $\ell \in L^n$ with $p\ell = m$. We consider $d\ell \in L^{n+1}$ for which we have $pd\ell = dp\ell = dm = 0$, so $d\ell \in \ker p = \operatorname{im} i$, thus there is an element $k \in K^{n+1}$ with $ik = d\ell$. We have $idk = dik = dd\ell = 0$. Since i is injective we have $dk = 0$, so $[k] \in H^{n+1}(K)$.

Now we put $\delta[m] := [k]$ or $\delta[p\ell] = [i^{-1} d\ell]$.

This method of diagram chasing can be used for the whole proof of the theorem. The reader is advised to do it at least once in his life with fingers on the diagram above. For the naturality imagine two copies of the diagram lying above each other with homomorphisms going up.

9.9. Five-Lemma. *Let*

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "\varphi_4"'] & A_5 \arrow[d, "\varphi_5"'] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \end{tikzcd}$$

be a commutative diagram of abelian groups with exact lines. If φ_1, φ_2, φ_4, and φ_5 are isomorphisms then also the middle φ_3 is an isomorphism.

Proof. Diagram chasing in this diagram leads to the result. The chase becomes simpler if one first replaces the diagram by the following equivalent one with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & A_2/\operatorname{im}\alpha_1 \arrow[r, "\alpha_2'"] \arrow[d, "\varphi_2'"', "\cong"] & A_3 \arrow[r, "\alpha_3'"] \arrow[d, "\varphi_3"'] & \ker\alpha_4 \arrow[r] \arrow[d, "\varphi_4'"', "\cong"] & 0 \\ 0 \arrow[r] & B_2/\operatorname{im}\beta_2 \arrow[r, "\beta_2'"] & B_3 \arrow[r, "\beta_3'"] & \ker\beta_4 \arrow[r] & 0. \end{tikzcd}$$ □

9.10. Theorem. Mayer-Vietoris sequence. *Let U and V be open subsets in a manifold M such that $M = U \cup V$. Then there is an exact sequence*

$$\dots \to H^k(M) \xrightarrow{\alpha_*} H^k(U) \oplus H^k(V) \xrightarrow{\beta_*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \to \cdots$$

It is natural in the triple (M, U, V) in the sense explained in 9.8. The homomorphisms α_* and β_* are algebra homomorphisms, but δ is not.

Proof. This follows from 9.6 and theorem 9.8. □

Since we shall need it later we will give now a detailed description of the connecting homomorphism δ. Let $\{f_U, f_V\}$ be a partition of unity with $\operatorname{supp} f_U \subset U$ and $\operatorname{supp} f_V \subset V$. Let $\omega \in \Omega^k(U \cap V)$ with $d\omega = 0$ so that $[\omega] \in H^k(U \cap V)$. Then $(f_V.\omega, -f_U.\omega) \in \Omega^k(U) \oplus \Omega^k(V)$ is mapped to ω by β and so we have by the prescrition in 9.8

$$\begin{aligned} \delta[\omega] &= [\alpha^{-1}\, d(f_V.\omega, -f_U.\omega)] = [\alpha^{-1}(df_V \wedge \omega, -df_U \wedge \omega)] \\ &= [df_V \wedge \omega] = -[df_U \wedge \omega)], \end{aligned}$$

where we have used the following fact: $f_U + f_V = 1$ implies that on $U \cap V$ we have $df_V = -df_U$ thus $df_V \wedge \omega = -df_U \wedge \omega$ and off $U \cap V$ both are 0.

9.11. Axioms for cohomology. The De Rham cohomology is uniquely determined by the following properties which we have already verified:

(1) $H^*(\quad)$ is a contravariant functor from the category of smooth manifolds and smooth mappings into the category of $\mathbb{Z}$-graded groups and graded homomorphisms.
(2) $H^k(\text{point}) = \mathbb{R}$ for $k = 0$ and $= 0$ for $k \neq 0$.
(3) If f and g are C^∞-homotopic then $H^*(f) = H^*(g)$.
(4) If $M = \bigsqcup_\alpha M_\alpha$ is a disjoint union of open subsets then $H^*(M) = \prod_\alpha H^*(M_\alpha)$.
(5) If U and V are open in M then there exists a connecting homomorphism $\delta : H^k(U \cap V) \to H^{k+1}(U \cup V)$ which is natural in the triple $(U \cup V, U, V)$ such that the following sequence is exact:

$$\dots \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(U \cup V) \to \cdots$$

There are lots of other cohomology theories for topological spaces like singular cohomology, Čech-cohomology, simplicial cohomology, Alexander-Spanier cohomology etc which satisfy the above axioms for manifolds when defined with real coefficients, so they all coincide with the De Rham cohomology on manifolds. See books on algebraic topology or sheaf theory for all this.

9.12. Example. If M is contractible (which is equivalent to the seemingly stronger concept of C^∞-contractibility, see the remark in 9.4) then $H^0(M) = \mathbb{R}$ since M is connected, and $H^k(M) = 0$ for $k \neq 0$, because the constant mapping $c : M \to \text{ point } \to M$ onto some fixed point of M is homotopic to Id_M, so $H^*(c) = H^*(Id_M) = Id_{H^*(M)}$ by 9.4. But we have

$$\begin{tikzcd} H^k(M) \arrow[rr, "H^k(c)"] \arrow[dr] & & H^k(M) \\ & H^k(\text{point}) \arrow[ur, hook] & \end{tikzcd}$$

More generally, two manifolds M and N are called to be smoothly homotopy equivalent if there exist smooth mappings $f : M \to N$ and $g : N \to M$ such that $g \circ f$ is homotopic to Id_M and $f \circ g$ is homotopic to Id_N. If this is the case both $H^*(f)$ and $H^*(g)$ are isomorphisms, since $H^*(g) \circ H^*(f) = Id_{H^*(M)}$ and $H^*(f) \circ H^*(g) = Id_{H^*(N)}$.

As an example consider a vector bundle (E, p, M) with zero section $0_E : M \to E$. Then $p \circ 0_E = Id_M$ whereas $0_E \circ p$ is homotopic to Id_E via $(t, u) \mapsto t.u$. Thus $H^*(E)$ is isomorphic to $H^*(M)$.

9.13. Example. The cohomology of spheres. *For $n \geq 1$ we have*

$$H^k(S^n) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } 1 \leq k \leq n-1 \\ \mathbb{R} & \text{for } k = n \\ 0 & \text{for } k > n \end{cases}$$

$$H^k(S^0) = H^k(\text{2 points}) = \begin{cases} \mathbb{R}^2 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

We may say: The cohomology of S^n has two generators as graded vector space, one in dimension 0 and one in dimension n. The Poincaré polynomial is given by $f_{S^n}(t) = 1 + t^n$.

Proof. The assertion for S^0 is obvious, and for S^1 it was proved in 9.3 so let $n \geq 2$. Then $H^0(S^n) = \mathbb{R}$ since it is connected, so let $k > 0$. Now fix a north pole $a \in S^n$, $0 < \varepsilon < 1$, and let

$$\begin{aligned} S^n &= \{x \in \mathbb{R}^{n+1} : |x|^2 = \langle x, x\rangle = 1\}, \\ U &= \{x \in S^n : \langle x, a\rangle > -\varepsilon\}, \\ V &= \{x \in S^n : \langle x, a\rangle < \varepsilon\}, \end{aligned}$$

so U and V are overlapping northern and southern hemispheres, respectively, which are diffeomorphic to an open ball and thus smoothly contractible. Their cohomology is thus described in 9.12. Clearly $U \cup V = S^n$ and $U \cap V \cong S^{n-1} \times (-\varepsilon, \varepsilon)$ which is obviously (smoothly) homotopy equivalent to S^{n-1}. By theorem 9.10 we have the following part of the Mayer-Vietoris sequence

$$\begin{tikzcd} H^k(U) \oplus H^k(V) \arrow[r] \arrow[d, equal] & H^k(U \cap V) \arrow[r, "\delta"] \arrow[d, equal] & H^{k+1}(S^n) \arrow[r] & H^{k+1}(U) \oplus H^{k+1}(V) \arrow[d, equal] \\ 0 & H^k(S^{n-1}) & & 0, \end{tikzcd}$$

where the vertical isomorphisms come from 9.12. So we have $H^k(S^{n-1}) \cong H^{k+1}(S^n)$ for $k > 0$ and $n \geq 2$.

Next we look at the initial segment of the Mayer-Vietoris sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^0(S^n) \arrow[r] \arrow[d, equal] & H^0(U \sqcup V) \arrow[r, "\beta"] \arrow[d, equal] & H^0(U \cap V) \arrow[r, "\delta"] \arrow[d, equal] & H^1(S^n) \arrow[r] & H^1(U \sqcup V) \arrow[d, equal] \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "\alpha"] & \mathbb{R}^2 \arrow[r] & \mathbb{R} & & 0 \end{tikzcd}$$

From exactness we have: in the lower line α is injective, so $\dim(\ker \beta) = 1$, so β is surjective and thus $\delta = 0$. This implies that $H^1(S^n) = 0$ for $n \geq 2$. Starting from $H^k(S^1)$ for $k > 0$ the result now follows by induction on n.

By looking more closely on on the initial segment of the Mayer-Vietoris sequence for $n = 1$ and taking into account the form of $\delta : H^0(S^0) \to H^1(S^1)$ we could even derive the result for S^1 without using 9.3. The reader is advised to try this. $\square$

9.14. Example. The Poincaré polynomial of the Stiefel manifold $V(k, n; \mathbb{R})$ of oriented orthonormal k-frames in $\mathbb{R}^n$ (see 15.5) is given by:

For:	$f_{V(k,n)} =$
$n = 2m,\ k = 2l + 1,\ l \geq 0:$	$(1 + t^{2m-1}) \prod_{i=1}^{l} (1 + t^{4m-4i-1})$
$n = 2m + 1,\ k = 2l,\ l \geq 1:$	$\prod_{i=1}^{l} (1 + t^{4m-4i+3})$
$n = 2m,\ k = 2l,\ m > l \geq 1:$	$(1 + t^{2m-2l})(1 + t^{2m-1}) \prod_{i=1}^{l-1} (1 + t^{4m-4i-1})$
$n = 2m + 1, k = 2l + 1,$ $m > l \geq 0:$	$(1 + t^{2m-2l}) \prod_{i=1}^{l-1} (1 + t^{4m-4i+3})$

Since $V(n - 1, n; \mathbb{R}) = SO(n; \mathbb{R})$ we get

$$f_{SO(2m;\mathbb{R})}(t) = (1 + t^{2m-1}) \prod_{i=1}^{m-1} (1 + t^{4i-1}),$$

$$f_{SO(2m+1,\mathbb{R})}(t) = \prod_{i=1}^{m} (1 + t^{4i-1}).$$

So the cohomology can be quite complicated. For a proof of these formulas using the Gysin sequence for sphere bundles see [Greub-Halperin-Vanstone II, 1973].

9.15. Relative De Rham cohomology. Let $N \subset M$ be a closed submanifold and let

$$\Omega^k(M, N) := \{\omega \in \Omega^k(M) : i^*\omega = 0\},$$

where $i : N \to M$ is the embedding. Since $i^* \circ d = d \circ i^*$ we get a graded differential subalgebra $(\Omega^*(M, N), d)$ of $(\Omega^*(M), d)$. Its cohomology, denoted by $H^*(M, N)$, is called the *relative De Rham cohomology* of the *manifold pair* (M, N).

9.16. Lemma. *In the setting of 9.15,*

$$0 \to \Omega^*(M,N) \hookrightarrow \Omega^*(M) \xrightarrow{i^*} \Omega^*(N) \to 0$$

is an exact sequence of differential graded algebras. Thus by 9.8 we the following long exact sequence in cohmology

$$\dots \to H^k(M,N) \to H^k(M) \to H^k(N) \xrightarrow{\delta} H^{k+1}(M,N) \to \dots$$

which is natural in the manifold pair (M,N). It is called the long exact cohomology sequence of the pair (M,N).

Proof. We only have to show that $i^* : \Omega^*(M) \to \Omega^*(N)$ is surjective. So we have to extend each $\omega \in \Omega^k(N)$ to the whole of M. We cover N by submanifold charts of M with respect to N. These and $M \setminus N$ cover M. On each of the submanifold charts one can easily extend the restriction of ω and one can glue all these extensions by a partition of unity which is subordinated to the cover of M. $\square$

10. Cohomology with compact supports and Poincaré duality

10.1. Cohomology with compact supports. Let $\Omega_c^k(M)$ denote the space of all k-forms with compact support on the manifold M. Since $\operatorname{supp}(d\omega) \subset \operatorname{supp}(\omega)$, $\operatorname{supp}(\mathcal{L}_X\omega) \subset \operatorname{supp}(X) \cap \operatorname{supp}(\omega)$, and $\operatorname{supp}(i_X\omega) \subset \operatorname{supp}(X) \cap \operatorname{supp}(\omega)$, all formulas of section 7 are also valid in $\Omega_c^*(M) = \bigoplus_{k=0}^{\dim M} \Omega_c^k(M)$. So $\Omega_c^*(M)$ is an ideal and a differential graded subalgebra of $\Omega^*(M)$. The cohomology of $\Omega_c^*(M)$

$$H_c^k(M) := \frac{\ker(d : \Omega_c^k(M) \to \Omega_c^{k+1}(M))}{\operatorname{im} d : \Omega_c^{k-1}(M) \to \Omega_c^k(M)},$$
$$H_c^*(M) := \bigoplus_{k=0}^{\dim M} H_c^k(M)$$

is called the *De Rham cohomology algebra with compact supports* of the manifold M. It has no unit if M is not compact.

10.2. Mappings. If $f : M \to N$ is a smooth mapping between manifolds and if $\omega \in \Omega_c^k(N)$ is a form with compact support, then $f^*\omega$ is a k-form on M, in general with noncompact support. So Ω_c^* is not a functor on the category of all smooth manifolds and all smooth mappings. But if we restrict the morphisms suitably, then Ω_c^* becomes a functor. There are two ways to do this:

(1) Ω_c^* is a contravariant functor on the category of all smooth manifolds and *proper smooth mappings* (f is called proper if f^{-1}(compact set) is a compact set) by the usual pullback operation.
(2) Ω_c^* is a covariant functor on the category of all smooth manifolds and embeddings of open submanifolds: for $i : U \hookrightarrow M$ and $\omega \in \Omega_c^k(U)$ just extend ω by 0 off U to get $i_*\omega \in \Omega_c^k(M)$. Clearly $i_* \circ d = d \circ i_*$.

10.3. Remark. 1. If a manifold M is a disjoint union, $M = \bigsqcup_\alpha M_\alpha$, then we have obviously $H_c^k(M) = \bigoplus_\alpha H_c^k(M_\alpha)$.

2. $H_c^0(M)$ is a direct sum of copies of $\mathbb{R}$, one for each compact connected component of M.

3. If M is compact, then $H_c^k(M) = H^k(M)$.

10.4. The Mayer-Vietoris sequence with compact supports. Let M be a smooth manifold, let U, $V \subset M$ be open subsets such that $M = U \cup V$. We consider the following embeddings:

$$\begin{tikzcd}
 & U\cap V \arrow[dl, hook', "j_U"'] \arrow[dr, hook, "j_V"] & \\
U \arrow[dr, hook, "i_U"'] & & V \arrow[dl, hook', "i_V"] \\
 & M. &
\end{tikzcd}$$

Theorem. *The following sequence of graded differential algebras is exact:*

$$0 \to \Omega^*_c(U\cap V) \xrightarrow{\beta_c} \Omega^*_c(U)\oplus\Omega^*_c(V) \xrightarrow{\alpha_c} \Omega^*_c(M) \to 0,$$

where $\beta_c(\omega) := ((j_U)_*\omega, (j_V)_*\omega)$ *and* $\alpha_c(\varphi,\psi) = (i_U)_*\varphi - (i_V)_*\psi$*. So by 9.8 we have the following long exact sequence*

$$\to H^{k-1}_c(M) \xrightarrow{\delta_c} H^k_c(U\cap V) \to H^k_c(U)\oplus H^k_c(V) \to H^k_c(M) \xrightarrow{\delta_c} H^{k+1}_c(U\cap V) \to$$

which is natural in the triple (M,U,V)*. It is called the Mayer Vietoris sequence with compact supports.*

The connecting homomorphism $\delta_c : H^k_c(M) \to H^{k+1}_c(U\cap V)$ is given by

$$\begin{aligned}\delta_c[\varphi] &= [\beta_c^{-1}\, d\,\alpha_c^{-1}(\varphi)] = [\beta_c^{-1}\, d(f_U\varphi, -f_V\varphi)]\\ &= [df_U\wedge\varphi \restriction U\cap V] = -[df_V\wedge\varphi\restriction U\cap V].\end{aligned}$$

Proof. The only part that is not completely obvious is that α_c is surjective. Let $\{f_U, f_V\}$ be a partition of unity with $\operatorname{supp}(f_U)\subset U$ and $\operatorname{supp}(f_V)\subset V$, and let $\varphi\in\Omega^k_c(M)$. Then $f_U\varphi\in\Omega^k_c(U)$ and $-f_V\varphi\in\Omega^k_c(V)$ satisfy $\alpha_c(f_U\varphi, -f_V\varphi) = (f_U+f_V)\varphi = \varphi$. $\square$

10.5. Proper homotopies. A smooth mapping $h : \mathbb{R}\times M\to N$ is called a *proper homotopy* if $h^{-1}(\text{ compact set })\cap([0,1]\times M)$ is compact. A continuous homotopy $h : [0,1]\times M\to N$ is a proper homotopy if and only if it is a proper mapping.

Lemma. *Let* $f, g : M\to N$ *be proper and proper homotopic, then* $f^* = g^* : H^k_c(N)\to H^k_c(M)$ *for all* k*.*

Proof. Recall the proof of lemma 9.4.
Claim. In the proof of 9.4 we have furthermore $\bar h : \Omega^k_c(N)\to\Omega^{k-1}_c(M)$.
Let $\omega\in\Omega^k_c(M)$ and let $K_1 := \operatorname{supp}(\omega)$, a compact set in M. Then $K_2 :=$

$h^{-1}(K_1) \cap ([0,1] \times M)$ is compact in $\mathbb{R} \times M$, and finally $K_3 := pr_2(K_2)$ is compact in M. If $x \notin K_3$ then we have

$$(\bar{h}\omega)_x = ((I_0^1 \circ i_T \circ h^*)\omega)_x = \int_0^1 (\operatorname{ins}_t^*(i_T h^* \omega))_x \, dt) = 0.$$

The rest of the proof is then again as in 9.4. $\square$

10.6. Lemma.

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \textit{for } k = n \\ 0 & \textit{else.} \end{cases}$$

First Proof. We embed $\mathbb{R}^n$ into its one point compactification $\mathbb{R}^n \cup \{\infty\}$ which is diffeomorphic to S^n, see 1.2. The embedding induces the exact sequence of complexes

$$0 \to \Omega_c(\mathbb{R}^n) \to \Omega(S^n) \to \Omega(S^n)_\infty \to 0,$$

where $\Omega(S^n)_\infty$ denotes the space of germs at the point $\infty \in S^n$. For germs at a point the lemma of Poincaré is valid, so we have $H^0(\Omega(S^n)_\infty) = \mathbb{R}$ and $H^k(\Omega(S^n)_\infty) = 0$ for $k > 0$. By theorem 9.8 there is a long exact sequence in cohomology whose beginning is:

$$\begin{array}{ccccccccccc} H_c^0(\mathbb{R}^n) & \longrightarrow & H^0(S^n) & \longrightarrow & H^0(\Omega(S^n)_\infty) & \xrightarrow{\delta} & H_c^1(\mathbb{R}^n) & \longrightarrow & H^1(S^n) & \longrightarrow & H^1(\Omega(S^n)_\infty) \\ \| & & \| & & \| & & & & & & \| \\ 0 & & \mathbb{R} & & \mathbb{R} & & & & & & 0 \end{array}$$

From this we see that $\delta = 0$ and consequently $H_c^1(\mathbb{R}^n) \cong H^1(S^n)$. Another part of this sequence for $k \geq 2$ is:

$$\begin{array}{ccccccc} H^{k-1}(\Omega(S^n)_\infty) & \xrightarrow{\delta} & H_c^k(\mathbb{R}^n) & \longrightarrow & H^k(S^n) & \longrightarrow & H^k(\Omega(S^n)_\infty) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

It implies $H_c^k(\mathbb{R}^n) \cong H^k(S^n)$ for all k. $\square$

10.7. Fiber integration. Let M be a manifold, $pr_1 : M \times \mathbb{R} \to M$. We define an operator called fiber integration

$$\int_{\text{fiber}} : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M)$$

as follows. Let t be the coordinate function on $\mathbb{R}$. A differential form with compact support on $M \times \mathbb{R}$ is a finite linear combination of two types of forms:

(1) $pr_1^*\varphi.f(x,t)$, shorter $\varphi.f$.
(2) $pr_1^*\varphi \wedge f(x,t)dt$, shorter $\varphi \wedge f dt$.

where $\varphi \in \Omega(M)$ and $f \in C_c^\infty(M \times \mathbb{R}, \mathbb{R})$. We then put

(1) $\int_{\text{fiber}} pr_1^*\varphi f := 0.$
(2) $\int_{\text{fiber}} pr_1^*\varphi \wedge f\,dt := \varphi \int_{-\infty}^{\infty} f(\quad, t)dt$

10.8. Lemma. *We have $d \circ \int_{\text{fiber}} = \int_{\text{fiber}} \circ d$. Thus $\int_{\text{fiber}}$ induces a mapping in cohomology*

$$\left(\int_{\text{fiber}}\right)_* : H_c^k(M \times \mathbb{R}) \to H_c^{k-1}(M),$$

which however is not an algebra homomorphism.

Proof. In case (1) we have

$$\begin{aligned}\int_{\text{fiber}} d(\varphi.f) &= \int_{\text{fiber}} d\varphi.f + (-1)^k \int_{\text{fiber}} \varphi.d_1 f + (-1)^k \int_{\text{fiber}} \varphi.\tfrac{\partial f}{\partial t}dt \\ &= (-1)^k \varphi \int_{-\infty}^{\infty} \tfrac{\partial f}{\partial t}dt = 0 \quad \text{ since } f \text{ has compact support} \\ &= d \int_{\text{fiber}} \varphi.f.\end{aligned}$$

In case (2) we get

$$\begin{aligned}\int_{\text{fiber}} d(\varphi \wedge f\,dt) &= \int_{\text{fiber}} d\varphi \wedge f\,dt + (-1)^k \int_{\text{fiber}} \varphi \wedge d_1 f \wedge dt \\ &= d\varphi \int_{-\infty}^{\infty} f(\quad, t)dt + (-1)^k \varphi \int_{-\infty}^{\infty} d_1 f(\quad, t)dt \\ &= d\left(\varphi \int_{\infty}^{\infty} f(\quad, t)dt\right) = d\int_{\text{fiber}} \varphi \wedge f\,dt. \quad \square\end{aligned}$$

10.9. In order to find a mapping in the converse direction we let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}$ with $\int_{-\infty}^{\infty} e(t)dt = 1$. We define $e_* : \Omega_c^k(M) \to \Omega_c^{k+1}(M \times \mathbb{R})$ by $e_*(\varphi) = \varphi \wedge e$. Then $de_*(\varphi) = d(\varphi \wedge e) = d\varphi \wedge e + 0 = e_*(d\varphi)$, so we have an induced mapping in cohomology $e_* : H_c^k(M) \to H_c^{k+1}(M \times \mathbb{R})$.

We have $\int_{\text{fiber}} \circ e_* = Id_{\Omega_c^k(M)}$, since

$$\int_{\text{fiber}} e_*(\varphi) = \int_{\text{fiber}} \varphi \wedge e(\quad)dt = \varphi \int_{\infty}^{\infty} e(t)dt = \varphi.$$

Next we define $K : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M \times \mathbb{R})$ by

(1) $K(\varphi.f) := 0$
(2) $K(\varphi \wedge f\,dt) = \varphi \int_{-\infty}^{t} f\,dt - \varphi.A(t) \int_{-\infty}^{\infty} f\,dt$, where $A(t) := \int_{-\infty}^{t} e(t)dt$.

10.10. Lemma. *Then we have*

$$Id_{\Omega^k_c(M\times\mathbb{R})} - e_* \circ \int_{\text{fiber}} = (-1)^{k-1}(d\circ K - K\circ d)$$

Proof. We have to check the two cases. In case (1) we have

$$\begin{aligned}
(Id - e_*\circ \int_{\text{fiber}})(\varphi.f) &= \varphi.f - 0,\\
(d\circ K - K\circ d)(\varphi.f) &= 0 - K(d\varphi.f + (-1)^k\varphi\wedge d_1 f + (-1)^k\varphi\wedge \tfrac{\partial f}{\partial t}dt)\\
&= -(-1)^k\left(\varphi\int_{-\infty}^t \tfrac{\partial f}{\partial t}dt - \varphi.A(t)\int_{-\infty}^\infty \tfrac{\partial f}{\partial t}dt\right)\\
&= (-1)^{k-1}\varphi.f + 0.
\end{aligned}$$

In case (2) we get

$$\begin{aligned}
(Id - e_*\circ \int_{\text{fiber}})(\varphi\wedge f\,dt) &= \varphi\wedge f\,dt - \varphi\int_{-\infty}^\infty f\,dt\wedge e,\\
(d\circ K - K\circ d)(\varphi\wedge f\,dt) &= d\left(\varphi\int_{-\infty}^t f\,dt - \varphi.A(t)\int_{-\infty}^\infty f\,dt\right)\\
&\qquad - K(d\varphi\wedge f\,dt + (-1)^{k-1}\varphi\wedge d_1 f\wedge dt)\\
&= (-1)^{k-1}\left(\varphi\wedge f\,dt - \varphi\wedge e\int_{-\infty}^\infty f\,dt\right) \qquad\square
\end{aligned}$$

10.11. Corollary. *The induced mappings $\left(\int_{\text{fiber}}\right)_*$ and e_* are inverse to each other, and thus isomorphism between $H^k_c(M\times\mathbb{R})$ and $H^{k-1}_c(M)$.*

Proof. This is clear from the chain homotopy 10.10. $\square$

10.12. Second Proof of 10.6. For $k\le n$ we have

$$\begin{aligned}
H^k_c(\mathbb{R}^n) &\cong H^{k-1}_c(\mathbb{R}^{n-1})\cong\dots\cong H^0_c(\mathbb{R}^{n-k})\\
&= \begin{cases} 0 & \text{for } k<n\\ H^0_c(\mathbb{R}^0)=\mathbb{R} & \text{for } k=n.\end{cases}
\end{aligned}$$

Note that the isomorphism $H^n_c(\mathbb{R}^n)\cong\mathbb{R}$ is given by integrating the differential form with compact support with respect to the standard orientation. This is well defined since by Stokes' theorem 8.11 we have $\int_{\mathbb{R}^n} d\omega = \int_\emptyset \omega = 0$, so the integral induces a mapping $\int_* : H^n_c(\mathbb{R}^n)\to\mathbb{R}$. $\square$

10.13. Example. We consider the open Möbius strip M in $\mathbb{R}^3$. Open means without boundary. Then M is contractible onto S^1, in fact M is the total space of a real line bundle over S^1. So from 9.12 we see that $H^k(M) \cong H^k(S^1) = \mathbb{R}$ for $k = 0, 1$ and $= 0$ for $k > 1$.

Now we claim that $H^k_c(M) = 0$ for all k. For that we cut the Möbius strip in two pieces which are glued at the end with one turn (make a drawing), so that $M = U \cup V$ where $U \cong \mathbb{R}^2$, $V \cong \mathbb{R}^2$, and $U \cap V \cong \mathbb{R}^2 \sqcup \mathbb{R}^2$, the disjoint union. We also know that $H^0_c(M) = 0$ since M is not compact and connected. Then the Mayer-Vietoris sequence (see 10.4) is given by

$$
\begin{array}{cl}
H^1_c(U) \oplus H^1_c(V) & = 0 \\
\downarrow & \\
H^1_c(M) & \\
\downarrow{\scriptstyle \delta} & \\
H^2_c(U \cap V) & = \mathbb{R} \oplus \mathbb{R} \\
\downarrow{\scriptstyle \beta_c} & \\
H^2_c(U) \oplus H^2_c(V) & = \mathbb{R} \oplus \mathbb{R} \\
\downarrow & \\
H^2_c(M) & \\
\downarrow & \\
H^3_c(U \cap V) & = 0.
\end{array}
$$

We shall show that the linear mapping β_c has rank 2. So we read from the sequence that $H^1_c(M) = 0$ and $H^2_c(M) = 0$. By dimension reasons $H^k(M) = 0$ for $k > 2$.

Let φ, $\psi \in \Omega^2_c(U \cap V)$ be two forms, supported in the two connected components, respectively, with integral 1 in the orientation induced from one on U. Then $\int_U \varphi = 1$, $\int_U \psi = 1$, but for some orientation on V we have $\int_V \varphi = 1$ and $\int_V \psi = -1$. So the matrix of the mapping β_c in these bases is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which has rank 2.

10.14. Mapping degree for proper mappings. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth proper mapping, then $f^* : \Omega^k_c(\mathbb{R}^n) \to \Omega^k_c(\mathbb{R}^n)$ is defined and is an algebra homomorphism. So also the induced mapping in cohomology with compact supports makes sense and by

$$\begin{array}{ccc} H^n_c(\mathbb{R}^n) & \xrightarrow{f^*} & H^n_c(\mathbb{R}^n) \\ {\scriptstyle\int_*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\int_*} \\ \mathbb{R} & \xrightarrow{\deg f} & \mathbb{R} \end{array}$$

a linear mapping $\mathbb{R} \to \mathbb{R}$, i. e. multiplication by a real number, is defined. This number $\deg f$ is called the *"mapping degree"* of f.

10.15. Lemma. *The mapping degree of proper mappings has the following properties:*

(1) *If f, $g : \mathbb{R}^n \to \mathbb{R}^n$ are proper, then* $\deg(f \circ g) = \deg(f). \deg(g)$.
(2) *If f and $g : \mathbb{R}^n \to \mathbb{R}^n$ are proper homotopic (see 10.5) then* $\deg(f) = \deg(g)$.
(3) $\deg(Id_{\mathbb{R}^n}) = 1$.
(4) *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is proper and not surjective then* $\deg(f) = 0$.

Proof. Only statement (4) needs a proof. Since f is proper, $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$: for K compact in $\mathbb{R}^n$ the inverse image $K_1 = f^{-1}(K)$ is compact, so $f(K_1) = f(\mathbb{R}^n) \cap K$ is compact, thus closed. By local compactness $f(\mathbb{R}^n)$ is closed.

Suppose that there exists $x \in \mathbb{R}^n \setminus f(\mathbb{R}^n)$, then there is an open neighborhood $U \subset \mathbb{R}^n \setminus f(\mathbb{R}^n)$. We choose a bump n-form α on $\mathbb{R}^n$ with support in U and $\int \alpha = 1$. Then $f^*\alpha = 0$, so $\deg(f) = 0$ since $[\alpha]$ is a generator of $H^n_c(\mathbb{R}^n)$. □

10.16. Regular values. Let $f : M \to N$ be a smooth mapping between manifolds.

(1) $x \in M$ is called a *"singular point"* of f if $T_x f$ is not surjective, and is called a *"regular point"* of f if $T_x f$ is surjective.
(2) $y \in N$ is called a *"regular value"* of f if $T_x f$ is surjective for all $x \in f^{-1}(y)$. If not y is called a singular value. Note that any $y \in N \setminus f(M)$ is a regular value.

Theorem. Sard, 1942. *The set of all singular values of a smooth mapping $f : M \to N$ is of Lebesgue measure 0 in N.*

So any smooth mapping has regular values. For the proof of this result we refer to [Hirsch, 1976].

10.17. Lemma. *For a proper smooth mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ the mapping degree is an integer, in fact for any regular value y of f we have*

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(\det(df(x))) \in \mathbb{Z}.$$

Proof. By 10.15.(4) we may assume that f is surjective. By Sard's theorem, see 10.16, there exists a regular value y of f. We have $f^{-1}(y) \neq \emptyset$, and for all $x \in f^{-1}(y)$ the tangent mapping $T_x f$ is surjective, thus an isomorphism. By the inverse mapping theorem f is locally a diffeomorphism from an open neighborhood of x onto a neighborhood of y. Thus $f^{-1}(y)$ is a discrete and compact set, say $f^{-1}(y) = \{x_1, \dots, x_k\} \subset \mathbb{R}^n$.

Now we choose pairwise disjoint open neighborhoods U_i of x_i and an open neighborhood V of y such that $f : U_i \to V$ is a diffeomorphism for each i. We choose an n-form α on $\mathbb{R}^n$ with support in V and $\int \alpha = 1$. So $f^*\alpha = \sum_i (f|U_i)^*\alpha$ and moreover

$$\begin{aligned}
\int_{U_i} (f|U_i)^*\alpha &= \operatorname{sign}(\det(df(x_i))) \int_V \alpha = \operatorname{sign}(\det(df(x_i))) \\
\deg(f) &= \int_{\mathbb{R}^n} f^*\alpha = \sum_i \int_{U_i} (f|U_i)^*\alpha \\
&= \sum_i^k \operatorname{sign}(\det(df(x_i))) \in \mathbb{Z}. \quad \square
\end{aligned}$$

10.18. Example. The last result for a proper smooth mapping $f : \mathbb{R} \to \mathbb{R}$ can be interpreted as follows: think of f as parametrizing the path of a car on an (infinite) street. A regular value of f is then a position on the street where the car never stops. Wait there and count the directions of the passes of the car: the sum is the mapping degree, the number of journeys from $-\infty$ to ∞. In dimension 1 it can be only -1, 0, or $+1$ (why?).

10.19. Poincaré duality. Let M be an oriented smooth manifold of dimension m without boundary. By Stokes' theorem the integral $\int : \Omega_c^m(M) \to \mathbb{R}$ vanishes on exact forms and induces the *"cohomologigal integral"*

$$\int_* H_c^m(M) \to \mathbb{R}. \tag{1}$$

It is surjective (use a bump m-form with small support). The *'Poincaré product'* is the bilinear form

$$
(2) \qquad \begin{aligned} P^k_M &: H^k(M) \times H^{m-k}_c(M) \to \mathbb{R}, \\ P^k_M([\alpha],[\beta]) &= \int_* [\alpha] \wedge [\beta] = \int_M \alpha \wedge \beta. \end{aligned}
$$

It is well defined since $d\gamma \wedge \beta = d(\gamma \wedge \beta)$ etc. If $j : U \to M$ is an orientation preserving embedding of an open submanifold then for $[\alpha] \in H^k(M)$ and for $[\beta] \in H^{m-k}_c(U)$ we may compute as follows:

$$
(3) \qquad \begin{aligned} P^k_U(j^*[\alpha],[\beta]) &= \int_* (j^*[\alpha]) \wedge [\beta] = \int_U j^*\alpha \wedge \beta \\ &= \int_U j^*(\alpha \wedge j_*\beta) = \int_{j(U)} \alpha \wedge j_*\beta \\ &= \int_M \alpha \wedge j_*\beta = P^k_M([\alpha], j_*[\beta]). \end{aligned}
$$

Now we define the Poincaré duality operator

$$
(4) \qquad \begin{aligned} D^k_M &: H^k(M) \to (H^{m-k}_c(M))^*, \\ \langle [\beta], D^k_M[\alpha] \rangle &= P^k_M([\alpha],[\beta]). \end{aligned}
$$

For example we have $D^0_{\mathbb{R}^n}(1) = (\int_{\mathbb{R}^n})_* \in (H^n_c(\mathbb{R}^n))^*$.

Let $M = U \cup V$ with U, V open in M, then we have the two Mayer Vietoris sequences from 9.10 and from 10.4

$$
\begin{gathered} \dots \to H^k(M) \xrightarrow{\alpha_*} H^k(U) \oplus H^k(V) \xrightarrow{\beta_*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \to \dots \\ \leftarrow H^{m-k}_c(M) \leftarrow H^{m-k}_c(U) \oplus H^{m-k}_c(V) \leftarrow H^{m-k}_c(U \cap V) \xleftarrow{\delta_c} H^{m-(k+1)}_c(M) \leftarrow \end{gathered}
$$

We take dual spaces and dual mappings in the second sequence and we replace δ in the first sequence by $(-1)^{k-1}\delta$ and get the following diagram which is commutative as we will see in a moment.

$$
(5)\qquad
\begin{tikzcd}
\dots \arrow[d, "(-1)^{k-2}\delta"'] & \dots \arrow[d, "\delta_c^*"] \\
H^k(M) \arrow[r, "D_M^k"] \arrow[d] & H_c^{m-k}(M)^* \arrow[d] \\
H^k(U)\oplus H^k(V) \arrow[r, "D_U^k \oplus D_V^k"] \arrow[d] & H_c^{m-k}(U)^* \oplus H_c^{m-k}(V)^* \arrow[d] \\
H^k(U\cap V) \arrow[r, "D_{U\cap V}"] \arrow[d, "(-1)^{k-1}\delta"'] & H_c^{m-k}(U\cap V)^* \arrow[d, "\delta_c^*"] \\
H^{k+1}(M) \arrow[r, "D_M^{k+1}"] \arrow[d] & H_c^{m-(k+1)}(M)^* \arrow[d] \\
\dots & \dots
\end{tikzcd}
$$

10.20. Lemma. *The diagram (5) in 10.19 commutes.*

Proof. The first and the second square from the top commute by 10.19.(3). So we have to check that the bottom one commutes. Let $[\alpha] \in H^k(U\cap V)$ and $[\beta] \in H_c^{m-(k+1)}(M)$, and let (f_U, f_V) be a partition of unity which is subordinated to the open cover (U, V) of M. Then we have

$$
\begin{aligned}
\langle[\beta], D_M^{k+1}(-1)^{k-1}\delta[\alpha]\rangle &= P_M^{k+1}((-1)^{k-1}\delta[\alpha],[\beta]) \\
&= P_M^{k+1}((-1)^{k-1}[df_V\wedge\alpha],[\beta]) \qquad \text{by 9.10} \\
&= (-1)^{k-1}\int_M df_V\wedge\alpha\wedge\beta. \\
\langle[\beta], \delta_c^* D_{U\cap V}^k[\alpha]\rangle &= \langle\delta_c[\beta], D_{U\cap V}^k[\alpha]\rangle = P_{U\cap V}^k([\alpha],\delta_c[\beta]) \\
&= P_{U\cap V}^k([\alpha],[df_U\wedge\beta] = -[df_V\wedge\beta]) \qquad \text{by 10.4} \\
&= -\int_{U\cap V}\alpha\wedge df_V\wedge\beta = -(-1)^k\int_M df_V\wedge\alpha\wedge\beta. \quad \square
\end{aligned}
$$

10.21. Theorem. Poincaré Duality. *If M is an oriented manifold of dimension m without boundary then the Poincaré duality mapping*

$$D^k_M : H^k(M) \to H^{m-k}_c(M)^*$$

is a linear isomomorphism for each k.

Proof. **Step 1.** Let $\mathcal{O}$ be an i-base for the open sets of M, i. e. $\mathcal{O}$ is a basis containing all finite intersections of sets in $\mathcal{O}$. Let $\mathcal{O}_f$ be the the set of all open sets in M which are finite unions of sets in $\mathcal{O}$. Let $\mathcal{O}_s$ be the set of all open sets in M which are at most countable disjoint unions of sets in $\mathcal{O}$. Then obviously $\mathcal{O}_f$ and $\mathcal{O}_s$ are again i-bases.

Step 2. Let $\mathcal{O}$ be an i-base for M. If $D_O : H(O) \to H_c(O)^*$ is an isomorphism for all $O \in \mathcal{O}$, then also for all $O \in \mathcal{O}_f$.

Let $U \in \mathcal{O}_f$, $U = O_1 \cup \dots \cup O_k$ for $O_i \in \mathcal{O}$. We consider O_1 and $V = O_2 \cup \dots \cup O_k$. Then $O_1 \cap V = (O_1 \cap O_2) \cup \dots \cup (O_1 \cap O_k)$ is again a union of elements of $\mathcal{O}$ since it is an i-base. Now we prove the claim by induction on k. The case $k = 1$ is trivial. By induction D_{O_1}, D_V, and $D_{O_1 \cap V}$ are isomorphisms, so D_U is also an isomorphism by the five-lemma 9.9 applied to the diagram 10.19.(5).

Step 3. If $\mathcal{O}$ is a basis of open sets in M such that D_O is an isomorphism for all $O \in \mathcal{O}$, then also for all $O \in \mathcal{O}_s$.

If $U \in \mathcal{O}_s$ we have $U = O_1 \sqcup O_2 \sqcup \ldots = \bigsqcup_{i=1}^\infty O_i$ for $O_i \in \mathcal{O}$. But then the diagram

$$\begin{tikzcd}
H(U) \arrow[rr, equal] \arrow[d, "D_U"'] & & \prod_{i=1}^\infty H(O_i) \arrow[d, "\prod D_{O_i}"] \\
H_c(U)^* \arrow[r, equal] & (\bigoplus_{i=1}^\infty H_c(O_i))^* \arrow[r, equal] & \prod_{i=1}^\infty H_c(O_i)^*
\end{tikzcd}$$

commutes and implies that D_U is an isomorphism.

Step 4. If D_O is an isomorphism for each $O \in \mathcal{O}$ where $\mathcal{O}$ is an i-base for the open sets of M then D_U is an isomorphism for each open set $U \subset M$.

For $((\mathcal{O}_f)_s)_f$ contains all open sets of M. This is a consequence of the proof that each manifold admits a finite atlas. Then the result follows from steps 2 and 3.

Step 5. $D_{\mathbb{R}^m} : H(\mathbb{R}^m) \to H_c(\mathbb{R}^m)^*$ is an isomorphism.

We have

$$H^k(\mathbb{R}^m) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases} \qquad H^k_c(\mathbb{R}^m) = \begin{cases} \mathbb{R} & \text{for } k = m \\ 0 & \text{for } k \neq m \end{cases}$$

The class [1] is a generator for $H^0(\mathbb{R}^m)$, and $[\alpha]$ is a generator for $H_c^m(\mathbb{R}^m)$ where α is any m-form with compact support and $\int_M \alpha = 1$. But then $P^0_{\mathbb{R}^m}([1],[\alpha]) = \int_{\mathbb{R}^m} 1.\alpha = 1$.

Step 6. For each open subset $U \subset \mathbb{R}^m$ the mapping D_U is an isomorphism.

The set $\{\{x \in \mathbb{R}^m : a^i < x^i < b^i \text{ for all } i\} : a^i < b^i\}$ is an i-base of $\mathbb{R}^m$. Each element O in it is diffeomorphic (with orientation preserved) to $\mathbb{R}^m$, so D_O is a diffeomorphism by step 5. From step 4 the result follows.

Step 7. D_M is an isomorphism for each oriented manifold M.

Let $\mathcal{O}$ be the the set of all open subsets of M which are diffeomorphic to an open subset of $\mathbb{R}^m$, i. e. all charts of a maximal atlas. Then $\mathcal{O}$ is an i-base for M, and D_O is an isomorphism for each $O \in \mathcal{O}$. By step 4 D_U is an isomorphism for each open U in M, thus also D_U. $\square$

10.22. Corollary. *For each oriented manifold M without boundary the bilinear pairings*

$$P_M : H^*(M) \times H_c^*(M) \to \mathbb{R},$$
$$P^k_M : H^k(M) \times H_c^{m-k}(M) \to \mathbb{R}$$

are not degenerate.

10.23. Corollary. *Let $j : U \to M$ be the embedding of an open submanifold of an oriented manifold M of dimension m without boundary. Then of the following two mappings one is an isomorphism if and only if the other one is:*

$$j^* : H^k(U) \leftarrow H^k(M),$$
$$j_* : H_c^{m-k}(U) \to H_c^{m-k}(M).$$

Proof. Use 10.19.(3), $P^k_U(j^*[\alpha],[\beta]) = P^k_M([\alpha], j_*[\beta])$. $\square$

10.24. Theorem. *Let M be an oriented connected manifold of dimension m without boundary. Then the integral*

$$\int_* : H_c^m(M) \to \mathbb{R}$$

is an isomorphism. So $\ker \int_M = d(\Omega_c^{m-1}(M)) \subset \Omega_c^m(M)$.

Proof. Considering m-forms with small support shows that the integral is surjective. By Poincaré duality 10.21 $\dim_{\mathbb{R}} H_c^m(M)^* = \dim_{\mathbb{R}} H^0(M) = 1$ since M is connected. $\square$

Definition. The uniquely defined cohomology class $\omega_M \in H_c^m(M)$ with integral $\int_M \omega_M = 1$ is called the *orientation class* of the manifold M.

10.25. Relative cohomology with compact supports. Let M be a smooth manifold and let N be a closed submanifold. Then the injection $i : N \to M$ is a proper smooth mapping. We consider the spaces

$$\Omega^k_c(M,N) := \{\omega \in \Omega^k_c(M) : \omega|N = i^*\omega = 0\}$$

whose direct sum is a graded differential subalgebra $(\Omega^*_c(M,N), d)$ of $(\Omega^*_c(M), d)$. Its cohomology, denoted by $H^*_c(M,N)$, is called the *relative De Rham cohomology with compact supports* of the *manifold pair* (M,N).

$$0 \to \Omega^*_c(M,N) \hookrightarrow \Omega^*_c(M) \xrightarrow{i^*} \Omega^*_c(N) \to 0$$

is an exact sequence of differential graded algebras. This is seen by the same proof as of 9.16 with some obvious changes. Thus by 9.8 we have the following long exact sequence in cohmology

$$\dots \to H^k_c(M,N) \to H^k_c(M) \to H^k_c(N) \xrightarrow{\delta} H^{k+1}_c(M,N) \to \dots$$

which is natural in the manifold pair (M,N). It is called the *long exact cohomology sequence with compact supports* of the pair (M,N).

10.26. Now let M be an oriented smooth manifold of dimension m with boundary ∂M. Then ∂M is a closed submanifold of M. Since for $\omega \in \Omega^{m-1}_c(M,\partial M)$ we have $\int_M d\omega = \int_{\partial M}\omega = \int_{\partial M} 0 = 0$, the integral of m-forms factors as follows

$$\begin{tikzcd}
\Omega^m_c(M,\partial M) \arrow[r, hook] \arrow[d] & \Omega^m_c(M) \arrow[r, "\int_M"] & \mathbb{R} \\
H^m_c(M,\partial M) \arrow[urr, dashed, "\int_*"] & &
\end{tikzcd}$$

to the *cohomological integral* $\int_* : H^m_c(M,\partial M) \to \mathbb{R}$.

Example. Let $I = [a,b]$ be a compact intervall, then $\partial I = \{a,b\}$. We have

$H^1(I) = 0$ since $f dt = d\int_a^t f(s)ds$. The long exact sequence in cohomology is

$$\begin{array}{c}
0 \\
\downarrow \\
H^0(I, \partial I) = 0 \\
\downarrow \\
H^0(I) = \mathbb{R} \\
\downarrow \\
H^0(\partial I) = \mathbb{R}^2 \\
\delta \downarrow \\
H^1(I, \partial I) \xrightarrow[\cong]{\int_*} \mathbb{R} \\
\downarrow \\
H^1(I) = 0 \\
\downarrow \\
H^1(\partial I) = 0.
\end{array}$$

The connecting homomorphism $\delta : H^0(\partial I) \to H^1(I, \partial I)$ is given by the following procedure: Let $(f(a), f(b)) \in H^0(\partial I)$, where $f \in C^\infty(I, \mathbb{R})$. Then

$$\delta(f(a), f(b)) = [df] = \int_* [df] = \int_a^b df = \int_a^b f'(t)dt = f(b) - f(a).$$

So the fundamental theorem of calculus can be interpreted as the connecting homomorphism for the long exact sequence of the realtive cohomology for the pair $(I, \partial I)$.

The general situation. Let M be an oriented smooth manifold with boundary ∂M. We consider the following piece of the long exact sequence in cohomology with compact supports of the pair $(M, \partial M)$:

$$\begin{tikzcd}
H_c^{m-1}(M) \arrow[r] & H_c^{m-1}(\partial M) \arrow[r, "\delta"] \arrow[d, "\int_*"'] & H_c^m(M, \partial M) \arrow[r] \arrow[d, "\int_*"] & H_c^m(M) \arrow[r] & 0 \\
& \mathbb{R} \arrow[r, equal] & \mathbb{R} & &
\end{tikzcd}$$

The connecting homomorphism is given by

$$\delta[\omega|\partial M] = [d\omega]_{H_c^m(M,\partial M)}, \quad \omega \in \Omega_c^{m-1}(M),$$

so commutation of the diagram above is equivalent to the validity of Stokes' theorem.

11. De Rham cohomology of compact manifolds

11.1. The oriented double cover. Let M be a manifold. We consider the orientation bundle $\operatorname{Or}(M)$ of M which we dicussed in 8.6, and we consider the subset $\operatorname{or}(M) := \{v \in \operatorname{Or}(M) : |v| = 1\}$. We shall see shortly that it is a submanifold of the total space $\operatorname{Or}(M)$, that it is orientable, and that $\pi_M : \operatorname{or}(M) \to M$ is a double cover of M. The manifold $\operatorname{or}(M)$ is called the *orientable double cover* of M.

We first check that the total space $\operatorname{Or}(M)$ of the orientation bundle is orientable. Let (U_α, u_α) be an atlas for M. Then the orientation bundle is given by the cocycle of transition functions

$$\tau_{\alpha\beta}(x) = \operatorname{sign} \varphi_{\alpha\beta}(x) = \operatorname{sign}\det d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x)).$$

Let (U_α, τ_α) be the induced vector bundle atlas for $\operatorname{Or}(M)$, see 6.3. We consider the mappings

$$\begin{tikzcd}
\operatorname{Or}(M)|U_\alpha \arrow[rr, "\tau_\alpha"] \arrow[dr, "\pi_M"'] & & U_\alpha \times \mathbb{R} \arrow[r, "u_\alpha \times Id"] \arrow[dl, "pr_1"] & u_\alpha(U_\alpha) \times \mathbb{R} \subset \mathbb{R}^{m+1} \\
& U_\alpha & &
\end{tikzcd}$$

and we use them as charts for $\operatorname{Or}(M)$. The chart changes $u_\beta(U_{\alpha\beta}) \times \mathbb{R} \to u_\alpha(U_{\alpha\beta}) \times \mathbb{R}$ are then given by

$$\begin{aligned}
(y,t) &\mapsto (u_\alpha \circ u_\beta^{-1}(y), \tau_{\alpha\beta}(u_\beta^{-1}(y))t) \\
&= (u_\alpha \circ u_\beta^{-1}(y), \operatorname{sign}\det d(u_\beta \circ u_\alpha^{-1})((u_\alpha \circ u_\beta^{-1})(y))t) \\
&= (u_\alpha \circ u_\beta^{-1}(y), \operatorname{sign}\det d(u_\alpha \circ u_\beta^{-1})(y)t)
\end{aligned}$$

The Jacobi matrix of this mapping is

$$\begin{pmatrix} d(u_\alpha \circ u_\beta^{-1})(y) & * \\ 0 & \operatorname{sign}\det d(u_\alpha \circ u_\beta^{-1})(y) \end{pmatrix}$$

which has positive determinant.

Now we let $Z := \{v \in \operatorname{Or}(M) : |v| \le 1\}$ which is a submanifold with boundary in $\operatorname{Or}(M)$ of the same dimension and thus orientable. Its boundary ∂Z coincides with $\operatorname{or}(M)$, which is thus orientable.

Next we consider the diffeomorphism $\varphi : \operatorname{or}(M) \to \operatorname{or}(M)$ which is induced by the multiplication with -1 in $\operatorname{Or}(M)$. We have $\varphi \circ \varphi = Id$ and $\pi_M^{-1}(x) = \{z, \varphi(z)\}$ for $z \in \operatorname{or}(M)$ and $\pi_M(z) = x$.

Suppose that the manifold M is connected. Then the oriented double cover $\operatorname{or}(M)$ has at most two connected components, since π_M is a two sheeted convering map. If $\operatorname{or}(M)$ has two components, then φ restricts to a diffeomorphism between them. The projection π_M, if restricted to one of the components, becomes invertible, so $\operatorname{Or}(M)$ admits a section which vanishes nowhere, thus M is orientable. So we see that $\operatorname{or}(M)$ is connected if and only if M is not orientable.

The pullback mapping $\varphi^* : \Omega(\operatorname{or}(M)) \to \Omega(\operatorname{or}(M))$ also satisfies $\varphi^* \circ \varphi^* = Id$. We put

$$\begin{aligned}
\Omega_+(\operatorname{or}(M)) &: = \{\omega \in \Omega(\operatorname{or}(M)) : \varphi^*\omega = \omega\},\\
\Omega_-(\operatorname{or}(M)) &: = \{\omega \in \Omega(\operatorname{or}(M)) : \varphi^*\omega = -\omega\}.
\end{aligned}$$

For each $\omega \in \Omega(\operatorname{or}(M))$ we have $\omega = \frac{1}{2}(\omega + \varphi^*\omega) + \frac{1}{2}(\omega - \varphi^*\omega) \in \Omega_+(\operatorname{or}(M)) \oplus \Omega_-(\operatorname{or}(M))$, so $\Omega(\operatorname{or}(M)) = \Omega_+(\operatorname{or}(M)) \oplus \Omega_-(\operatorname{or}(M))$. Since $d \circ \varphi^* = \varphi^* \circ d$ these two subspaces are invariant under d, thus we conclude that

$$H^k(\operatorname{or}(M)) = H^k(\Omega_+(\operatorname{or}(M))) \oplus H^k(\Omega_-(\operatorname{or}(M))). \tag{1}$$

Since $\pi_M^* : \Omega(M) \to \Omega(\operatorname{or}(M))$ is an embedding with image $\Omega_+(\operatorname{or}(M))$ we see that the induced mapping $\pi_M^* : H^k(M) \to H^k(\operatorname{or}(M))$ is also an embedding with image $H^k(\Omega_+(\operatorname{or}(M)))$.

11.2. Theorem. *For a compact manifold M we have $\dim_{\mathbb{R}} H^*(M) < \infty$.*

Proof. Step 1. If M is orientable we have by Poincaré duality 10.21

$$H^k(M) \xrightarrow[\cong]{D^k_M} (H^{m-k}_c(M))^* = (H^{m-k}(M))^* \xleftarrow[\cong]{(D^{m-k}_M)^*} (H^k_c(M))^{**},$$

so $H^k(M)$ is finite dimensional since otherwise $\dim(H^k(M))^* > \dim H^k(M)$.

Step 2. Let M be not orientable. Then from 11.1 we see that the oriented double cover $\operatorname{or}(M)$ of M is compact, oriented, and connected, and we have $\dim H^k(M) = \dim H^k(\Omega_+(\operatorname{or}(M))) \le \dim H^k(\operatorname{or}(M)) < \infty$. $\square$

11.3. Theorem. *Let M be a connected manifold of dimension m. Then*

$$H^m(M) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ is compact and orientable,}\\ 0 & \text{else.}\end{cases}$$

Proof. If M is compact and orientable by 10.24 we the integral $\int_* : H^m(M) \to \mathbb{R}$ is an isomorphism.

Next let M be compact but not orientable. Then the oriented double cover $\operatorname{or}(M)$ is connected, compact and oriented. Let $\omega \in \Omega^m(\operatorname{or}(M))$ be an m-form which vanishes nowhere. Then also $\varphi^*\omega$ is nowhere zero where $\varphi : \operatorname{or}(M) \to \operatorname{or}(M)$ is the covering transformation from 11.1. So $\varphi^*\omega = f\omega$ for a function $f \in C^\infty(\operatorname{or}(M), \mathbb{R})$ which vanishes nowhere. So $f > 0$ or $f < 0$. If $f > 0$ then $\alpha := \omega + \varphi^*\omega = (1 + f)\omega$ is again nowhere 0 and $\varphi^*\alpha = \alpha$, so $\alpha = \pi_M^*\beta$ for an m-form β on M without zeros. So M is orientable, a contradiction. Thus $f < 0$ and φ changes the orientation.

The m-form $\gamma := \omega - \varphi^*\omega = (1 - f)\omega$ has no zeros, so $\int_{\operatorname{or}(M)} \gamma > 0$ if we orient $\operatorname{or}(M)$ using ω, thus the cohomology class $[\gamma] \in H^m(\operatorname{or}(M))$ is not zero. But $\varphi^*\gamma = -\gamma$ so $\gamma \in \Omega_-(\operatorname{or}(M))$, thus $H^m(\Omega_-(\operatorname{or}(M))) \neq 0$. By the first part of the proof we have $H^m(\operatorname{or}(M)) = \mathbb{R}$ and from 11.1 we get $H^m(\operatorname{or}(M)) = H^m(\Omega_-(\operatorname{or}(M)))$, so $H^m(M) = H^m(\Omega_+(\operatorname{or}(M))) = 0$.

Finally let us suppose that M is not compact. If M is orientable we have by Poincaré duality 10.21 and by 10.3.(2) that $H^m(M) \cong H^0_c(M)^* = 0$.

If M is not orientable then $\operatorname{or}(M)$ is connected by 11.1 and not compact, so $H^m(M) = H^m(\Omega_+(\operatorname{or}(M))) \subset H^m(\operatorname{or}(M)) = 0$. $\square$

11.4. Corollary. *Let M be a connected manifold which is not orientable. Then* $\operatorname{or}(M)$ *is orientable and the Poincaré duality pairing of* $\operatorname{or}(M)$ *satisfies*

$$\begin{aligned}
&P^k_{\operatorname{or}(M)}(H^k_+(\operatorname{or}(M)), (H^{m-k}_c)_+(\operatorname{or}(M))) = 0\\
&P^k_{\operatorname{or}(M)}(H^k_-(\operatorname{or}(M)), (H^{m-k}_c)_-(\operatorname{or}(M))) = 0\\
&H^k_+(\operatorname{or}(M)) \cong (H^{m-k}_c)_-(\operatorname{or}(M))^*\\
&H^k_-(\operatorname{or}(M)) \cong (H^{m-k}_c)_+(\operatorname{or}(M))^*
\end{aligned}$$

Proof. From 11.1 we know that $\operatorname{or}(M)$ is connected and orientable. So $\mathbb{R} = H^0(\operatorname{or}(M)) \cong H^m_c(\operatorname{or}(M))^*$.

Now we orient $\operatorname{or}(M)$ and choose a positive bump m-form ω with compact support on $\operatorname{or}(M)$ so that $\int_{\operatorname{or}(M)} \omega > 0$. From the proof of 11.3 we know that the covering transformation $\varphi : \operatorname{or}(M) \to \operatorname{or}(M)$ changes the orientation, so $\varphi^*\omega$ is negatively oriented, $\int_{\operatorname{or}(M)} \varphi^*\omega < 0$. Then $\omega - \varphi^*\omega \in \Omega^m_-(\operatorname{or}(M))$ and $\int_{\operatorname{or}(M)}(\omega - \varphi^*\omega) > 0$, so $(H^m_c)_-(\operatorname{or}(M)) = \mathbb{R}$ and $(H^m_c)_+(\operatorname{or}(M)) = 0$.

Since φ^* is an algebra homomorphism we have

$$\begin{aligned}
&\Omega^k_+(\operatorname{or}(M)) \wedge (\Omega^{m-k}_c)_+(\operatorname{or}(M)) \subset (\Omega^m_c)_+(\operatorname{or}(M)),\\
&\Omega^k_-(\operatorname{or}(M)) \wedge (\Omega^{m-k}_c)_-(\operatorname{or}(M)) \subset (\Omega^m_c)_+(\operatorname{or}(M)).
\end{aligned}$$

From $(H_c^m)_+(\mathrm{or}(M)) = 0$ the first two results follows. The last two assertions then follow from this and $H^k(\mathrm{or}(M)) = H_+^k(\mathrm{or}(M)) \oplus H_-^k(\mathrm{or}(M))$ and the analogous decomposition of $H_c^k(\mathrm{or}(M))$. □

11.5. Theorem. *For the real projective spaces we have*

$$\begin{aligned} &H^0(\mathbb{RP}^n) = \mathbb{R} \\ &H^k(\mathbb{RP}^n) = 0 \qquad \textit{for } 1 \le k < n, \\ &H^n(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & \textit{for odd } n, \\ 0 & \textit{for even } n. \end{cases} \end{aligned}$$

Proof. The projection $\pi : S^n \to \mathbb{RP}^n$ is a smooth covering mapping with 2 sheets, the covering transformation is the antipodal mapping $A : S^n \to S^n$, $x \mapsto -x$. We put $\Omega_+(S^n) = \{\omega \in \Omega(S^n) : A^*\omega = \omega\}$ and $\Omega_-(S^n) = \{\omega \in \Omega(S^n) : A^*\omega = -\omega\}$. The pullback $\pi^* : \Omega(\mathbb{RP}^n) \to \Omega(S^n)$ is an embedding onto $\Omega_+(S^n)$.

Let Δ be the determinant function on the oriented Euclidean space $\mathbb{R}^{n+1}$. We identify T_xS^n with $\{x\}^\perp$ in $\mathbb{R}^{n+1}$ and we consider the n-form $\omega_{S^n} \in \Omega^n(S^n)$ which is given by $(\omega_{S^n})_x(X_1, \dots, X_n) = \Delta(x, X_1, \dots, X_n)$. Then we have

$$\begin{aligned} (A^*\omega_{S^n})_x(X_1, \dots, X_n) &= (\omega_{S^n})_{A(x)}(T_xA.X_1, \dots, T_xA.X_n) \\ &= (\omega_{S^n})_{-x}(-X_1, \dots, -X_n) \\ &= \Delta(-x, -X_1, \dots, -X_n) \\ &= (-1)^{n+1}\Delta(x, X_1, \dots, X_n) \\ &= (-1)^{n+1}(\omega_{S^n})_x(X_1, \dots, X_n) \end{aligned}$$

Since ω_{S^n} is invariant under the action of the group $SO(n+1, \mathbb{R})$ it must be the Riemannian volume form, so

$$\int_{S^n} \omega_{S^n} = \mathrm{vol}(S^n) = \frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})} = \begin{cases} \frac{2\pi^k}{(k-1)!} & \text{for } n = 2k-1 \\ \frac{2^k\pi^{k-1}}{1\cdot 3\cdot 5\dots(2k-3)} & \text{for } n = 2k-2 \end{cases}$$

Thus $[\omega_{S^n}] \in H^n(S^n)$ is a generator for the cohomology. We have $A^*\omega_{S^n} = (-1)^{n+1}\omega_{S^n}$, so

$$\omega_{S^n} \in \begin{cases} \Omega_+^n(S^n) & \text{for odd } n, \\ \Omega_-^n(S^n) & \text{for even } n. \end{cases}$$

Thus $H^n(\mathbb{RP}^n) = H^n(\Omega_+(S^n))$ equals $H^n(S^n) = \mathbb{R}$ for odd n and equals 0 for even n.

Since $\mathbb{RP}^n$ is connected we have $H^0(\mathbb{RP}^n) = \mathbb{R}$. For $1 \le k < n$ we have $H^k(\mathbb{RP}^n) = H^k(\Omega_+(S^n)) \subset H^k(S^n) = 0$. □

11.6. Corollary. *Let M be a compact manifold. Then for all Betti numbers we have $b_k(M) := \dim_{\mathbb{R}} H^k(M) < \infty$. If M is compact and orientable of dimension m we have $b_k(M) = b_{m-k}(M)$.*

Proof. This follows from 11.2 and from Poincaré duality 10.21. □

11.7. Euler-Poincaré characteristic. If M is compact then all Betti numbers are finite, so the Euler Poincaré characteristic (see also 9.2)

$$\chi_M = \sum_{k=0}^{\dim M} (-1)^k b_k(M) = f_M(-1)$$

is defined.

Theorem. *Let M be a compact and orientable manifold of dimension m. Then we have:*

(1) *If m is odd then $\chi_M = 0$.*
(2) *If $m = 2n$ for odd n then $\chi_M \equiv b_n(M) \equiv 0 (\mod 2)$.*
(3) *If $m = 4k$ then $\chi_M \equiv b_{2k}(M) \equiv$ signature$(P^{2k}_M)(\mod 2)$.*

Proof. From 11.6 we have $b_q(M) = b_{m-q}(M)$. So $\chi_M = \sum_{q=0}^m (-1)^q b_q = \sum_{q=0}^m (-1)^q b_{m-q} = (-1)^m \chi_M$ which implies (1).

If $m = 2n$ we have $\chi_M = \sum_{q=0}^{2n} (-1)^q b_q = 2\sum_{q=0}^{n-1}(-1)^q b_q + (-1)^n b_n$, so $\chi_M \equiv b_n(\mod 2)$. In general we have for a compact oriented manifold

$$P^q_M([\alpha],[\beta]) = \int_M \alpha \wedge \beta = (-1)^{q(m-q)} \int_M \beta \wedge \alpha = (-1)^{q(m-q)} P^{m-q}_M([\beta],[\alpha]).$$

For odd n and $m = 2n$ we see that P^n_M is a skew symmetric non degenerate bilinear form on $H^q(M)$, so b_n must be even (see 4.7 or ?? below) which implies (2).

(3). If $m = 4k$ then P^{2k}_M is a non degenerate symmetric bilinear form on $H^{2k}(M)$, an inner product. By the *signature* of a non degenerate symmetric inner product one means the number of positive eigenvalues minus the number of negative eigenvalues, so the number $\dim H^{2k}(M)_+ - \dim H^{2k}(M)_- =: a_+ - a_-$, but since $H^{2k}(M)_+ \oplus H^{2k}(M)_- = H^{2k}(M)$ we have $a_+ + a_- = b_{2k}$, so $a_+ - a_- = b_{2k} - 2a_- \equiv b_{2k}(\mod 2)$. □

11.8. The mapping degree. Let M and N be smooth compact oriented manifolds, both of the same dimension m. Then for any smooth mapping f :

$M \to N$ there is a real number $\deg f$, called the *degree* of f, which is given in the bottom row of the diagram

$$\begin{tikzcd} H^m(M) \arrow[d, "\int_*"', "\cong"] & H^m(N) \arrow[l, "H^m(f)"'] \arrow[d, "\int_*"', "\cong"] \\ \mathbb{R} & \mathbb{R} \arrow[l, "\deg f"'] \end{tikzcd}$$

where the vertical arrows are isomorphisms by 10.24, and where $\deg f$ is the linear mapping given by multiplication with that number. So we also the defining relation

$$\int_M f^*\omega = \deg f \int_N \omega \qquad \text{for all } \omega \in \Omega^m(N).$$

11.9. Lemma. *The mapping degree* deg *has the following properties:*

(1) $\deg(f \circ g) = \deg f \cdot \deg g$, $\deg(Id_M) = 1$.
(2) *If* $f,\, g : M \to N$ *are (smoothly) homotopic then* $\deg f = \deg g$.
(3) *If* $\deg f \neq 0$ *then* f *is surjective.*
(4) *If* $f : M \to M$ *is a diffeomorphism then* $\deg f = 1$ *if* f *respects the orientation and* $\deg f = -1$ *if* f *reverses the orientation.*

Proof. (1) and (2) are clear. (3). If $f(M) \neq N$ we choose a bump m-form ω on N with support in the open set $N \setminus f(M)$. Then $f^*\omega = 0$ so we have $0 = \int_M f^*\omega = \deg f \int_N \omega$. Since $\int_N \omega \neq 0$ we get $\deg f = 0$.

(4) follows either directly from the definition of the integral 8.7 of from 11.11 below. □

11.10. Examples on spheres. Let $f \in O(n+1, \mathbb{R})$ and restrict it to a mapping $f : S^n \to S^n$. Then $\deg f = \det f$. This follows from the description of the volume form on S^n given in the proof of 11.5.

Let $f,\, g : S^n \to S^n$ be smooth mappings. If $f(x) \neq -g(x)$ for all $x \in S^n$ then the mappings f and g are smoothly homotopic: The homotopy moves $f(x)$ along the shorter arc of the geodesic (big circle) to $g(x)$. So $\deg f = \deg g$.

If $f(x) \neq -x$ for all $x \in S^n$ then f is homotopic to Id_{S^n}, so $\deg f = 1$.

If $f(x) \neq x$ for all $x \in S^n$ then f is homotopic to $-Id_{S^n}$, so $\deg f = (-1)^{n+1}$.

The *hairy ball theorem* says that on S^n for even n each vector field vanishes somewhere. This can be seen as follows. The tangent bundle of the sphere is

$$TS^n = \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |x|^2 = 1, \langle x, y\rangle = 0\},$$

so a vector field without zeros is a mapping $x \mapsto (x, g(x))$ with $g(x) \perp x$; then $f(x) := g(x)/|g(x)|$ defines a smooth mapping $f : S^n \to S^n$ with $f(x) \perp x$ for all

x. So $f(x) \neq x$ for all x, thus $\deg f = (-1)^{n+1} = -1$. But also $f(x) \neq -x$ for all x, so $\deg f = 1$, a contradiction.

Finally we consider the unit circle $S^1 \xrightarrow{i} \mathbb{C} = \mathbb{R}^2$. Its volume form is given by $\omega := i^*(x\,dy - y\,dx) = i^* \frac{x\,dy - y\,dx}{x^2+y^2}$; obviously we have $\int_{S^1} xdy - ydx = 2\pi$. Now let $f : S^1 \to S^1$ be smooth, $f(t) = (x(t), y(t))$ for $0 \leq t \leq 2\pi$. Then

$$\deg f = \frac{1}{2\pi} \int_{S^1} f^*(xdy - ydx)$$

is the winding number about 0 from compex analysis.

11.11. The mapping degree is an integer. Let $f : M \to N$ be a smooth mapping between compact oriented manifolds of dimension m. Let $b \in N$ be a regular value for f which exists by Sard's theorem, see 10.16. Then for each $x \in f^{-1}(b)$ the tangent mapping $T_x f$ mapping is invertible, so f is diffeomorphism near x. Thus $f^{-1}(b)$ is a finite set, since M is compact. We define the mapping $\varepsilon : M \to \{-1, 0, 1\}$ by

$$\varepsilon(x) = \begin{cases} 0 & \text{if } T_x f \text{ is not invertible} \\ 1 & \text{if } T_x f \text{ is invertible and respects orientations} \\ -1 & \text{if } T_x f \text{ is invertible and changes orientations.} \end{cases}$$

11.12. Theorem. *In the setting of 11.11, if $b \in N$ is a regular value for f, then*

$$\deg f = \sum_{x \in f^{-1}(b)} \varepsilon(x).$$

In particular $\deg f$ *is always an integer.*

Proof. The proof is the same as for lemma 10.17 with obvious changes. $\square$

12. Lie groups III. Analysis on Lie groups

Invariant integration on Lie groups

12.1. Invariant differential forms on Lie groups. Let G be a real Lie group of dimension n with Lie algebra $\mathfrak{g}$. Then the tangent bundle of G is a trivial vector bundle, see 5.16, so G is orientable. Recall from section 4 the notation: $\mu : G \times G \to G$ is the multiplication, $\mu_x : G \to G$ is left translation by x, and $\mu^y : G \to G$ is right translation. $\nu : G \to G$ is the inversion.

A differential form $\omega \in \Omega^n(G)$ is called *left invariant* if $\mu_x^* \omega = \omega$ for all $x \in G$. Then ω is uniquely determined by its value $\omega_e \in \Lambda^n T^* G = \Lambda^n \mathfrak{g}^*$. For each determinant function Δ on $\mathfrak{g}$ there is a unique left invariant n-form L_Δ on G which is given by

$$
\begin{aligned}
(1) \qquad (L_\Delta)_x(X_1, \dots, X_n) &:= \Delta(T_x(\mu_{x^{-1}}).X_1, \dots, T_x(\mu_{x^{-1}}).X_n),\\
(L_\Delta)_x &= T_x(\mu_{x^{-1}})^* \Delta.
\end{aligned}
$$

Likewise there is a unique right invariant n-form R_Δ which is given by

$$
(2) \qquad (R_\Delta)_x(X_1, \dots, X_n) := \Delta(T_x(\mu^{x^{-1}}).X_1, \dots, T_x(\mu^{x^{-1}}).X_n).
$$

12.2. Lemma. *We have for all $a \in G$*

$$
\begin{aligned}
(1) \qquad (\mu^a)^* L_\Delta &= \det(Ad(a^{-1})) L_\Delta,\\
(2) \qquad (\mu_a)^* R_\Delta &= \det(Ad(a)) R_\Delta,\\
(3) \qquad (R_\Delta)_a &= \det(Ad(a)) (L_\Delta)_a.
\end{aligned}
$$

Proof. We compute as follows:

$$
\begin{aligned}
((\mu^a)^* L_\Delta)_x(X_1, \dots, X_n) &= (L_\Delta)_{xa}(T_x(\mu^a).X_1, \dots, T_x(\mu^a).X_n)\\
&= \Delta(T_{xa}(\mu_{(xa)^{-1}}).T_x(\mu^a).X_1, \dots, T_{xa}(\mu_{(xa)^{-1}}).T_x(\mu^a).X_n)\\
&= \Delta(T_a(\mu_{a^{-1}}).T_{xa}(\mu_{x^{-1}}).T_x(\mu^a).X_1, \dots, T_a(\mu_{a^{-1}}).T_{xa}(\mu_{x^{-1}}).T_x(\mu^a).X_n)\\
&= \Delta(T_a(\mu_{a^{-1}}).T_e(\mu^a).T_x(\mu_{x^{-1}}).X_1, \dots, T_a(\mu_{a^{-1}}).T_e(\mu^a).T_x(\mu_{x^{-1}}).X_n)\\
&= \Delta(Ad(a^{-1}).T_x(\mu_{x^{-1}}).X_1, \dots, Ad(a^{-1}).T_x(\mu_{x^{-1}}).X_n)\\
&= \det(Ad(a^{-1}))\Delta(T_x(\mu_{x^{-1}}).X_1, \dots, T_x(\mu_{x^{-1}}).X_n)
\end{aligned}
$$

$$
\begin{aligned}
&= \det(Ad(a^{-1}))(L_\Delta)_x(X_1,\dots,X_n).\\
((\mu_a)^*R_\Delta)_x(X_1,\dots,X_n) &= (R_\Delta)_{ax}(T_x(\mu_a).X_1,\dots,T_x(\mu_a).X_n)\\
&= \Delta(T_{ax}(\mu^{(ax)^{-1}}).T_x(\mu_a).X_1,\dots,T_{ax}(\mu^{(ax)^{-1}}).T_x(\mu_a).X_n)\\
&= \Delta(T_a(\mu^{a^{-1}}).T_{ax}(\mu^{x^{-1}}).T_x(\mu_a).X_1,\dots,T_a(\mu^{a^{-1}}).T_{ax}(\mu^{x^{-1}}).T_x(\mu_a).X_n)\\
&= \Delta(T_a(\mu^{a^{-1}}).T_e(\mu_a).T_x(\mu^{x^{-1}}).X_1,\dots,T_a(\mu^{a^{-1}}).T_e(\mu_a).T_x(\mu^{x^{-1}}).X_n)\\
&= \Delta(Ad(a).T_x(\mu^{x^{-1}}).X_1,\dots,Ad(a).T_x(\mu^{x^{-1}}).X_n)\\
&= \det(Ad(a))\Delta(T_x(\mu^{x^{-1}}).X_1,\dots,T_x(\mu^{x^{-1}}).X_n)\\
&= \det(Ad(a))(R_\Delta)_x(X_1,\dots,X_n).\\
\det(Ad(a))(L_\Delta)_a(X_1,\dots,X_n)&\\
&= \det(Ad(a))\Delta(T_a(\mu_{a^{-1}}).X_1,\dots,T_a(\mu_{a^{-1}}).X_n)\\
&= \Delta(Ad(a).T_a(\mu_{a^{-1}}).X_1,\dots,Ad(a).T_a(\mu_{a^{-1}}).X_n)\\
&= \Delta(T_a(\mu^{a^{-1}}).T_e(\mu_a).T_a(\mu_{a^{-1}}).X_1,\dots,T_a(\mu^{a^{-1}}).T_e(\mu_a).T_a(\mu_{a^{-1}}).X_n)\\
&= \Delta(T_a(\mu^{a^{-1}}).X_1,\dots,T_a(\mu^{a^{-1}}).X_n) = (R_\Delta)_a(X_1,\dots,X_n). \quad \square
\end{aligned}
$$

12.3. Corollary and Definition. *The Lie group G admits a left and right invariant n-form if and only if* $\det(Ad(a)) = 1$ *for all* $a \in G$.

The Lie group G is called unimodular if $|\det(Ad(a))| = 1$ *for all* $a \in G$.

Proof. This is obvious from lemma 12.2. □

12.4. Haar measure. We orient the Lie group G by a left invariant n-form L_Δ. If $f \in C_c^\infty(G,\mathbb{R})$ is a smooth function with compact support on G then the integral $\int_G fL_\Delta$ is defined and we have

$$\int_G (\mu_a^* f)L_\Delta = \int_G \mu_a^*(fL_\Delta) = \int_G fL_\Delta,$$

because $\mu_a : G \to G$ is an orientation preserving diffeomorphism of G. Thus $f \mapsto \int_G fL_\Delta$ is a left invariant integration on G, which is also denoted by $\int_G f(x)d_Lx$, and which gives rise to a left invariant measure on G, the so called *Haar measure*. It is unique up to a multiplicative constant, since $\dim(\Lambda^n\mathfrak{g}^*) = 1$. In the other notation the left invariance looks like

$$\int_G f(ax)d_Lx = \int_G f(x)d_Lx \text{ for all } f \in C_c^\infty(G,\mathbb{R}), a \in G.$$

From lemma 12.2.(1) we have

$$\begin{aligned}\int_G ((\mu^a)^* f) L_\Delta &= \det(Ad(a^{-1})) \int_G (\mu^a)^*(f L_\Delta)\\ &= |\det(Ad(a^{-1}))| \int_G f L_\Delta,\end{aligned}$$

since the mapping μ^a is orientation preserving if and only if $\det(Ad(a)) > 0$. So a left Haar measure is also a right invariant one if and only if the Lie group G is unimodular.

12.5. Lemma. *Each compact Lie group is unimodular.*

Proof. The mapping $\det \circ Ad : G \to GL(1, \mathbb{R})$ is a homomorphism of Lie groups, so its image is a compact subgroup of $GL(1, \mathbb{R})$. Thus $\det(Ad(G))$ equals $\{1\}$ or $\{1, -1\}$. In both cases we have $|\det(Ad(a))| = 1$ for all $a \in G$. $\square$

Analysis for mappings between Lie groups

12.6. Definition. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let $f : G \to H$ be a smooth mapping. Then we define the mapping $Df : G \to L(\mathfrak{g}, \mathfrak{h})$ by

$$Df(x) := T_{f(x)}((\mu^{f(x)})^{-1}).T_x f.T_e(\mu^x) = \delta f(x).T_e(\mu^x),$$

and we call it the *right trivialized derivative* of f.

12.7. Lemma. *The chain rule: For smooth $g : K \to G$ and $f : G \to H$ we have*

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x).$$

The product rule: For $f, h \in C^\infty(G, H)$ we have

$$D(fh)(x) = Df(x) + Ad(f(x)) Dh(x).$$

Proof. We compute as follows:

$$\begin{aligned}D(f \circ g)(X) &= T(\mu^{f(g(x))^{-1}}).T_x(f \circ g).T_e(\mu^x)\\ &= T(\mu^{f(g(x))^{-1}}).T_{g(x)}(f).T_e(\mu^{g(x)}).T(\mu^{g(x)^{-1}}).T_x(g).T_e(\mu^x)\\ &= Df(g(x)).Dg(x).\end{aligned}$$

$$\begin{aligned}
D(fh)(x) &= T(\mu^{(f(x)h(x))^{-1}}).T_x(\mu\circ(f,h)).T_e(\mu^x)\\
&= T(\mu^{(f(x)^{-1}}).T(\mu^{h(x))^{-1}}).T_{f(x),h(x)}\mu.(T_xf.T_e(\mu^x),T_xh.T_e(\mu^x))\\
&= T(\mu^{(f(x)^{-1}}).T(\mu^{h(x))^{-1}}).\Big(T(\mu^{h(x)}).T_xf.T_e(\mu^x)+T(\mu_{f(x)}).T_xh.T_e(\mu^x)\Big)\\
&= T(\mu^{(f(x)^{-1}}).T_xf.T_e(\mu^x)+T(\mu^{(f(x)^{-1}}).T(\mu_{f(x)}).T(\mu^{h(x))^{-1}}).T_xh.T_e(\mu^x)\\
&= Df(x)+Ad(f(x)).Dh(x). \quad\square
\end{aligned}$$

12.8. Inverse function theorem. *Let $f : G\to H$ be smooth and for some $x\in G$ let $Df(x):\mathfrak{g}\to\mathfrak{h}$ be invertible. Then f is a diffeomorphism from a suitable neighborhood of x in G onto a neighborhood of $f(x)$ in H, and for the derivative we have $D(f^{-1})(f(x))=(Df(x))^{-1}$.*

Proof. This follows from the usual inverse function theorem. $\square$

12.9. Lemma. *Let $f\in C^\infty(G,G)$ and let $\Delta\in\Lambda^{\dim G}\mathfrak{g}^*$ be a determinant function on $\mathfrak{g}$. Then we have for all $x\in G$,*

$$(f^*R_\Delta)_x=\det(Df(x))(R_\Delta)_x.$$

Proof. Let $\dim G=n$. We compute as follows

$$\begin{aligned}
(f^*R_\Delta)_x(X_1,\dots,X_n) &= (R_\Delta)_{f(x)}(T_xf.X_1,\dots,T_xf.X_n)\\
&= \Delta(T(\mu^{f(x)^{-1}}).T_xf.X_1,\dots)\\
&= \Delta(T(\mu^{f(x)^{-1}}).T_xf.T(\mu^x).T(\mu^{x^{-1}}).X_1,\dots)\\
&= \Delta(Df(x).T(\mu^{x^{-1}}).X_1,\dots)\\
&= \det(Df(x))\Delta(T(\mu^{x^{-1}}).X_1,\dots)\\
&= \det(Df(x))(R_\Delta)_x(X_1,\dots,X_n). \quad\square
\end{aligned}$$

12.10. Theorem. Transformation formula for multiple integrals. *Let $f:G\to G$ be a diffeomorphism, let $\Delta\in\Lambda^{\dim G}\mathfrak{g}^*$. Then for any $g\in C_c^\infty(G,\mathbb{R})$ we have*

$$\int_G g(f(x))|\det(Df(x))|d_Rx=\int_G g(y)d_Ry,$$

where d_Rx is the right Haar measure, given by R_Δ.

Proof. We consider the locally constant function $\varepsilon(x)=\operatorname{sign}\det(Df(x))$ which is 1 on those connected components where f respects the orientation and is -1

on the other components. Then the integral is the sum of all integrals over the connected components and we may investigate each one separately, so let us restrict attention to the component G_0 of the identity. By a right translation (which does not change the integrals) we may assume that $f(G_0) = G_0$. So finally let us assume without loss of generality that G is connected, so that ε is constant. Then by lemma 12.9 we have

$$\begin{aligned}\int_G gR_\Delta &= \varepsilon \int_G f^*(gR_\Delta) = \varepsilon \int_G f^*(g)f^*(R_\Delta)\\ &= \int_G (g\circ f)\varepsilon\det(Df)R_\Delta = \int_G (g\circ f)|\det(Df)|R_\Delta. \quad\square\end{aligned}$$

12.11. Theorem. *Let G be a compact and connected Lie group, let $f \in C^\infty(G,G)$ and $\Delta \in \Lambda^{\dim G}\mathfrak{g}^*$. Then we have for $g \in C^\infty(G,\mathbb{R})$,*

$$\begin{aligned}\deg f \int_G gR_\Delta &= \int_G (g\circ f)\det(Df)R_\Delta, \quad \textit{or}\\ \deg f \int_G g(y)d_Ry &= \int_G g(f(x))\det(Df(x))d_Rx.\end{aligned}$$

Here $\deg f$*, the mapping degree of f, see 11.8, is an integer.*

Proof. From lemma 12.9 we have $f^*R_\Delta = \det(Df)R_\Delta$. Using this and the defining relation from 11.8 for $\deg f$ we may compute as follows:

$$\begin{aligned}\deg f \int_G gR_\Delta &= \int_G f^*(gR_\Delta) = \int_G f^*(g)f^*(R_\Delta)\\ &= \int_G (g\circ f)\det(Df)R_\Delta. \quad\square\end{aligned}$$

12.12. Examples. Let G be a compact connected Lie group.

1. If $f = \mu^a : G \to G$ then $D(\mu^a)(x) = Id_{\mathfrak{g}}$. From theorem 12.11 we get $\int_G gR_\Delta = \int_G (g\circ\mu^a)R_\Delta$, the right invariance of the right Haar measure.

2. If $f = \mu_a : G \to G$ then $D(\mu_a)(x) = T(\mu^{(ax)^{-1}}).T_x(\mu_a).T_e(\mu^x) = Ad(a)$. So the last two results give $\int_G gR_\Delta = \int_G (g\circ\mu_a)|\det Ad(a)|R_\Delta$ which we already know from 12.4.

3. If $f(x) = x^2 = \mu(x,x)$ we have

$$\begin{aligned}Df(x) &= T_{x^2}(\mu^{x^{-2}}).T_{(x,x)}\mu.(T_e(\mu^x), T_e(\mu^x))\\ &= T_x(\mu^{x^{-1}}).T_{x^2}(\mu^{x^{-1}})\left(T_x(\mu_x).T_e(\mu^x) + T_x(\mu^x).T_e(\mu^x)\right)\\ &= Ad(x) + Id_{\mathfrak{g}}.\end{aligned}$$

Let us now suppose that $\int_G R_\Delta = 1$, then we get

$$\deg((\quad)^2) = \deg((\quad)^2)\int_G R_\Delta = \int_G \det(Id_{\mathfrak g} + Ad(x))d_Rx$$
$$\int_G g(x^2)\det(Id_{\mathfrak g} + Ad(x))d_Rx = \int_G \det(Id_{\mathfrak g} + Ad(x))d_Rx \int_G g(x)d_Rx.$$

4. Let $f(x) = x^k$ for $k \in \mathbb{N}$, $\int_G d_Rx = 1$. Then we claim that

$$D((\quad)^k)(x) = \sum_{i=0}^{k-1} Ad(x^i).$$

This follows from induction, starting from example 3 above, since

$$\begin{aligned} D((\quad)^k)(x) &= D(Id_G(\quad)^{k-1})(x) \\ &= D(Id_G)(x) + Ad(x).D((\quad)^{k-1})(x) \qquad \text{by 12.7} \\ &= Id_{\mathfrak g} + Ad(x)(\sum_{i=0}^{k-2} Ad(x^i)) = \sum_{i=0}^{k-1} Ad(x^i). \end{aligned}$$

We conclude that

$$\deg(\quad)^k = \int_G \det\left(\sum_{i=0}^{k} Ad(x^i)\right) d_Rx.$$

If G is abelian we have $\deg(\quad)^k = k^{\dim G}$ since then $Ad(x) = Id_{\mathfrak g}$.

5. Let $f(x) = \nu(x) = x^{-1}$. Then we have $D\nu(x) = T\mu^{\nu(x)^{-1}}.T_x\nu.T_e\mu^x = -Ad(x^{-1})$. Using this we see that the result in 4. holds also for negative k, if the summation is interpreted in the right way:

$$D((\quad)^{-k})(x) = \sum_{i=0}^{-k+1} Ad(x^i) = -\sum_{i=0}^{k-1} Ad(x^{-i}).$$

Cohomology of compact connected Lie groups

12.13. Let G be a connected Lie group with Lie algebra $\mathfrak g$. The De Rham cohomology of G is the cohomology of the graded differential algebra $(\Omega(G), d)$. We will investigate now what is contributed by the subcomplex of the left invariant differential forms.

Definition. A differential form $\omega \in \Omega(G)$ is called *left invariant differential form* if $\mu_a^*\omega = \omega$ for all $a \in G$. We denote by $\Omega_L(G)$ the subspace of all left invariant forms. Clearly the mapping

$$\begin{gathered} L : \Lambda\mathfrak{g}^* \to \Omega_L(G), \\ (L_\omega)_x(X_1, \dots, X_k) = \omega(T(\mu_{x^{-1}}).X_1, \dots, T(\mu_{x^{-1}}).X_k), \end{gathered}$$

is a linear isomorphism. Since $\mu_a^* \circ d = d \circ \mu_a^*$ the space $(\Omega_L(G), d)$ is a graded differential subalgebra of $(\Omega(G), d)$.

We shall also need the representation $\widetilde{Ad} : G \to GL(\Lambda\mathfrak{g}^*)$ which is given by $\widetilde{Ad}(a) = \Lambda(Ad(a^{-1})^*)$ or

$$(\widetilde{Ad}(a)\omega)(X_1, \dots, X_k) = \omega(Ad(a^{-1}).X_1, \dots, Ad(a^{-1}).X_k).$$

12.14. Lemma. *1. Via the isomorphism $L : \Lambda\mathfrak{g}^* \to \Omega_L(G)$ the exterior differential d has the following form on $\Lambda\mathfrak{g}^*$:*

$$d\omega(X_0, \dots, X_k) = \sum_{0 \le i < j \le k} (-1)^{i+j}\omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots \widehat{X_j}, \dots, X_k),$$

where $\omega \in \Lambda^k\mathfrak{g}^*$ and $X_i \in \mathfrak{g}$.

2. For $X \in \mathfrak{g}$ we have $i(L(X))\Omega_L(G) \subset \Omega_L(G)$ and $\mathcal{L}_{L(X)}\Omega_L(G) \subset \Omega_L(G)$. Thus we have induced mappings

$$\begin{gathered} i_X : \Lambda^k\mathfrak{g}^* \to \Lambda^{k-1}\mathfrak{g}^*, \\ (i_X\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}); \\ \mathcal{L}_X : \Lambda^k\mathfrak{g}^* \to \Lambda^k\mathfrak{g}^*, \\ (\mathcal{L}_X\omega)(X_1, \dots, X_k) = \sum_{i=1}^k (-1)^i\omega([X, X_i], X_1, \dots, \widehat{X_i}, \dots X_k). \end{gathered}$$

3. These mappings satisfy all the properties from section 7, in particular

$$\begin{aligned} &\mathcal{L}_X = i_X \circ d + d \circ i_X, && \text{see 7.9.(2)}, \\ &\mathcal{L}_X \circ d = d \circ \mathcal{L}_X, && \text{see 7.9.(5)}, \\ &[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}, && \text{see 7.6.(3)}. \\ &[\mathcal{L}_X, i_Y] = i_{[X,Y]}, && \text{see 7.7.(2)}. \end{aligned}$$

4. The representation $\widetilde{Ad} : G \to GL(\Lambda\mathfrak{g}^*)$ has the following derivative: $T_e\widetilde{Ad}.X = \mathcal{L}_X$.

Proof. For $\omega \in \Lambda^k\mathfrak{g}^*$ and $X_i \in \mathfrak{g}$ the function

$$\begin{aligned}(L_\omega)_x(L_{X_0}(x),\dots,L_{X_k}(x)) &= \omega(T(\mu_{x^{-1}}).L_{X_1}(x),\dots)\\ &= \omega(T(\mu_{x^{-1}}).T(\mu_x).X_1,\dots)\\ &= \omega(X_1,\dots,X_k)\end{aligned}$$

is constant in x. This implies already that $i(L_X)\Omega_L(G) \subset \Omega_L(G)$ and the form of i_X in 2. Then by 7.8.(2) we have

$$\begin{aligned}(d\omega)(X_0,\dots,X_k) &= (dL_\omega)(L_{X_0},\dots,L_{X_k})(e)\\ &= \sum_{i=0}^k (-1)^i L_{X_i}(e)(\omega(X_0,\dots\widehat{X}_i,\dots X_k))\\ &\quad + \sum_{0\le i<j\le k} (-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots X_k),\end{aligned}$$

from which assertion 1 follows since the first summand is 0. Similarly we have

$$\begin{aligned}(\mathcal{L}_X\omega)(X_1,\dots,X_k) &= (\mathcal{L}_{L(X)}L_\omega)(L_{X_1},\dots,L_{X_k})(e)\\ &= L_X(e)(\omega(X_1,\dots,X_k)) + \sum_{i=1}^k (-1)^i\omega([X,X_i],X_1,\dots,\widehat{X}_i,\dots X_k).\end{aligned}$$

Again the first summand is 0 and the second result of (2) follows.

3. This is obvious.

4. For X and $X_i \in \mathfrak{g}$ and for $\omega \in \Lambda^k\mathfrak{g}^*$ we have

$$\begin{aligned}((T_e\widetilde{Ad}.X)\omega)(X_1,\dots,X_k) &= \tfrac{\partial}{\partial t}\big|_0 (\widetilde{Ad}(\exp(tX))\omega)(X_1,\dots,X_k)\\ &= \tfrac{\partial}{\partial t}\big|_0 \omega(Ad(\exp(-tX)).X_1,\dots,Ad(\exp(-tX)).X_k)\\ &= \sum_{i=1}^k \omega(X_1,\dots,X_{i-1},-ad(X)X_i,X_{i+1},\dots X_k)\\ &= \sum_{i=1}^k (-1)^i\omega([X,X_i],X_1,\dots,\widehat{X}_i,\dots X_k)\\ &= (\mathcal{L}_X\omega)(X_1,\dots,X_k). \quad \square\end{aligned}$$

12.15. Lemma of Maschke. *Let G be a compact Lie group, let*

$$(0 \to)V_1 \xrightarrow{i} V_2 \xrightarrow{p} V_3 \to 0$$

be an exact sequence of G-modules and homomorphisms such that each V_i is a complete locally convex vector space and the representation of G on each V_i consists of continuous linear mappings with $g \mapsto g.v$ continuous $G \to V_i$ for each $v \in V_i$. Then also the sequence

$$(0 \to)V_1^G \xrightarrow{i} V_2^G \xrightarrow{p^G} V_3^G \to 0$$

is exact, where $V_i^G := \{v \in V_i : g.v = v$ for all $g \in G\}$.

Proof. We prove first that p^G is surjective. Let $v_3 \in V_3^G \subset V_3$. Since $p : V_2 \to V_3$ is surjective there is an $v_2 \in V_2$ with $p(v_2) = v_3$. We consider the element $\tilde{v}_2 := \int_G x.v_2 d_L x$; the integral makes sense since $x \mapsto x.v_2$ is a continuous mapping $G \to V_2$, G is compact, and Riemann sums converge in the locally convex topology of V_2. We assume that $\int_G d_L x = 1$. Then we have $a.\tilde{v}_2 = a.\int_G x.v_2 d_L x = \int_G (ax).v_2 d_L x = \int_G x.v_2 d_L x = \tilde{v}_2$ by the left invariance of the integral, see 12.4, where one uses continuous linear functionals to reduce to the scalar valued case. So $\tilde{v}_2 \in V_2^G$ and since p is a G-homomorphism we get

$$\begin{aligned} p^G(\tilde{v}_2) &= p(\tilde{v}_2) = p(\int_G x.v_2 d_L x) \\ &= \int_G p(x.v_2) d_L x = \int_G x.p(v_2) d_L x \\ &= \int x.v_3 d_L x = \int_G v_3 d_L x = v_3. \end{aligned}$$

So p^G is surjective.

Now we prove that the sequence is exact at V_2^G. Clearly $p^G \circ i^G = (p \circ i)|V_1^G = 0$. Suppose conversely that $v_2 \in V_2^G$ with $p^G(v_2) = p(v_2) = 0$. Then there is an $v_1 \in V_1$ with $i(v_1) = v_2$. Consider $\tilde{v}_1 := \int_G x.v_1 d_L x$. As above we see that $\tilde{v}_1 \in V_1^G$ and that $i^G(\tilde{v}_1) = v_2$. □

12.16. Theorem (Chevalley, Eilenberg). *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then we have:*

(1) $H^*(G) = H^*(\Lambda\mathfrak{g}^*, d) := H^*(\mathfrak{g})$.
(2) $H^*(\mathfrak{g}) = H^*(\Lambda\mathfrak{g}^*, d) = (\Lambda\mathfrak{g}^*)^{\mathfrak{g}} = \{\omega \in \Lambda\mathfrak{g}^* : \mathcal{L}_X\omega = 0$ *for all* $X \in \mathfrak{g}\}$, *the space of all $\mathfrak{g}$-invariant forms on $\mathfrak{g}$.*

The algebra $H^*(\mathfrak{g}) = H(\Lambda\mathfrak{g}^*, d)$ is called the *cohomology of the Lie algebra* $\mathfrak{g}$.

Proof. (Following [Pitie, 1976].)

(1). Let $Z^k(G) = \ker(d : \Omega^k(G) \to \Omega^{k+1}(G))$, and let us consider the following exact sequence of vector spaces:

$$\Omega^{k-1}(G) \xrightarrow{d} Z^k(G) \to H^k(G) \to 0 \tag{3}$$

The group G acts on $\Omega(G)$ by $a \mapsto \mu_{a^{-1}}^*$, this action commutes with d and induces thus an action of G of $Z^k(G)$ and also on $H^k(G)$. On the space $\Omega(G)$ we may consider the compact C^∞-topology (uniform convergence on the compact G, in all derivatives separately). In this topology d is continuous and $Z^k(G)$ is closed, and the action of G is pointwise continuous. So the assumptions of the lemma of Maschke 12.15 are satisfied and we conclude that the following sequence is also exact:

$$\Omega_L^{p-1}(G) \xrightarrow{d} Z^k(G)^G \to H^k(G)^G \to 0 \tag{4}$$

Since G is connected, for each $a \in G$ we may find a smooth curve $c : [0,1] \to G$ with $c(0) = e$ and $c(1) = a$. Then $(t,x) \mapsto \mu_{c(t)^{-1}}(x) = c(t)^{-1}x$ is a smooth homotopy between Id_G and $\mu_{a^{-1}}$, so by 9.4 the two mappings induce the same mapping in homology; we have $\mu_{a^{-1}}^* = Id : H^k(G) \to H^k(G)$ for each $a \in G$. Thus $H^k(G)^G = H^k(G)$. Furthermore $Z^k(G)^G = \ker(d : \Omega_L^k(G) \to \Omega_L^{k+1}(G))$, so from the exact sequence (4) we may conclude that

$$H^k(G) = H^k(G)^G = \frac{\ker(d : \Omega_L^k(G) \to \Omega_L^{k+1}(G))}{\operatorname{im}(d : \Omega_L^{k-1}(G) \to \Omega_L^k(G))} = H^k(\Lambda\mathfrak{g}^*, d).$$

(2). From 12.14.3 we have $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$, so by 12.14.4 we conclude that $\widetilde{Ad}(a) \circ d = d \circ \widetilde{Ad}(a) : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ since G is connected. Thus the the sequence

$$\Lambda^{k-1}\mathfrak{g}^* \xrightarrow{d} Z^k(\mathfrak{g}^*) \to H^k(\Lambda\mathfrak{g}^*, d) \to 0, \tag{5}$$

is an exact sequence of G-modules and G-homomorphisms, where $Z^k(\mathfrak{g}^*) = \ker(d : \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*)$. All spaces are finite dimensional, so the lemma of Maschke 12.15 is applicable and we may conclude that also the following sequence is exact:

$$(\Lambda^{k-1}\mathfrak{g}^*)^G \xrightarrow{d} Z^k(\mathfrak{g}^*)^G \to H^k(\Lambda\mathfrak{g}^*, d)^G \to 0, \tag{6}$$

The space $H^k(\Lambda\mathfrak{g}^*, d)^G$ consist of all cohomology classes α with $\widetilde{Ad}(a)\alpha = \alpha$ for all $a \in G$. Since G is connected, by 12.14.4 these are exactly the α with $\mathcal{L}_X\alpha = 0$ for all $X \in \mathfrak{g}$. For $\omega \in \Lambda\mathfrak{g}^*$ with $d\omega = 0$ we have by 12.14.3 that $\mathcal{L}_X\omega = i_X d\omega + di_X\omega = di_X\omega$, so that $\mathcal{L}_X\alpha = 0$ for all $\alpha \in H^k(\Lambda\mathfrak{g}^*, d)$. Thus we get $H^k(\Lambda\mathfrak{g}^*, d) = H^k(\Lambda\mathfrak{g}^*, d)^G$. Also we have $(\Lambda\mathfrak{g}^*)^G = (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ so that the exact sequence (6) tranlates to

$$H^k(\mathfrak{g}) = H^k(\Lambda\mathfrak{g}^*, d) = H^k((\Lambda\mathfrak{g}^*)^{\mathfrak{g}}, d). \tag{7}$$

Now let $\omega \in (\Lambda^k\mathfrak{g}^*)^{\mathfrak{g}} = \{\varphi : \mathcal{L}_X\varphi = 0 \text{ for all } X \in \mathfrak{g}\}$ and consider the inversion $\nu : G \to G$. Then we have for $\omega \in \Lambda^k\mathfrak{g}^*$ and $X_i \in \mathfrak{g}$:

$$\begin{aligned}
(\nu^* L_\omega)_a&(T_e(\mu_a).X_1, dots, T_e(\mu_a).X_k) = \\
&= (L_\omega)_{a^{-1}}(T_a\nu.T_e(\mu_a).X_1, dots, T_a\nu.T_e(\mu_a).X_k) \\
&= (L_\omega)_{a^{-1}}(-T(\mu^{a^{-1}}).T(\mu_{a^{-1}}).T_e(\mu_a).X_1, \dots) \\
&= (L_\omega)_{a^{-1}}(-T_e(\mu^{a^{-1}}).X_1, \dots, -T_e(\mu^{a^{-1}}).X_k) \\
&= (-1)^k\omega(T\mu_a.T\mu^{a^{-1}}.X_1, \dots, T\mu_a.T\mu^{a^{-1}}.X_k) \\
&= (-1)^k\omega(Ad(a).X_1, dots, Ad(a).X_k) \\
&= (-1)^k(\widetilde{Ad}(a^{-1})\omega)(X_1, dots, X_k) \\
&= (-1)^k\omega(X_1, \dots, X_k) \qquad \text{since } \omega \in (\Lambda^k\mathfrak{g}^*)^{\mathfrak{g}} \\
&= (-1)^k(L_\omega)_a(T_e(\mu_a).X_1, dots, T_e(\mu_a).X_k).
\end{aligned}$$

So for $\omega \in (\Lambda^k\mathfrak{g}^*)^{\mathfrak{g}}$ we have $\nu^*L_\omega = (-1)^k L_\omega$ and thus also $(-1)^{k+1}L_{d\omega} = \nu^*dL_\omega = d\nu^*L_\omega = (-1)^k dL_\omega = (-1)^k L_{d\omega}$ which implies $d\omega = 0$. Hence we have $d|(\Lambda\mathfrak{g}^*)^{\mathfrak{g}} = 0$.

From (7) we how get $H^k(\mathfrak{g}) = H^k((\Lambda\mathfrak{g}^*)^{\mathfrak{g}}, 0) = (\Lambda^k\mathfrak{g}^*)^{\mathfrak{g}}$ as required. $\square$

12.17. Corollary. *Let G be a compact connected Lie group. Then its Poincaré polynomial is given by*

$$f_G(t) = \int_G \det(Ad(x) + tId_{\mathfrak{g}})d_Lx.$$

Proof. Let $\dim G = n$. By definition 9.2 and by Poincaré duality 11.6 we have

$$f_G(t) = \sum_{k=0}^n b_k(G)t^k = \sum_{k=0}^n b_k(G)t^{n-k} = \sum_{k=0}^n \dim_{\mathbb{R}} H^k(G)t^{n-k}.$$

On the other hand we hand we have

$$\begin{aligned}
\int_G \det(Ad(x)+tId_{\mathfrak g})d_Lx &= \int_G \det(Ad(x^{-1})^*+tId_{\mathfrak g^*})d_Lx \\
&= \int_G \sum_{k=0}^n \operatorname{Trace}(\Lambda^k Ad(x^{-1})^*)\, t^{n-k} d_Lx \qquad \text{by 12.19 below} \\
&= \sum_{k=0}^n \int_G \operatorname{Trace}(\widetilde{Ad}(x)|\Lambda^k\mathfrak g^*)d_Lx\, t^{n-k}.
\end{aligned}$$

If $\rho : G \to GL(V)$ is a finite dimensional representation of G then the operator $\int_G \rho(x)d_Lx : V \to V$ is just a projection onto V^G, the space of fixed points of the represetation, see the proof of the lemma of Maschke 12.14. The trace of a projection is the dimension of the image. So

$$\begin{aligned}
\int_G \operatorname{Trace}(\widetilde{Ad}(a)|\Lambda^k\mathfrak g^*)d_Lx &= \operatorname{Trace}\left(\int_G (\widetilde{Ad}(a)|\Lambda^k\mathfrak g^*)d_Lx\right) \\
&= \dim(\Lambda^k\mathfrak g^*)^G = \dim H^k(G). \quad \square
\end{aligned}$$

12.18. Let $\mathbb T^n = (S^1)^n$ be the n-dimensional torus, let $\mathfrak t^n$ be its Lie algebra. The bracket is zero since the torus is an abelian group. From theorem 12.16 we have then that $H^*(\mathbb T^n) = (\Lambda(\mathfrak t^n)^*)^{\mathfrak t^n} = \Lambda(\mathfrak t^n)^*$, so the Poincaré Polynomial is $f_{\mathbb T^n}(t) = (1+t)^n$.

12.19. Lemma. *Let V be an n-dimensional vector space and let $A : V \to V$ be a linear mapping. Then we have*

$$\det(A + tId_V) = \sum_{k=0}^n t^{n-k} \operatorname{Trace}(\Lambda^k A).$$

Proof. By $\Lambda^k A : \Lambda^k V \to \Lambda^k V$ we mean the mapping $v_1\wedge\dots\wedge v_k \mapsto Av_1\wedge\dots\wedge Av_k$. Let $e_1,\dots,e_n$ be a basis of V. By the definition of the determinant we have

$$\begin{aligned}
\det(A+tId_V)(e_1\wedge\dots\wedge e_n) &= (Ae_1+te_1)\wedge\dots\wedge(Ae_n+te_n) \\
&= \sum_{k=0}^n t^{n-k} \sum_{i_1<\dots<i_k} e_1\wedge\dots\wedge Ae_{i_1}\wedge\dots\wedge Ae_{i_k}\wedge\dots\wedge e_n.
\end{aligned}$$

The multivectors $(e_{i_1} \wedge \dots \wedge e_{i_k})_{i_1 < \dots < i_k}$ are a basis of $\Lambda^k V$ and we can thus write

$$(\Lambda^k A)(e_{i_1} \wedge \dots \wedge e_{i_k}) = Ae_{i_1} \wedge \dots \wedge Ae_{i_k} = \sum_{j_1 < \dots < j_k} A^{j_1 \dots j_k}_{i_1 \dots i_k} e_{j_1} \wedge \dots \wedge e_{j_k},$$

where $(A^{j_1 \dots j_k}_{i_1 \dots i_k})$ is the matrix of $\Lambda^k A$ in this basis. We see that

$$e_1 \wedge \dots \wedge Ae_{i_1} \wedge \dots \wedge Ae_{i_k} \wedge \dots \wedge e_n = A^{i_1 \dots i_k}_{i_1 \dots i_k} e_1 \wedge \dots \wedge e_n.$$

Consequently we have

$$\begin{aligned} \det(A + tId_V)e_1 \wedge \dots \wedge e_n &= \sum_{k=0}^{n} t^{n-k} \sum_{i_1 < \dots < i_k} A^{i_1 \dots i_k}_{i_1 \dots i_k} e_1 \wedge \dots \wedge e_n \\ &= \sum_{k=0}^{n} t^{n-k} \operatorname{Trace}(\Lambda^{\mathrm{k}} \mathrm{A})\, e_1 \wedge \dots \wedge e_n, \end{aligned}$$

which implies the result. $\square$

13. Derivations on the Algebra of Differential Forms and the Frölicher-Nijenhuis Bracket

13.1. Derivations. In this section let M be a smooth manifold. We consider the graded commutative algebra $\Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M) = \bigoplus_{k=-\infty}^{\infty} \Omega^k(M)$ of differential forms on M, where we put $\Omega^k(M) = 0$ for $k < 0$ and $k > \dim M$. The denote by $\operatorname{Der}_k \Omega(M)$ the space of all *(graded) derivations* of degree k, i.e. all linear mappings $D : \Omega(M) \to \Omega(M)$ with $D(\Omega^\ell(M)) \subset \Omega^{k+\ell}(M)$ and $D(\varphi \wedge \psi) = D(\varphi) \wedge \psi + (-1)^{k\ell}\varphi \wedge D(\psi)$ for $\varphi \in \Omega^\ell(M)$.

Lemma. *Then the space* $\operatorname{Der} \Omega(M) = \bigoplus_k \operatorname{Der}_k \Omega(M)$ *is a graded Lie algebra with the graded commutator* $[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$ *as bracket. This means that the bracket is graded anticommutative, and satisfies the graded Jacobi identity*

$$[D_1, D_2] = -(-1)^{k_1 k_2}[D_2, D_1],$$
$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1 k_2}[D_2, [D_1, D_3]]$$

(so that $ad(D_1) = [D_1, \quad]$ *is itself a derivation of degree* k_1*).*

Proof. Plug in the definition of the graded commutator and compute. $\square$

In section 7 we have already met some graded derivations: for a vector field X on M the derivation i_X is of degree -1, $\mathcal{L}_X$ is of degree 0, and d is of degree 1. Note also that the important formula $\mathcal{L}_X = d\, i_X + i_X\, d$ translates to $\mathcal{L}_X = [i_X, d]$.

13.2. Algebraic derivations. A derivation $D \in \operatorname{Der}_k \Omega(M)$ is called *algebraic* if $D \mid \Omega^0(M) = 0$. Then $D(f.\omega) = f.D(\omega)$ for $f \in C^\infty(M, \mathbb{R})$, so D is of tensorial character by 7.3. So D induces a derivation $D_x \in \operatorname{Der}_k \Lambda T_x^*M$ for each $x \in M$. It is uniquely determined by its restriction to 1-forms $D_x|T_x^*M : T_x^*M \to \Lambda^{k+1}T^*M$ which we may view as an element $K_x \in \Lambda^{k+1}T_x^*M \otimes T_xM$ depending smoothly on $x \in M$. To express this dependence we write $D = i_K = i(K)$, where $K \in C^\infty(\Lambda^{k+1}T^*M \otimes TM) =: \Omega^{k+1}(M; TM)$. Note the defining equation: $i_K(\omega) = \omega \circ K$ for $\omega \in \Omega^1(M)$. We call $\Omega(M, TM) = \bigoplus_{k=0}^{\dim M} \Omega^k(M, TM)$ the space of all *vector valued differential forms.*

Theorem. (1) *For* $K \in \Omega^{k+1}(M, TM)$ *the formula*

$$(i_K\omega)(X_1, \dots, X_{k+\ell}) =$$
$$= \tfrac{1}{(k+1)!\,(\ell-1)!} \sum_{\sigma \in \mathcal{S}_{k+\ell}} \operatorname{sign} \sigma\, .\omega(K(X_{\sigma 1}, \dots, X_{\sigma(k+1)}), X_{\sigma(k+2)}, \dots)$$

for $\omega \in \Omega^\ell(M)$*,* $X_i \in \mathfrak{X}(M)$ *(or* T_xM*) defines an algebraic graded derivation* $i_K \in \operatorname{Der}_k \Omega(M)$ *and any algebraic derivation is of this form.*

(2) *By* $i([K,L]^\wedge) := [i_K, i_L]$ *we get a bracket* $[\quad,\quad]^\wedge$ *on* $\Omega^{*+1}(M,TM)$ *which defines a graded Lie algebra structure with the grading as indicated, and for* $K \in \Omega^{k+1}(M,TM)$*,* $L \in \Omega^{\ell+1}(M,TM)$ *we have*

$$[K,L]^\wedge = i_K L - (-1)^{k\ell} i_L K$$

where $i_K(\omega \otimes X) := i_K(\omega) \otimes X$*.*

$[\quad,\quad]^\wedge$ is called the *algebraic bracket* or the *Nijenhuis-Richardson bracket*, see [Nijenhuis-Richardson, 1967].

Proof. Since ΛT_x^*M is the free graded commutative algebra generated by the vector space T_x^*M any $K \in \Omega^{k+1}(M,TM)$ extends to a graded derivation. By applying it to an exterior product of 1-forms one can derive the formula in (1). The graded commutator of two algebraic derivations is again algebraic, so the injection $i : \Omega^{*+1}(M,TM) \to \operatorname{Der}_*(\Omega(M))$ induces a graded Lie bracket on $\Omega^{*+1}(M,TM)$ whose form can be seen by applying it to a 1-form. $\square$

13.3. Lie derivations. The exterior derivative d is an element of $\operatorname{Der}_1 \Omega(M)$. In view of the formula $\mathcal{L}_X = [i_X, d] = i_X\, d + d\, i_X$ for vector fields X, we define for $K \in \Omega^k(M;TM)$ the *Lie derivation* $\mathcal{L}_K = \mathcal{L}(K) \in \operatorname{Der}_k \Omega(M)$ by $\mathcal{L}_K := [i_K, d]$.

Then the mapping $\mathcal{L} : \Omega(M,TM) \to \operatorname{Der} \Omega(M)$ is injective, since $\mathcal{L}_K f = i_K df = df \circ K$ for $f \in \mathcal{C}^\infty(M,\mathbb{R})$.

Theorem. *For any graded derivation* $D \in \operatorname{Der}_k \Omega(M)$ *there are unique* $K \in \Omega^k(M;TM)$ *and* $L \in \Omega^{k+1}(M;TM)$ *such that*

$$D = \mathcal{L}_K + i_L.$$

We have $L = 0$ *if and only if* $[D,d] = 0$*.* D *is algebraic if and only if* $K = 0$*.*

Proof. Let $X_i \in \mathfrak{X}(M)$ be vector fields. Then $f \mapsto (Df)(X_1,\dots,X_k)$ is a derivation $C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$, so by 3.3 there is a unique vector field $K(X_1,\dots,X_k) \in \mathfrak{X}(M)$ such that

$$(Df)(X_1,\dots,X_k) = K(X_1,\dots,X_k)f = df(K(X_1,\dots,X_k)).$$

Clearly $K(X_1,\dots,X_k)$ is $C^\infty(M,\mathbb{R})$-linear in each X_i and alternating, so K is tensorial by 7.3, $K \in \Omega^k(M;TM)$.

The defining equation for K is $Df = df \circ K = i_K df = \mathcal{L}_K f$ for $f \in C^\infty(M,\mathbb{R})$. Thus $D - \mathcal{L}_K$ is an algebraic derivation, so $D - \mathcal{L}_K = i_L$ by 13.2 for unique $L \in \Omega^{k+1}(M;TM)$.

Since we have $[d,d] = 2d^2 = 0$, by the graded Jacobi identity we obtain $0 = [i_K,[d,d]] = [[i_K,d],d] + (-1)^{k-1}[d,[i_K,d]] = 2[\mathcal{L}_K, d]$. The mapping $K \mapsto [i_K,d] = \mathcal{L}_K$ is injective, so the last assertions follow. $\square$

13.4. Applying $i(Id_{TM})$ on a k-fold exterior product of 1-forms we see that $i(Id_{TM})\omega = k\omega$ for $\omega \in \Omega^k(M)$. Thus we have $\mathcal{L}(Id_{TM})\omega = i(Id_{TM})d\omega - d\,i(Id_{TM})\omega = (k+1)d\omega - kd\omega = d\omega$. Thus $\mathcal{L}(Id_{TM}) = d$.

13.5. Let $K \in \Omega^k(M;TM)$ and $L \in \Omega^\ell(M;TM)$. Then clearly $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$, so we have

$$[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K, L])$$

for a uniquely defined $[K, L] \in \Omega^{k+\ell}(M;TM)$. This vector valued form $[K, L]$ is called the *Frölicher-Nijenhuis bracket* of K and L.

Theorem. *The space $\Omega(M;TM) = \bigoplus_{k=0}^{\dim M} \Omega^k(M;TM)$ with its usual grading is a graded Lie algebra for the Frölicher-Nijenhuis bracket. So we have*

$$[K, L] = -(-1)^{k\ell}[L, K]$$
$$[K_1, [K_2, K_3]] = [[K_1, K_2], K_3] + (-1)^{k_1 k_2}[K_2, [K_1, K_3]]$$

$Id_{TM} \in \Omega^1(M;TM)$ is in the center, i.e. $[K, Id_{TM}] = 0$ for all K.

$\mathcal{L} : (\Omega(M;TM), [\quad,\quad]) \to \operatorname{Der}\Omega(M)$ is an injective homomorphism of graded Lie algebras. For vector fields the Frölicher-Nijenhuis bracket coincides with the Lie bracket.

Proof. $df \circ [X, Y] = \mathcal{L}([X, Y])f = [\mathcal{L}_X, \mathcal{L}_Y]f$. The rest is clear. □

13.6. Lemma. *For $K \in \Omega^k(M;TM)$ and $L \in \Omega^{\ell+1}(M;TM)$ we have*

$$\begin{aligned}
&[\mathcal{L}_K, i_L] = i([K, L]) - (-1)^{k\ell}\mathcal{L}(i_L K), \text{ or}\\
&[i_L, \mathcal{L}_K] = \mathcal{L}(i_L K) - (-1)^k\, i([L, K]).
\end{aligned}$$

This generalizes 7.7.2.

Proof. For $f \in C^\infty(M, \mathbb{R})$ we have $[i_L, \mathcal{L}_K]f = i_L\, i_K\, df - 0 = i_L(df \circ K) = df \circ (i_L K) = \mathcal{L}(i_L K)f$. So $[i_L, \mathcal{L}_K] - \mathcal{L}(i_L K)$ is an algebraic derivation.

$$\begin{aligned}
[[i_L, \mathcal{L}_K], d] &= [i_L, [\mathcal{L}_K, d]] - (-1)^{k\ell}[\mathcal{L}_K, [i_L, d]] =\\
&= 0 - (-1)^{k\ell}\mathcal{L}([K, L]) = (-1)^k[i([L, K]), d].
\end{aligned}$$

Since $[\quad, d]$ kills the '$\mathcal{L}$'s' and is injective on the 'i's', the algebraic part of $[i_L, \mathcal{L}_K]$ is $(-1)^k\, i([L, K])$. □

13.7. Module structure. The space $\operatorname{Der}\Omega(M)$ is a graded module over the graded algebra $\Omega(M)$ with the action $(\omega \wedge D)\varphi = \omega \wedge D(\varphi)$, because $\Omega(M)$ is graded commutative.

Theorem. *Let the degree of ω be q, of φ be k, and of ψ be ℓ. Let the other degrees be as indicated. Then we have:*

$$\begin{aligned}
&(1) && [\omega\wedge D_1, D_2] = \omega\wedge[D_1,D_2] - (-1)^{(q+k_1)k_2} D_2(\omega)\wedge D_1.\\
&(2) && i(\omega\wedge L) = \omega\wedge i(L)\\
&(3) && \omega\wedge\mathcal{L}_K = \mathcal{L}(\omega\wedge K) + (-1)^{q+k-1} i(d\omega\wedge K).\\
&(4) && [\omega\wedge L_1, L_2]^\wedge = \omega\wedge[L_1,L_2]^\wedge - \\
& && \qquad - (-1)^{(q+\ell_1-1)(\ell_2-1)} i(L_2)\omega\wedge L_1.\\
&(5) && [\omega\wedge K_1, K_2] = \omega\wedge[K_1,K_2] - (-1)^{(q+k_1)k_2}\mathcal{L}(K_2)\omega\wedge K_1\\
& && \qquad + (-1)^{q+k_1} d\omega\wedge i(K_1)K_2.\\
&(6) && [\varphi\otimes X, \psi\otimes Y] = \varphi\wedge\psi\otimes[X,Y]\\
& && \qquad - \big(i_Y d\varphi\wedge\psi\otimes X - (-1)^{k\ell} i_X d\psi\wedge\varphi\otimes Y\big)\\
& && \qquad - \big(d(i_Y\varphi\wedge\psi)\otimes X - (-1)^{k\ell} d(i_X\psi\wedge\varphi)\otimes Y\big)\\
& && \qquad = \varphi\wedge\psi\otimes[X,Y] + \varphi\wedge\mathcal{L}_X\psi\otimes Y - \mathcal{L}_Y\varphi\wedge\psi\otimes X\\
& && \qquad + (-1)^k\left(d\varphi\wedge i_X\psi\otimes Y + i_Y\varphi\wedge d\psi\otimes X\right).
\end{aligned}$$

Proof. For (1) , (2) , (3) write out the definitions. For (4) compute $i([\omega\wedge L_1, L_2]^\wedge)$. For (5) compute $\mathcal{L}([\omega\wedge K_1, K_2])$. For (6) use (5) . $\square$

13.8. Theorem. *For $K\in\Omega^k(M;TM)$ and $\omega\in\Omega^\ell(M)$ the Lie derivative of ω along K is given by the following formula, where the X_i are vector fields on M.*

$$\begin{aligned}
(\mathcal{L}_K\omega)&(X_1,\dots,X_{k+\ell}) = \\
&= \tfrac{1}{k!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\ \mathcal{L}(K(X_{\sigma1},\dots,X_{\sigma k}))(\omega(X_{\sigma(k+1)},\dots,X_{\sigma(k+\ell)}))\\
&\quad + \tfrac{-1}{k!\,(\ell-1)!}\sum_\sigma \operatorname{sign}\sigma\ \omega([K(X_{\sigma1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots)\\
&\quad + \tfrac{(-1)^{k-1}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\ \omega(K([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(k+2)},\dots).
\end{aligned}$$

Proof. It suffices to consider $K=\varphi\otimes X$. Then by 13.7.3 we have $\mathcal{L}(\varphi\otimes X) = \varphi\wedge\mathcal{L}_X - (-1)^{k-1}d\varphi\wedge i_X$. Now use the global formulas of section 7 to expand this. $\square$

13.9. Theorem. *For $K \in \Omega^k(M;TM)$ and $L \in \Omega^\ell(M;TM)$ we have for the Frölicher-Nijenhuis bracket $[K,L]$ the following formula, where the X_i are vector fields on M.*

$$\begin{aligned}
&[K,L](X_1,\dots,X_{k+\ell}) = \\
&\quad = \tfrac{1}{k!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\; [K(X_{\sigma1},\dots,X_{\sigma k}),L(X_{\sigma(k+1)},\dots,X_{\sigma(k+\ell)})] \\
&\qquad + \tfrac{-1}{k!\,(\ell-1)!}\sum_\sigma \operatorname{sign}\sigma\; L([K(X_{\sigma1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots) \\
&\qquad + \tfrac{(-1)^{k\ell}}{(k-1)!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\; K([L(X_{\sigma1},\dots,X_{\sigma \ell}),X_{\sigma(\ell+1)}],X_{\sigma(\ell+2)},\dots) \\
&\qquad + \tfrac{(-1)^{k-1}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\; L(K([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(k+2)},\dots) \\
&\qquad + \tfrac{(-1)^{(k-1)\ell}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\; K(L([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(\ell+2)},\dots).
\end{aligned}$$

Proof. It suffices to consider $K = \varphi\otimes X$ and $L = \psi\otimes Y$, then for $[\varphi\otimes X,\psi\otimes Y]$ we may use 13.7.6 and evaluate that at $(X_1,\dots,X_{k+\ell})$. After some combinatorial computation we get the right hand side of the above formula for $K=\varphi\otimes X$ and $L=\psi\otimes Y$. $\square$

There are more illuminating ways to prove this formula, see [Michor, 1987].

13.10. Local formulas. In a local chart (U,u) on the manifold M we put $K \mid U = \sum K^i_\alpha d^\alpha\otimes\partial_i$, $L\mid U = \sum L^j_\beta d^\beta\otimes\partial_j$, and $\omega\mid U = \sum\omega_\gamma d^\gamma$, where $\alpha = (1\le\alpha_1<\alpha_2<\dots<\alpha_k\le\dim M)$ is a form index, $d^\alpha = du^{\alpha_1}\wedge\ldots\wedge du^{\alpha_k}$, $\partial_i = \frac{\partial}{\partial u^i}$ and so on.

Plugging $X_j = \partial_{i_j}$ into the global formulas 13.2, 13.8, and 13.9, we get the following local formulas:

$$\begin{aligned}
i_K\omega \mid U &= \sum K^i_{\alpha_1\dots\alpha_k}\omega_{i\alpha_{k+1}\dots\alpha_{k+\ell-1}}\, d^\alpha \\
[K,L]^\wedge \mid U &= \sum\Big(K^i_{\alpha_1\dots\alpha_k}\, L^j_{i\alpha_{k+1}\dots\alpha_{k+\ell}} \\
&\qquad\qquad - (-1)^{(k-1)(\ell-1)} L^i_{\alpha_1\dots\alpha_\ell}\, K^j_{i\alpha_{\ell+1}\dots\alpha_{k+\ell}}\Big)\, d^\alpha\otimes\partial_j \\
\mathcal{L}_K\omega \mid U &= \sum\Big(K^i_{\alpha_1\dots\alpha_k}\,\partial_i\omega_{\alpha_{k+1}\dots\alpha_{k+\ell}} \\
&\qquad\qquad + (-1)^k(\partial_{\alpha_1}K^i_{\alpha_2\dots\alpha_{k+1}})\,\omega_{i\alpha_{k+2}\dots\alpha_{k+\ell}}\Big) d^\alpha
\end{aligned}$$

$$\begin{aligned}[K,L]\mid U = \sum\Big(&K^i_{\alpha_1\dots\alpha_k}\,\partial_i L^j_{\alpha_{k+1}\dots\alpha_{k+\ell}}\\
&-(-1)^{k\ell}L^i_{\alpha_1\dots\alpha_\ell}\,\partial_i K^j_{\alpha_{\ell+1}\dots\alpha_{k+\ell}}\\
&-kK^j_{\alpha_1\dots\alpha_{k-1}i}\,\partial_{\alpha_k}L^i_{\alpha_{k+1}\dots\alpha_{k+\ell}}\\
&+(-1)^{k\ell}\ell L^j_{\alpha_1\dots\alpha_{\ell-1}i}\,\partial_{\alpha_\ell}K^i_{\alpha_{\ell+1}\dots\alpha_{k+\ell}}\Big)\,d^\alpha\otimes\partial_j\end{aligned}$$

13.11. Theorem. *For $K_i \in \Omega^{k_i}(M;TM)$ and $L_i \in \Omega^{k_i+1}(M;TM)$ we have*

$$\begin{aligned}(1)\quad [\mathcal{L}_{K_1}+i_{L_1},\mathcal{L}_{K_2}+i_{L_2}] = \mathcal{L}\left([K_1,K_2]+i_{L_1}K_2-(-1)^{k_1k_2}i_{L_2}K_1\right)&\\
+\,i\left([L_1,L_2]^\wedge+[K_1,L_2]-(-1)^{k_1k_2}[K_2,L_1]\right).&\end{aligned}$$

Each summand of this formula looks like a semidirect product of graded Lie algebras, but the mappings

$$\begin{aligned}i&:\Omega(M;TM)\to \operatorname{End}(\Omega(M;TM),[\quad,\quad])\\
ad&:\Omega(M;TM)\to \operatorname{End}(\Omega(M;TM),[\quad,\quad]^\wedge)\end{aligned}$$

do not take values in the subspaces of graded derivations. We have instead for $K\in\Omega^k(M;TM)$ and $L\in\Omega^{\ell+1}(M;TM)$ the following relations:

$$\begin{aligned}(2)\quad i_L[K_1,K_2] &= [i_LK_1,K_2]+(-1)^{k_1\ell}[K_1,i_LK_2]\\
&\quad-\Big((-1)^{k_1\ell}i([K_1,L])K_2-(-1)^{(k_1+\ell)k_2}i([K_2,L])K_1\Big)\\
(3)\quad [K,[L_1,L_2]^\wedge] &= [[K,L_1],L_2]^\wedge+(-1)^{kk_1}[L_1,[K,L_2]]^\wedge-\\
&\quad-\Big((-1)^{kk_1}[i(L_1)K,L_2]-(-1)^{(k+k_1)k_2}[i(L_2)K,L_1]\Big)\end{aligned}$$

The algebraic meaning of the relations of this theorem and its consequences in group theory have been investigated in [Michor, 1989]. The corresponding product of groups is well known to algebraists under the name ‘Zappa-Szep’-product.

Proof. Equation (1) is an immediate consequence of 13.6. Equations (2) and (3) follow from (1) by writing out the graded Jacobi identity, or as follows: Consider $\mathcal{L}(i_L[K_1,K_2])$ and use 13.6 repeatedly to obtain $\mathcal{L}$ of the right hand side of (2). Then consider $i([K,[L_1,L_2]^\wedge])$ and use again 13.6 several times to obtain i of the right hand side of (3). $\square$

13.12. Corollary (of 8.9). *For K, $L \in \Omega^1(M;TM)$ we have*

$$\begin{aligned}[K,L](X,Y) &= [KX,LY] - [KY,LX] \\ &\quad - L([KX,Y] - [KY,X]) \\ &\quad - K([LX,Y] - [LY,X]) \\ &\quad + (LK + KL)[X,Y].\end{aligned}$$

13.13. Curvature. Let $P \in \Omega^1(M;TM)$ be a fiber projection, i.e. $P \circ P = P$. This is the most general case of a (first order) *connection*. We may call $\ker P$ the *horizontal space* and $\operatorname{im} P$ the *vertical space* of the connection. If P is of constant rank, then both are sub vector bundles of TM. If $\operatorname{im} P$ is some primarily fixed sub vector bundle or (tangent bundle of) a foliation, P can be called a connection for it. Special cases of this will be treated extensively later on. The following result is immediate from 13.12.

Lemma. *We have*

$$[P,P] = 2R + 2\bar{R},$$

where R, $\bar{R} \in \Omega^2(M;TM)$ are given by $R(X,Y) = P[(Id-P)X,(Id-P)Y]$ and $\bar{R}(X,Y) = (Id-P)[PX,PY]$.

If P has constant rank, then R is the obstruction against integrability of the horizontal bundle $\ker P$, and $\bar{R}$ is the obstruction against integrability of the vertical bundle $\operatorname{im} P$. Thus we call R the *curvature* and $\bar{R}$ the *cocurvature* of the connection P. We will see later, that for a principal fiber bundle R is just the negative of the usual curvature.

13.14. Lemma (Bianchi identity). *If $P \in \Omega^1(M;TM)$ is a connection (fiber projection) with curvature R and cocurvature $\bar{R}$, then we have*

$$\begin{aligned}&[P, R + \bar{R}] = 0 \\ &[R,P] = i_R\bar{R} + i_{\bar{R}}R.\end{aligned}$$

Proof. We have $[P,P] = 2R + 2\bar{R}$ by 13.13 and $[P,[P,P]] = 0$ by the graded Jacobi identity. So the first formula follows. We have $2R = P \circ [P,P] = i_{[P,P]}P$. By 13.11.2 we get $i_{[P,P]}[P,P] = 2[i_{[P,P]}P,P] - 0 = 4[R,P]$. Therefore $[R,P] = \frac{1}{4}i_{[P,P]}[P,P] = i(R+\bar{R})(R+\bar{R}) = i_R\bar{R} + i_{\bar{R}}R$ since R has vertical values and kills vertical vectors, so $i_RR = 0$; likewise for $\bar{R}$. $\square$

13.15. Naturality of the Frölicher-Nijenhuis bracket. Let $f : M \to N$ be a smooth mapping between manifolds. Two vector valued forms $K \in \Omega^k(M;TM)$ and $K' \in \Omega^k(N;TN)$ are called *f-related* or *f-dependent*, if for all $X_i \in T_xM$ we have

$$K'_{f(x)}(T_xf \cdot X_1, \dots, T_xf \cdot X_k) = T_xf \cdot K_x(X_1, \dots, X_k). \tag{1}$$

Theorem.

(2) *If K and K' as above are f-related then $i_K \circ f^* = f^* \circ i_{K'} : \Omega(N) \to \Omega(M)$.*

(3) *If $i_K \circ f^* \mid B^1(N) = f^* \circ i_{K'} \mid B^1(N)$, then K and K' are f-related, where B^1 denotes the space of exact 1-forms.*

(4) *If K_j and K'_j are f-related for $j = 1, 2$, then $i_{K_1}K_2$ and $i_{K'_1}K'_2$ are f-related, and also $[K_1, K_2]^\wedge$ and $[K'_1, K'_2]^\wedge$ are f-related.*

(5) *If K and K' are f-related then $\mathcal{L}_K \circ f^* = f^* \circ \mathcal{L}_{K'} : \Omega(N) \to \Omega(M)$.*

(6) *If $\mathcal{L}_K \circ f^* \mid \Omega^0(N) = f^* \circ \mathcal{L}_{K'} \mid \Omega^0(N)$, then K and K' are f-related.*

(7) *If K_j and K'_j are f-related for $j = 1, 2$, then their Frölicher-Nijenhuis brackets $[K_1, K_2]$ and $[K'_1, K'_2]$ are also f-related.*

Proof. (2) By 13.2 we have for $\omega \in \Omega^q(N)$ and $X_i \in T_xM$:

$$\begin{aligned}
&(i_K f^*\omega)_x(X_1, \dots, X_{q+k-1}) = \\
&\quad = \tfrac{1}{k!\,(q-1)!} \sum_\sigma \operatorname{sign}\sigma\, (f^*\omega)_x(K_x(X_{\sigma 1}, \dots, X_{\sigma k}), X_{\sigma(k+1)}, \dots) \\
&\quad = \tfrac{1}{k!\,(q-1)!} \sum_\sigma \operatorname{sign}\sigma\, \omega_{f(x)}(T_xf \cdot K_x(X_{\sigma 1}, \dots), T_xf \cdot X_{\sigma(k+1)}, \dots) \\
&\quad = \tfrac{1}{k!\,(q-1)!} \sum_\sigma \operatorname{sign}\sigma\, \omega_{f(x)}(K'_{f(x)}(T_xf \cdot X_{\sigma 1}, \dots), T_xf \cdot X_{\sigma(k+1)}, \dots) \\
&\quad = (f^* i_{K'}\omega)_x(X_1, \dots, X_{q+k-1})
\end{aligned}$$

(3) follows from this computation, since the df, $f \in C^\infty(M, \mathbb{R})$ separate points.

(4) follows from the same computation for K_2 instead of ω, the result for the bracket then follows from 13.2.2.

(5) The algebra homomorphism f^* intertwines the operators i_K and $i_{K'}$ by (2), and f^* commutes with the exterior derivative d. Thus f^* intertwines the commutators $[i_K, d] = \mathcal{L}_K$ and $[i_{K'}, d] = \mathcal{L}_{K'}$.

(6) For $g \in \Omega^0(N)$ we have $\mathcal{L}_K\, f^*\, g = i_K\, d\, f^*\, g = i_K\, f^*\, dg$ and $f^*\, \mathcal{L}_{K'}\, g = f^*\, i_{K'}\, dg$. By (3) the result follows.

(7) The algebra homomorphism f^* intertwines $\mathcal{L}_{K_j}$ and $\mathcal{L}_{K'_j}$, so also their graded commutators which equal $\mathcal{L}([K_1, K_2])$ and $\mathcal{L}([K'_1, K'_2])$, respectively. Now use (6) . $\square$

13.16. Let $f : M \to N$ be a local diffeomorphism. Then we can consider the pullback operator $f^* : \Omega(N; TN) \to \Omega(M; TM)$, given by

$$(1) \qquad (f^*K)_x(X_1, \dots, X_k) = (T_xf)^{-1}K_{f(x)}(T_xf \cdot X_1, \dots, T_xf \cdot X_k).$$

Note that this is a special case of the pullback operator for sections of natural vector bundles in 6.15. Clearly K and f^*K are then f-related.

Theorem. *In this situation we have:*

(2) $f^*[K, L] = [f^*K, f^*L]$.
(3) $f^*\, i_K L = i_{f^*K} f^*L$.
(4) $f^*[K, L]^\wedge = [f^*K, f^*L]^\wedge$.
(5) *For a vector field $X \in \mathfrak{X}(M)$ and $K \in \Omega(M; TM)$ by 6.15 the Lie derivative $\mathcal{L}_X K = \frac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^X_t)^*K$ is defined. Then we have $\mathcal{L}_X K = [X, K]$, the Frölicher-Nijenhuis-bracket.*

We may say that the Frölicher-Nijenhuis bracket, $[\ \ ,\ \ ]^\wedge$, etc. are *natural bilinear concomitants*.

Proof. (2) – (4) are obvious from 13.15. (5) Obviously $\mathcal{L}_X$ is $\mathbb{R}$-linear, so it suffices to check this formula for $K = \psi \otimes Y$, $\psi \in \Omega(M)$ and $Y \in \mathfrak{X}(M)$. But then

$$\begin{aligned}\mathcal{L}_X(\psi \otimes Y) &= \mathcal{L}_X\psi \otimes Y + \psi \otimes \mathcal{L}_X Y \quad \text{by 6.16}\\ &= \mathcal{L}_X\psi \otimes Y + \psi \otimes [X, Y]\\ &= [X, \psi \otimes Y] \quad \text{by 13.7.6.} \quad \square\end{aligned}$$

13.17. Remark. At last we mention the best known application of the Frölicher-Nijenhuis bracket, which also led to its discovery. A vector valued 1-form $J \in \Omega^1(M; TM)$ with $J \circ J = -Id$ is called a *almost complex structure*; if it exists, $\dim M$ is even and J can be viewed as a fiber multiplication with $\sqrt{-1}$ on TM. By 13.12 we have

$$[J, J](X, Y) = 2([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]).$$

The vector valued form $\frac{1}{2}[J, J]$ is also called the *Nijenhuis tensor* of J. For it the following result is true:

A manifold M with a almost complex structure J is a complex manifold (i.e., there exists an atlas for M with holomorphic chart-change mappings) if and only if $[J, J] = 0$. See [Newlander-Nirenberg, 1957].

14. Fiber Bundles and Connections

14.1. Definition. A *(fiber) bundle* (E, p, M, S) consists of manifolds E, M, S, and a smooth mapping $p : E \to M$; furthermore each $x \in M$ has an open neighborhood U such that $E \mid U := p^{-1}(U)$ is diffeomorphic to $U \times S$ via a fiber respecting diffeomorphism:

$$\begin{tikzcd}
E \mid U \arrow[rr, "\psi"] \arrow[dr, "p"'] & & U \times S \arrow[dl, "pr_1"] \\
& U. &
\end{tikzcd}$$

E is called the *total space*, M is called the *base space* or *basis*, p is a surjective submersion, called the *projection*, and S is called *standard fiber*. (U, ψ) as above is called a *fiber chart*.

A collection of fiber charts (U_α, ψ_α), such that (U_α) is an open cover of M, is called a *"fiber bundle atlas"*. If we fix such an atlas, then $\psi_\alpha \circ {\psi_\beta}^{-1}(x, s) = (x, \psi_{\alpha\beta}(x, s))$, where $\psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times S \to S$ is smooth and $\psi_{\alpha\beta}(x, \ )$ is a diffeomorphism of S for each $x \in U_{\alpha\beta} := U_\alpha \cap U_\beta$. We may thus consider the mappings $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(S)$ with values in the group $\mathrm{Diff}(S)$ of all diffeomorphisms of S; their differentiability is a subtle question, which will not be discussed in this book, but see [Michor, 1988]. In either form these mappings $\psi_{\alpha\beta}$ are called the *transition functions* of the bundle. They satisfy the *cocycle condition*: $\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x)$ for $x \in U_{\alpha\beta\gamma}$ and $\psi_{\alpha\alpha}(x) = Id_S$ for $x \in U_\alpha$. Therefore the collection $(\psi_{\alpha\beta})$ is called a *cocycle of transition functions*.

Given an open cover (U_α) of a manifold M and a cocycle of transition functions $(\psi_{\alpha\beta})$ we may construct a fiber bundle (E, p, M, S) similarly as in 6.3.

14.2. Lemma. *Let $p : N \to M$ be a surjective submersion (a fibered manifold) which is proper, so that $p^{-1}(K)$ is compact in E for each compact $K \subset M$, and let M be connected. Then (N, p, M) is a fiber bundle.*

Proof. We have to produce a fiber chart at each $x_0 \in M$. So let (U, u) be a chart centered at x_0 on M such that $u(U) \cong \mathbb{R}^m$. For each $x \in U$ let $\xi_x(y) := (T_y u)^{-1}.u(x)$, then $\xi_x \in \mathfrak{X}(U)$, depending smoothly on $x \in U$, such that $u(\mathrm{Fl}^{\xi_x}_t u^{-1}(z)) = z + t.u(x)$, so each ξ_x is a complete vector field on U. Since p is a submersion, with the help of a partition of unity on $p^{-1}(U)$ we may construct vector fields $\eta_x \in \mathfrak{X}(p^{-1}(U))$ which depend smoothly on $x \in U$ and are p-related to ξ_x: $Tp.\eta_x = \xi_x \circ p$. Thus $p \circ \mathrm{Fl}^{\eta_x}_t = \mathrm{Fl}^{\xi_x}_t \circ p$ by 3.14, so $\mathrm{Fl}^{\eta_x}_t$ is fiber respecting, and since p is proper and ξ_x is complete, η_x has a global flow too.

Denote $p^{-1}(x_0)$ by S. Then $\varphi : U \times S \to p^{-1}(U)$, defined by $\varphi(x, y) = \mathrm{Fl}_1^{\eta_x}(y)$, is a diffeomorphism and is fiber respecting, so (U, φ^{-1}) is a fiber chart. Since M is connected, the fibers $p^{-1}(x)$ are all diffeomorphic.

14.3. Let (E, p, M, S) be a fiber bundle; we consider the fiber linear tangent mapping $Tp : TE \to TM$ and its kernel $\ker Tp =: VE$ which is called the *vertical bundle* of E. The following is special case of 13.13.

Definition. A *connection* on the fiber bundle (E, p, M, S) is a vector valued 1-form $\Phi \in \Omega^1(E; VE)$ with values in the vertical bundle VE such that $\Phi \circ \Phi = \Phi$ and $\mathrm{Im}\Phi = VE$; so Φ is just a projection $TE \to VE$.

Then $\ker \Phi$ is of constant rank, so by 6.6 $\ker \Phi$ is a sub vector bundle of TE, it is called the space of *horizontal vectors* or the *horizontal bundle* and it is denoted by HE. Clearly $TE = HE \oplus VE$ and $T_uE = H_uE \oplus V_uE$ for $u \in E$.

Now we consider the mapping $(Tp, \pi_E) : TE \to TM \times_M E$. Then by definition $(Tp, \pi_E)^{-1}(0_{p(u)}, u) = V_uE$, so $(Tp, \pi_E) \mid HE : HE \to TM \times_M E$ is fiber linear over E and injective, so by reason of dimensions it is a fiber linear isomorphism: Its inverse is denoted by

$$C := ((Tp, \pi_E) \mid HE)^{-1} : TM \times_M E \to HE \hookrightarrow TE.$$

So $C : TM \times_M E \to TE$ is fiber linear over E and is a right inverse for (Tp, π_E). C is called the *horizontal lift* associated to the connection Φ.

Note the formula $\Phi(\xi_u) = \xi_u - C(Tp.\xi_u, u)$ for $\xi_u \in T_uE$. So we can equally well describe a connection Φ by specifying C. Then we call Φ the *vertical projection* (no confusion with 6.11 will arise) and $\chi := \mathrm{id}_{TE} - \Phi = C \circ (Tp, \pi_E)$ will be called the *horizontal projection*.

14.4. Curvature. If $\Phi : TE \to VE$ is a connection on the bundle (E, p, M, S), then as in 13.13 the curvature R of Φ is given by

$$2R = [\Phi, \Phi] = [Id - \Phi, Id - \Phi] = [\chi, \chi] \in \Omega^2(E; VE)$$

(The cocurvature $\bar{R}$ vanishes since the vertical bundle VE is integrable). We have $R(X, Y) = \frac{1}{2}[\Phi, \Phi](X, Y) = \Phi[\chi X, \chi Y]$, so R is an obstruction against integrability of the horizontal subbundle. Note that for vector fields $\xi, \eta \in \mathfrak{X}(M)$ and their horizontal lifts $C\xi, C\eta \in \mathfrak{X}(E)$ we have $R(C\xi, C\eta) = [C\xi, C\eta] - C([\xi, \eta])$. Since the vertical bundle VE is integrable, by 13.14 we have the *Bianchi identity* $[\Phi, R] = 0$.

14.5. Pullback. Let (E, p, M, S) be a fiber bundle and consider a smooth mapping $f : N \to M$. Since p is a submersion, f and p are transversal in the sense of 2.18 and thus the pullback $N \times_{(f,M,p)} E$ exists. It will be called the *pullback* of the fiber bundle E by f and we will denote it by f^*E. The following diagram sets up some further notation for it:

$$\begin{tikzcd} f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\ N \arrow[r, "f"] & M. \end{tikzcd}$$

Proposition. *In the situation above we have:*

(1) *(f^*E, f^*p, N, S) is again a fiber bundle, and p^*f is a fiber wise diffeomorphism.*
(2) *If $\Phi \in \Omega^1(E; TE)$ is a connection on the bundle E, then the vector valued form $f^*\Phi$, given by $(f^*\Phi)_u(X) := T_u(p^*f)^{-1}.\Phi.T_u(p^*f).X$ for $X \in T_uE$, is a connection on the bundle f^*E. The forms $f^*\Phi$ and Φ are p^*f-related in the sense of 13.15.*
(3) *The curvatures of $f^*\Phi$ and Φ are also p^*f-related.*

Proof. (1). If (U_α, ψ_α) is a fiber bundle atlas of (E, p, M, S) in the sense of 14.1, then $(f^{-1}(U_\alpha), (f^*p, pr_2 \circ \psi_\alpha \circ p^*f))$ is visibly a fiber bundle atlas for (f^*E, f^*p, N, S), by the formal universal properties of a pullback 2.19. (2) is obvious. (3) follows from (2) and 13.15.7. $\square$

14.6. Let us suppose that a connection Φ on the bundle (E, p, M, S) has zero curvature. Then by 14.4 the horizontal bundle is integrable and gives rise to the *horizontal foliation* by 3.25.2. Each point $u \in E$ lies on a unique leaf $L(u)$ such that $T_vL(u) = H_vE$ for each $v \in L(u)$. The restriction $p \mid L(u)$ is locally a diffeomorphism, but in general it is neither surjective nor is it a covering onto its image. This is seen by devising suitable horizontal foliations on the trivial bundle $pr_2 : \mathbb{R} \times S^1 \to S^1$.

14.7. Local description. Let Φ be a connection on (E, p, M, S). Let us fix a fiber bundle atlas (U_α) with transition functions $(\psi_{\alpha\beta})$, and let us consider the connection $((\psi_\alpha)^{-1})^*\Phi \in \Omega^1(U_\alpha \times S; U_\alpha \times TS)$, which may be written in the form

$$((\psi_\alpha)^{-1})^*\Phi)(\xi_x, \eta_y) =: -\Gamma^\alpha(\xi_x, y) + \eta_y \text{ for } \xi_x \in T_xU_\alpha \text{ and } \eta_y \in T_yS,$$

since it reproduces vertical vectors. The Γ^α are given by

$$(0_x, \Gamma^\alpha(\xi_x, y)) := -T(\psi_\alpha).\Phi.T(\psi_\alpha)^{-1}.(\xi_x, 0_y).$$

We consider Γ^α as an element of the space $\Omega^1(U_\alpha;\mathfrak{X}(S))$, a 1-form on U^α with values in the infinite dimensional Lie algebra $\mathfrak{X}(S)$ of all vector fields on the standard fiber. The Γ^α are called the *Christoffel forms* of the connection Φ with respect to the bundle atlas (U_α, ψ_α).

Lemma. *The transformation law for the Christoffel forms is*

$$T_y(\psi_{\alpha\beta}(x,\quad)).\Gamma^\beta(\xi_x, y) = \Gamma^\alpha(\xi_x, \psi_{\alpha\beta}(x,y)) - T_x(\psi_{\alpha\beta}(\quad, y)).\xi_x.$$

The curvature R of Φ satisfies

$$(\psi_\alpha^{-1})^* R = d\Gamma^\alpha + [\Gamma^\alpha, \Gamma^\alpha]_{\mathfrak{X}(S)}.$$

Here $d\Gamma^\alpha$ is the exterior derivative of the 1-form $\Gamma^\alpha \in \Omega^1(U_\alpha;\mathfrak{X}(S))$ with values in the complete locally convex space $\mathfrak{X}(S)$. We will later also use the Lie derivative of it and the usual formulas apply: consult [Frölicher, Kriegl, 1988] for calculus in infinite dimensional spaces.

The formula for the curvature is the *Maurer-Cartan* formula which in this general setting appears only in the level of local description.

Proof. From $(\psi_\alpha \circ (\psi_\beta)^{-1})(x,y) = (x, \psi_{\alpha\beta}(x,y))$ we get that $T(\psi_\alpha \circ (\psi_\beta)^{-1}).(\xi_x, \eta_y) = (\xi_x, T_{(x,y)}(\psi_{\alpha\beta}).(\xi_x,\eta_y))$ and thus:

$$\begin{aligned}
&T(\psi_\beta^{-1}).(0_x, \Gamma^\beta(\xi_x, y)) = -\Phi(T(\psi_\beta^{-1})(\xi_x, 0_y)) = \\
&= -\Phi(T(\psi_\alpha^{-1}).T(\psi_\alpha \circ \psi_\beta^{-1}).(\xi_x, 0_y)) = \\
&= -\Phi(T(\psi_\alpha^{-1})(\xi_x, T_{(x,y)}(\psi_{\alpha\beta})(\xi_x, 0_y))) = \\
&= -\Phi(T(\psi_\alpha^{-1})(\xi_x, 0_{\psi_{\alpha\beta}(x,y)})) - \Phi(T(\psi_\alpha^{-1})(0_x, T_{(x,y)}\psi_{\alpha\beta}(\xi_x, 0_y)) = \\
&= T(\psi_\alpha^{-1}).(0_x, \Gamma^\alpha(\xi_x, \psi_{\alpha\beta}(x,y))) - T(\psi_\alpha^{-1})(0_x, T_x(\psi_{\alpha\beta}(\quad, y)).\xi_x).
\end{aligned}$$

This implies the transformation law.

For the curvature R of Φ we have by 14.4 and 14.5.3

$$\begin{aligned}
&(\psi_\alpha^{-1})^* R\,((\xi^1, \eta^1), (\xi^2, \eta^2)) = \\
&\quad = (\psi_\alpha^{-1})^*\Phi\,[(Id - (\psi_\alpha^{-1})^*\Phi)(\xi^1, \eta^1), (Id - (\psi_\alpha^{-1})^*\Phi)(\xi^2, \eta^2)] = \\
&\quad = (\psi_\alpha^{-1})^*\Phi[(\xi^1, \Gamma^\alpha(\xi^1)), (\xi^2, \Gamma^\alpha(\xi^2))] = \\
&\quad = (\psi_\alpha^{-1})^*\Phi\left([\xi^1, \xi^2], \xi^1\Gamma^\alpha(\xi^2) - \xi^2\Gamma^\alpha(\xi^1) + [\Gamma^\alpha(\xi^1), \Gamma^\alpha(\xi^2)]\right) = \\
&\quad = -\Gamma^\alpha([\xi^1, \xi^2]) + \xi^1\Gamma^\alpha(\xi^2) - \xi^2\Gamma^\alpha(\xi^1) + [\Gamma^\alpha(\xi^1), \Gamma^\alpha(\xi^2)] = \\
&\quad = d\Gamma^\alpha(\xi^1, \xi^2) + [\Gamma^\alpha(\xi^1), \Gamma^\alpha(\xi^2)]_{\mathfrak{X}(S)}. \quad \square
\end{aligned}$$

14.8. Theorem (Parallel transport). *Let Φ be a connection on a bundle (E, p, M, S) and let $c : (a, b) \to M$ be a smooth curve with $0 \in (a, b)$, $c(0) = x$.*

Then there is a neighborhood U of $E_x \times \{0\}$ in $E_x \times (a, b)$ and a smooth mapping $\operatorname{Pt}_c : U \to E$ such that:

(1) *$p(\operatorname{Pt}(c, u_x, t)) = c(t)$ if defined, and $\operatorname{Pt}(c, u_x, 0) = u_x$.*
(2) *$\Phi(\frac{d}{dt} \operatorname{Pt}(c, u_x, t)) = 0$ if defined.*
(3) *Reparametrisation invariance: If $f : (a', b') \to (a, b)$ is smooth with $0 \in (a', b')$, then $\operatorname{Pt}(c, u_x, f(t)) = \operatorname{Pt}(c \circ f, \operatorname{Pt}(c, u_x, f(0)), t)$ if defined.*
(4) *U is maximal for properties (1) and (2).*
(5) *In a certain sense Pt depends smoothly also on c.*

First proof. In local bundle coordinates $\Phi(\frac{d}{dt} \operatorname{Pt}(c, u_x, t)) = 0$ is an ordinary differential equation of first order, nonlinear, with initial condition $\operatorname{Pt}(c, u_x, 0) = u_x$. So there is a maximally defined local solution curve which is unique. All further properties are consequences of uniqueness.

Second proof. Consider the pullback bundle $(c^*E, c^*p, (a, b), S)$ and the pullback connection $c^*\Phi$ on it. It has zero curvature, since the horizontal bundle is 1-dimensional. By 14.6 the horizontal foliation exists and the parallel transport just follows a leaf and we may map it back to E, in detail: $\operatorname{Pt}(c, u_x, t) = p^*c((c^*p \mid L(u_x))^{-1}(t))$.

Third proof. Consider a fiber bundle atlas (U_α, ψ_α) as in 14.7. Then we have $\psi_\alpha(\operatorname{Pt}(c, \psi_\alpha^{-1}(x, y), t)) = (c(t), \gamma(y, t))$, where

$$0 = \left((\psi_\alpha^{-1})^*\Phi\right)\left(\tfrac{d}{dt}c(t), \tfrac{d}{dt}\gamma(y, t)\right) = -\Gamma^\alpha\left(\tfrac{d}{dt}c(t), \gamma(y, t)\right) + \tfrac{d}{dt}\gamma(y, t),$$

so $\gamma(y, t)$ is the integral curve (evolution line) through $y \in S$ of the time dependent vector field $\Gamma^\alpha\left(\frac{d}{dt}c(t)\right)$ on S. This vector field visibly depends smoothly on c. Clearly local solutions exist and all properties follow. For (5) we refer to [Michor, 1983]. $\square$

14.9. A connection Φ on (E, p, M, S) is called a *complete connection*, if the parallel transport Pt_c along any smooth curve $c : (a, b) \to M$ is defined on the whole of $E_{c(0)} \times (a, b)$. The third proof of theorem 14.8 shows that on a fiber bundle with compact standard fiber any connection is complete.

The following is a sufficient condition for a connection Φ to be complete:

> There exists a fiber bundle atlas (U_α, ψ_α) and complete Riemannian metrics g_α on the standard fiber S such that each Christoffel form $\Gamma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(S))$ takes values in the linear subspace of g_α-bounded vector fields on S

For in the third proof of theorem 14.8 above the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t))$ on S is g_α-bounded for compact time intervals. So by continuation the solution exists over $c^{-1}(U_\alpha)$, and thus globally.

A complete connection is called an *Ehresmann connection* in [Greub - Halperin - Vanstone I, p 314], where it is also indicated how to prove the following result.

Theorem. *Each fiber bundle admits complete connections.*

Proof. Let $\dim M = m$. Let (U_α, ψ_α) be a fiber bundle atlas as in 14.1. By topological dimension theory [Nagata, 1965] the open cover (U_α) of M admits a refinement such that any $m+2$ members have empty intersection, see also 1.1. Let (U_α) itself have this property. Choose a smooth partition of unity (f_α) subordinated to (U_α). Then the sets $V_\alpha := \{\, x : f_\alpha(x) > \frac{1}{m+2} \,\} \subset U_\alpha$ form still an open cover of M since $\sum f_\alpha(x) = 1$ and at most $m+1$ of the $f_\alpha(x)$ can be nonzero. By renaming assume that each V_α is connected. Then we choose an open cover (W_α) of M such that $\overline{W_\alpha} \subset V_\alpha$.

Now let g_1 and g_2 be complete Riemannian metrics on M and S, respectively (see [Nomizu - Ozeki, 1961] or [Morrow, 1970]). For not connected Riemannian manifolds complete means that each connected component is complete. Then $g_1|U_\alpha \times g_2$ is a Riemannian metric on $U_\alpha \times S$ and we consider the metric $g := \sum f_\alpha \psi_\alpha^*(g_1|U_\alpha \times g_2)$ on E. Obviously $p : E \to M$ is a Riemannian submersion for the metrics g and g_1. We choose now the connection $\Phi : TE \to VE$ as the orthonormal projection with respect to the Riemannian metric g.

Claim. Φ is a complete connection on E.

Let $c : [0,1] \to M$ be a smooth curve. We choose a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that $c([t_i, t_{i+1}]) \subset V_{\alpha_i}$ for suitable α_i. It suffices to show that $\operatorname{Pt}(c(t_i + \quad), u_{c(t_i)}, t)$ exists for all $0 \le t \le t_{i+1} - t_i$ and all $u_{c(t_i)}$, for all i — then we may piece them together. So we may assume that $c : [0,1] \to V_\alpha$ for some α. Let us now assume that for some $(x,y) \in V_\alpha \times S$ the parallel transport $\operatorname{Pt}(c, \psi_\alpha(x,y), t)$ is defined only for $t \in [0, t')$ for some $0 < t' < 1$. By the third proof of 14.8 we have $\operatorname{Pt}(c, \psi_\alpha(x,y), t) = \psi_\alpha^{-1}(c(t), \gamma(t))$, where $\gamma : [0, t') \to S$ is the maximally defined integral curve through $y \in S$ of the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t), \quad)$ on S. We put $g_\alpha := (\psi_\alpha^{-1})^* g$, then $(g_\alpha)_{(x,y)} = (g_1)_x \times (\sum_\beta f_\beta(x) \psi_{\beta\alpha}(x, \quad)^* g_2)_y$. Since $pr_1 : (V_\alpha \times S, g_\alpha) \to (V_\alpha, g_1|V_\alpha)$ is a Riemannian submersion and since the connection $(\psi_\alpha^{-1})^*\Phi$ is also given by orthonormal projection onto the vertical bundle, we get

$$\infty > g_1\text{-length}_0^{t'}(c) = g_\alpha\text{-length}(c, \gamma) = \int_0^{t'} |(c'(t), \tfrac{d}{dt}\gamma(t))|_{g_\alpha}\, dt =$$

$$= \int_0^{t'} \sqrt{|c'(t)|_{g_1}^2 + \textstyle\sum_\beta f_\beta(c(t))(\psi_{\alpha\beta}(c(t), -)^* g_2)(\tfrac{d}{dt}\gamma(t), \tfrac{d}{dt}\gamma(t))}\, dt \ge$$

$$\geq \int_0^{t'} \sqrt{f_\alpha(c(t))}\,|\tfrac{d}{dt}\gamma(t)|_{g_2}\,dt \geq \frac{1}{\sqrt{m+2}}\int_0^{t'} |\tfrac{d}{dt}\gamma(t)|_{g_2} dt.$$

So g_2-lenght(γ) is finite and since the Riemannian metric g_2 on S is complete, $\lim_{t\to t'}\gamma(t) =: \gamma(t')$ exists in S and the integral curve γ can be continued. $\square$

14.10. Holonomy groups and Lie algebras. Let (E,p,M,S) be a fiber bundle with a complete connection Φ, and let us assume that M is connected. We choose a fixed base point $x_0 \in M$ and we identify E_{x_0} with the standard fiber S. For each closed piecewise smooth curve $c : [0,1] \to M$ through x_0 the parallel transport $\operatorname{Pt}(c, \quad, 1) =: \operatorname{Pt}(c,1)$ (pieced together over the smooth parts of c) is a diffeomorphism of S. All these diffeomorphisms form together the group $\operatorname{Hol}(\Phi, x_0)$, the *holonomy group* of Φ at x_0, a subgroup of the diffeomorphism group $\operatorname{Diff}(S)$. If we consider only those piecewise smooth curves which are homotopic to zero, we get a subgroup $\operatorname{Hol}_0(\Phi, x_0)$, called the *restricted holonomy group* of the connection Φ at x_0.

Now let $C : TM \times_M E \to TE$ be the horizontal lifting as in 14.3, and let R be the curvature (14.4) of the connection Φ. For any $x \in M$ and $X_x \in T_xM$ the horizontal lift $C(X_x) := C(X_x, \quad) : E_x \to TE$ is a vector field along E_x. For X_x and $Y_x \in T_xM$ we consider $R(CX_x, CY_x) \in \mathfrak{X}(E_x)$. Now we choose any piecewise smooth curve c from x_0 to x and consider the diffeomorphism $\operatorname{Pt}(c,t) : S = E_{x_0} \to E_x$ and the pullback $\operatorname{Pt}(c,1)^*R(CX_x, CY_x) \in \mathfrak{X}(S)$. Let us denote by $\operatorname{hol}(\Phi, x_0)$ the closed linear subspace, generated by all these vector fields (for all $x \in M$, X_x, $Y_x \in T_xM$ and curves c from x_0 to x) in $\mathfrak{X}(S)$ with respect to the compact C^∞-topology, and let us call it the *holonomy Lie algebra* of Φ at x_0.

Lemma. $\operatorname{hol}(\Phi, x_0)$ *is a Lie subalgebra of* $\mathfrak{X}(S)$.

Proof. For $X \in \mathfrak{X}(M)$ we consider the local flow Fl^{CX}_t of the horizontal lift of X. It restricts to parallel transport along any of the flow lines of X in M. Then for vector fields on M the expression

$$\begin{aligned}
&\tfrac{d}{dt}|_0 (\operatorname{Fl}^{CX}_s)^*(\operatorname{Fl}^{CY}_t)^*(\operatorname{Fl}^{CX}_{-s})^*(\operatorname{Fl}^{CZ}_z)^* R(CU,CV) \restriction E_{x_0}\\
&\qquad = (\operatorname{Fl}^{CX}_s)^*[CY, (\operatorname{Fl}^{CX}_{-s})^*(\operatorname{Fl}^{CZ}_z)^* R(CU,CV)] \restriction E_{x_0}\\
&\qquad = [(\operatorname{Fl}^{CX}_s)^* CY, (\operatorname{Fl}^{CZ}_z)^* R(CU,CV)] \restriction E_{x_0}
\end{aligned}$$

is in $\operatorname{hol}(\Phi, x_0)$, since it is closed in the compact C^∞-topology and the derivative can be written as a limit. Thus

$$[(\operatorname{Fl}^{CX}_s)^*[CY_1, CY_2], (\operatorname{Fl}^{CZ}_z)^* R(CU,CV)] \restriction E_{x_0} \in \operatorname{hol}(\Phi, x_0)$$

by the Jacobi identity and

$$[(\mathrm{Fl}_s^{CX})^* C[Y_1, Y_2], (\mathrm{Fl}_z^{CZ})^* R(CU, CV)] \restriction E_{x_0} \in \mathrm{hol}(\Phi, x_0),$$

so also their difference

$$[(\mathrm{Fl}_s^{CX})^* R(CY_1, CY_2), (\mathrm{Fl}_z^{CZ})^* R(CU, CV)] \restriction E_{x_0}$$

is in $\mathrm{hol}(\Phi, x_0)$. $\square$

14.11. The following theorem is a generalization of the theorem of Ambrose and Singer on principal connections. The reader who does not know principal connections is advised to read parts of sections 15 and 16 first. We include this result here in order not to disturb the development in section 16 later.

Theorem. *Let Φ be a complete connection on the fibre bundle (E, p, M, S) and let M be connected. Suppose that for some (hence any) $x_0 \in M$ the holonomy Lie algebra $\mathrm{hol}(\Phi, x_0)$ is finite dimensional and consists of complete vector fields on the fiber E_{x_0}*

Then there is a principal bundle (P, p, M, G) with finite dimensional structure group G, an irreducible connection ω on it and a smooth action of G on S such that the Lie algebra $\mathfrak{g}$ of G equals the holonomy Lie algebra $\mathrm{hol}(\Phi, x_0)$, the fibre bundle E is isomorphic to the associated bundle $P[S]$, and Φ is the connection induced by ω. The structure group G equals the holonomy group $\mathrm{Hol}(\Phi, x_0)$. P and ω are unique up to isomorphism.

By a theorem of [Palais, 1957] a finite dimensional Lie subalgebra of $\mathfrak{X}(E_{x_0})$ like $\mathrm{hol}(\Phi, x_0)$ consists of complete vector fields if and only if it is generated by complete vector fields as a Lie algebra.

Proof. Let us again identify E_{x_0} and S. Then $\mathfrak{g} := \mathrm{hol}(\Phi, x_0)$ is a finite dimensional Lie subalgebra of $\mathfrak{X}(S)$, and since each vector field in it is complete, there is a finite dimensional connected Lie group G_0 of diffeomorphisms of S with Lie algebra $\mathfrak{g}$, see [Palais, 1957].

Claim 1. G_0 contains $\mathrm{Hol}_0(\Phi, x_0)$, the restricted holonomy group.

Let $f \in \mathrm{Hol}_0(\Phi, x_0)$, then $f = \mathrm{Pt}(c, 1)$ for a piecewise smooth closed curve c through x_0, which is nullhomotopic. Since the parallel transport is essentially invariant under reparametrisation, 14.8, we can replace c by $c \circ g$, where g is smooth and flat at each corner of c. So we may assume that c itself is smooth. Since c is homotopic to zero, by approximation we may assume that there is a smooth homotopy $H : \mathbb{R}^2 \to M$ with $H_1|[0, 1] = c$ and $H_0|[0, 1] = x_0$. Then $f_t := \mathrm{Pt}(H_t, 1)$ is a curve in $\mathrm{Hol}_0(\Phi, x_0)$ which is smooth as a mapping $\mathbb{R} \times S \to S$;

this can be seen by using the proof of claim 2 below or as in the proof of 16.7.3. We will continue the proof of claim 1 below.
Claim 2. $(\frac{d}{dt} f_t) \circ f_t^{-1} =: Z_t$ is in $\mathfrak{g}$ for all t.
To prove claim 2 we consider the pullback bundle $H^*E \to \mathbb{R}^2$ with the induced connection $H^*\Phi$. It is sufficient to prove claim 2 there. Let $X = \frac{d}{ds}$ and $Y = \frac{d}{dt}$ be the constant vector fields on $\mathbb{R}^2$, so $[X, Y] = 0$. Then $\operatorname{Pt}(c, s) = \operatorname{Fl}_s^{CX} |S$ and so on. We put

$$f_{t,s} = \operatorname{Fl}_{-s}^{CX} \circ \operatorname{Fl}_{-t}^{CY} \circ \operatorname{Fl}_s^{CX} \circ \operatorname{Fl}_t^{CY} : S \to S,$$

so $f_{t,1} = f_t$. Then we have in the vector space $\mathfrak{X}(S)$

$$\begin{aligned}
(\tfrac{d}{dt} f_{t,s}) \circ f_{t,s}^{-1} &= -(\operatorname{Fl}_s^{CX})^* CY + (\operatorname{Fl}_s^{CX})^* (\operatorname{Fl}_t^{CY})^* (\operatorname{Fl}_{-s}^{CX})^* CY, \\
(\tfrac{d}{dt} f_{t,1}) \circ f_{t,1}^{-1} &= \int_0^1 \tfrac{d}{ds} \left((\tfrac{d}{dt} f_{t,s}) \circ f_{t,s}^{-1} \right) ds \\
&= \int_0^1 \Big(-(\operatorname{Fl}_s^{CX})^* [CX, CY] + (\operatorname{Fl}_s^{CX})^* [CX, (\operatorname{Fl}_t^{CY})^* (\operatorname{Fl}_{-s}^{CX})^* CY] \\
&\qquad - (\operatorname{Fl}_s^{CX})^* (\operatorname{Fl}_t^{CY})^* (\operatorname{Fl}_{-s}^{CX})^* [CX, CY] \Big)\; ds.
\end{aligned}$$

Since $[X, Y] = 0$ we have $[CX, CY] = \Phi[CX, CY] = R(CX, CY)$ and

$$\begin{aligned}
(\operatorname{Fl}_t^{CX})^* CY &= C\left((\operatorname{Fl}_t^X)^* Y \right) + \Phi\left((\operatorname{Fl}_t^{CX})^* CY \right) \\
&= CY + \int_0^t \tfrac{d}{dt} \Phi (\operatorname{Fl}_t^{CX})^* CY \; dt \\
&= CY + \int_0^t \Phi (\operatorname{Fl}_t^{CX})^* [CX, CY] \; dt \\
&= CY + \int_0^t \Phi (\operatorname{Fl}_t^{CX})^* R(CX, CY) \; dt \\
&= CY + \int_0^t (\operatorname{Fl}_t^{CX})^* R(CX, CY) \; dt.
\end{aligned}$$

The flows $(\operatorname{Fl}^C X_s)^*$ and its derivative at 0 $\mathcal{L}_{CX} = [CX, \quad]$ do not lead out of $\mathfrak{g}$, thus all parts of the integrand above are in $\mathfrak{g}$ and so $(\frac{d}{dt} f_{t,1}) \circ f_{t,1}^{-1}$ is in $\mathfrak{g}$ for all t and claim 2 follows.

Now claim 1 can be shown as follows. There is a unique smooth curve $g(t)$ in G_0 satisfying $T_e(\rho_{g(t)}) Z_t = Z_t . g(t) = \frac{d}{dt} g(t)$ and $g(0) = e$; via the action of

G_0 on S the curve $g(t)$ is a curve of diffeomorphisms on S, generated by the time dependent vector field Z_t, so $g(t) = f_t$ and $f = f_1$ is in G_0. So we get $\operatorname{Hol}_0(\Phi, x_0) \subseteq G_0$.

Claim 3. $\operatorname{Hol}_0(\Phi, x_0)$ equals G_0.

In the proof of claim 1 we have seen that $\operatorname{Hol}_0(\Phi, x_0)$ is a smoothly arcwise connected subgroup of G_0, so it is a connected Lie subgroup by the results cited in 5.6. It suffices thus to show that the Lie algebra $\mathfrak{g}$ of G_0 is contained in the Lie algebra of $\operatorname{Hol}_0(\Phi, x_0)$, and for that it is enough to show, that for each ξ in a linearly spanning subset of $\mathfrak{g}$ there is a smooth mapping $f : [-1, 1] \times S \to S$ such that the associated curve $\check{f}$ lies in $\operatorname{Hol}_0(\Phi, x_0)$ with $\check{f}'(0) = 0$ and $\check{f}''(0) = \xi$.

By definition we may assume $\xi = \operatorname{Pt}(c, 1)^* R(CX_x, CY_x)$ for X_x, $Y_x \in T_xM$ and a smooth curve c in M from x_0 to x. We extend X_x and Y_x to vector fields X and $Y \in \mathfrak{X}(M)$ with $[X, Y] = 0$ near x. We may also suppose that $Z \in \mathfrak{X}(M)$ is a vector field which extends $c'(t)$ along $c(t)$: if c is simple we approximate it by an embedding and can consequently extend $c'(t)$ to such a vector field. If c is not simple we do this for each simple piece of c and have then several vector fields Z instead of one below. So we have

$$\begin{aligned}
\xi &= (\operatorname{Fl}^{CZ}_1)^* R(CX, CY) = (\operatorname{Fl}^{CZ}_1)^*[CX, CY] \quad \text{since } [X, Y](x) = 0\\
&= (\operatorname{Fl}^{CZ}_1)^* \tfrac{1}{2} \tfrac{d^2}{dt^2}|_{t=0} (\operatorname{Fl}^{CY}_{-t} \circ \operatorname{Fl}^{CX}_{-t} \circ \operatorname{Fl}^{CY}_t \circ \operatorname{Fl}^{CX}_t) \quad \text{by 3.16}\\
&= \tfrac{1}{2} \tfrac{d^2}{dt^2}|_{t=0} (\operatorname{Fl}^{CZ}_{-1} \circ \operatorname{Fl}^{CY}_{-t} \circ \operatorname{Fl}^{CX}_{-t} \circ \operatorname{Fl}^{CY}_t \circ \operatorname{Fl}^{CX}_t \circ \operatorname{Fl}^{CZ}_1),
\end{aligned}$$

where the parallel transport in the last equation first follows c from x_0 to x, then follows a small closed parallelogram near x in M (since $[X, Y] = 0$ near x) and then follows c back to x_0. This curve is clearly nullhomotopic.

Step 4. Now we make $\operatorname{Hol}(\Phi, x_0)$ into a Lie group which we call G, by taking $\operatorname{Hol}_0(\Phi, x_0) = G_0$ as its connected component of the identity. Then the quotient $\operatorname{Hol}(\Phi, x_0)/ \operatorname{Hol}_0(\Phi, x_0)$ is a countable group, since the fundamental group $\pi_1(M)$ is countable (by Morse theory M is homotopy equivalent to a countable CW-complex).

Step 5. Construction of a cocycle of transition functions with values in G. Let $(U_\alpha, u_\alpha : U_\alpha \to \mathbb{R}^m)$ be a locally finite smooth atlas for M such that each $u_\alpha : U_\alpha \to \mathbb{R}^m)$ is surjective. Put $x_\alpha := u_\alpha^{-1}(0)$ and choose smooth curves $c_\alpha : [0, 1] \to M$ with $c_\alpha(0) = x_0$ and $c_\alpha(1) = x_\alpha$. For each $x \in U_\alpha$ let $c^x_\alpha : [0, 1] \to M$ be the smooth curve $t \mapsto u_\alpha^{-1}(t.u_\alpha(x))$, then c^x_α connects x_α and x and the mapping $(x, t) \mapsto c^x_\alpha(t)$ is smooth $U_\alpha \times [0, 1] \to M$. Now we define a fibre bundle atlas $(U_\alpha, \psi_\alpha : E|U_\alpha \to U_\alpha \times S)$ by $\psi_\alpha^{-1}(x, s) = \operatorname{Pt}(c^x_\alpha, 1) \operatorname{Pt}(c_\alpha, 1)\, s$. Then ψ_α is smooth since $\operatorname{Pt}(c^x_\alpha, 1) = \operatorname{Fl}^{CX_x}_1$ for a local vector field X_x depending smoothly on x. Let us investigate the transition functions.

$$\begin{aligned}\psi_\alpha\psi_\beta^{-1}(x,s) &= \big(x, \operatorname{Pt}(c_\alpha,1)^{-1}\operatorname{Pt}(c_\alpha^x,1)^{-1}\operatorname{Pt}(c_\beta^x,1)\operatorname{Pt}(c_\beta,1)\,s\big)\\ &= \big(x, \operatorname{Pt}(c_\beta.c_\beta^x.(c_\alpha^x)^{-1}.(c_\alpha)^{-1},4)\,s\big)\\ &=: (x,\psi_{\alpha\beta}(x)\,s),\ \text{ where } \psi_{\alpha\beta}:U_{\alpha\beta}\to G.\end{aligned}$$

Clearly $\psi_{\beta\alpha}: U_{\beta\alpha}\times S\to S$ is smooth which implies that $\psi_{\beta\alpha}:U_{\beta\alpha}\to G$ is also smooth. $(\psi_{\alpha\beta})$ is a cocycle of transition functions and we use it to glue a principal bundle with structure group G over M which we call (P,p,M,G). From its construction it is clear that the associated bundle $P[S]=P\times_G S$ equals (E,p,M,S).

Step 6. Lifting the connection Φ to P.

For this we have to compute the Christoffel symbols of Φ with respect to the atlas of step 5. To do this directly is quite difficult since we have to differentiate the parallel transport with respect to the curve. Fortunately there is another way. Let $c:[0,1]\to U_\alpha$ be a smooth curve. Then we have

$$\begin{aligned}&\psi_\alpha(\operatorname{Pt}(c,t)\psi_\alpha^{-1}(c(0),s)) =\\ &\quad= \Big(c(t), \operatorname{Pt}((c_\alpha)^{-1},1)\operatorname{Pt}((c_\alpha^{c(0)})^{-1},1)\operatorname{Pt}(c,t)\operatorname{Pt}(c_\alpha^{c(0)},1)\operatorname{Pt}(c_\alpha,1)s\Big)\\ &\quad= (c(t),\gamma(t).s),\end{aligned}$$

where $\gamma(t)$ is a smooth curve in the holonomy group G. Let $\Gamma^\alpha\in\Omega^1(U_\alpha,\mathfrak{X}(S))$ be the Christoffel symbol of the connection Φ with respect to the chart (U_α,ψ_α). From the third proof of theorem 14.8 we have

$$\psi_\alpha(\operatorname{Pt}(c,t)\psi_\alpha^{-1}(c(0),s)) = (c(t),\bar\gamma(t,s)),$$

where $\bar\gamma(t,s)$ is the integral curve through s of the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t))$ on S. But then we get

$$\begin{aligned}\Gamma^\alpha(\tfrac{d}{dt}c(t))(\bar\gamma(t,s)) &= \tfrac{d}{dt}\bar\gamma(t,s) = \tfrac{d}{dt}(\gamma(t).s) = (\tfrac{d}{dt}\gamma(t)).s,\\ \Gamma^\alpha(\tfrac{d}{dt}c(t)) &= (\tfrac{d}{dt}\gamma(t))\circ\gamma(t)^{-1}\in\mathfrak{g}.\end{aligned}$$

So Γ^α takes values in the Lie sub algebra of fundamental vector fields for the action of G on S. By theorem 11.9 below the connection Φ is thus induced by a principal connection ω on P. Since by 11.8 the principal connection ω has the 'same' holonomy group as Φ and since this is also the structure group of P, the principal connection ω is irreducible, see 11.7. $\square$

15. Principal Fiber Bundles and G-Bundles

15.1. Definition. Let G be a Lie group and let (E,p,M,S) be a fiber bundle as in 14.1. A *G-bundle structure* on the fiber bundle consists of the following data:

(1) A left action $\ell : G\times S\to S$ of the Lie group on the standard fiber.
(2) A fiber bundle atlas (U_α,ψ_α) whose transition functions $(\psi_{\alpha\beta})$ act on S via the G-action: There is a family of smooth mappings $(\varphi_{\alpha\beta}:U_{\alpha\beta}\to G)$ which satisfies the cocycle condition $\varphi_{\alpha\beta}(x)\varphi_{\beta\gamma}(x)=\varphi_{\alpha\gamma}(x)$ for $x\in U_{\alpha\beta\gamma}$ and $\varphi_{\alpha\alpha}(x)=e$, the unit in the group, such that $\psi_{\alpha\beta}(x,s)=\ell(\varphi_{\alpha\beta}(x),s)=\varphi_{\alpha\beta}(x).s$.

A fiber bundle with a G-bundle structure is called a *G-bundle*. A fiber bundle atlas as in (2) is called a *G-atlas* and the family $(\varphi_{\alpha\beta})$ is also called a cocycle of transition functions, but now for the G-bundle.

To be more precise, two G-atlases are said to be equivalent (to describe the same G-bundle), if their union is also a G-atlas. This translates as follows to the two cocycles of transition functions, where we assume that the two coverings of M are the same (by passing to the common refinement, if necessary): $(\varphi_{\alpha\beta})$ and $(\varphi'_{\alpha\beta})$ are called *cohomologous* if there is a family $(\tau_\alpha : U_\alpha\to G)$ such that $\varphi_{\alpha\beta}(x)=\tau_\alpha(x)^{-1}.\varphi'_{\alpha\beta}(x).\tau_\beta(x)$ holds for all $x\in U_{\alpha\beta}$, compare with 6.3.

In (2) one should specify only an equivalence class of G-bundle structures or only a cohomology class of cocycles of G-valued transition functions. The proof of 6.3 now shows that from any open cover (U_α) of M, some cocycle of transition functions $(\varphi_{\alpha\beta}:U_{\alpha\beta}\to G)$ for it, and a left G-action on a manifold S, we may construct a G-bundle, which depends only on the cohomology class of the cocycle. By some abuse of notation we write (E,p,M,S,G) for a fiber bundle with specified G-bundle structure.

Examples. The tangent bundle of a manifold M is a fiber bundle with structure group $GL(m)$. More general a vector bundle (E,p,M,V) as in 6.1 is a fiber bundle with standard fiber the vector space V and with $GL(V)$-structure.

15.2. Definition. A *principal (fiber) bundle* (P,p,M,G) is a G-bundle with typical fiber a Lie group G, where the left action of G on G is just the left translation.

So by 15.1 we are given a bundle atlas $(U_\alpha,\varphi_\alpha : P|U_\alpha\to U_\alpha\times G)$ such that we have $\varphi_\alpha\varphi_\beta^{-1}(x,a)=(x,\varphi_{\alpha\beta}(x).a)$ for the cocycle of transition functions $(\varphi_{\alpha\beta}:U_{\alpha\beta}\to G)$. This is now called a *principal bundle atlas*. Clearly the principal bundle is uniquely specified by the cohomology class of its cocycle of transition functions.

Each principal bundle admits a unique right action $r : P \times G \to P$, called the *principal right action*, given by $\varphi_\alpha(r(\varphi_\alpha^{-1}(x,a),g)) = (x, ag)$. Since left and right translation on G commute, this is well defined. As in 5.10 we write $r(u,g) = u.g$ when the meaning is clear. The principal right action is visibly free and for any $u_x \in P_x$ the partial mapping $r_{u_x} = r(u_x, \quad) : G \to P_x$ is a diffeomorphism onto the fiber through u_x, whose inverse is denoted by $\tau_{u_x} : P_x \to G$. These inverses together give a smooth mapping $\tau : P \times_M P \to G$, whose local expression is $\tau(\varphi_\alpha^{-1}(x,a), \varphi_\alpha^{-1}(x,b)) = a^{-1}.b$. This mapping is also uniquely determined by the implicit equation $r(u_x, \tau(u_x, v_x)) = v_x$, thus we also have $\tau(u_x.g, u'_x.g') = g^{-1}.\tau(u_x, u'_x).g'$ and $\tau(u_x, u_x) = e$.

When considering principal bundles the reader should think of frame bundles as the foremost examples for this book. They will be treated in 15.11 below.

15.3. Lemma. *Let $p : P \to M$ be a surjective submersion (a fibered manifold), and let G be a Lie group which acts freely on P such that the orbits of the action are exactly the fibers $p^{-1}(x)$ of p. Then (P, p, M, G) is a principal fiber bundle.*

Proof. Let the action be a right one by using the group inversion if necessary. Let $s_\alpha : U_\alpha \to P$ be local sections (right inverses) for $p : P \to M$ such that (U_α) is an open cover of M. Let $\varphi_\alpha^{-1} : U_\alpha \times G \to P|U_\alpha$ be given by $\varphi_\alpha^{-1}(x,a) = s_\alpha(x).a$, which is obviously injective with invertible tangent mapping, so its inverse $\varphi_\alpha : P|U_\alpha \to U_\alpha \times G$ is a fiber respecting diffeomorphism. So $(U_\alpha, \varphi_\alpha)$ is already a fiber bundle atlas. Let $\tau : P \times_M P \to G$ be given by the implicit equation $r(u_x, \tau(u_x, u'_x)) = u'_x$, where r is the right G-action. τ is smooth by the implicit function theorem and clearly we have $\tau(u_x, u'_x.g) = \tau(u_x, u'_x).g$ and $\varphi_\alpha(u_x) = (x, \tau(s_\alpha(x), u_x))$. Thus we have $\varphi_\alpha\varphi_\beta^{-1}(x,g) = \varphi_\alpha(s_\beta(x).g) = (x, \tau(s_\alpha(x), s_\beta(x).g)) = (x, \tau(s_\alpha(x), s_\beta(x)).g)$ and $(U_\alpha, \varphi_\alpha)$ is a principal bundle atlas. $\square$

15.4. Remarks. In the proof of Lemma 15.3 we have seen, that a principal bundle atlas of a principal fiber bundle (P, p, M, G) is already determined if we specify a family of smooth sections of P, whose domains of definition cover the base M.

Lemma 15.3 can serve as an equivalent definition for a principal bundle. But this is true only if an implicit function theorem is available, so in topology or in infinite dimensional differential geometry one should stick to our original definition.

From the Lemma itself it follows, that the pullback f^*P over a smooth mapping $f : M' \to M$ is again a principal fiber bundle.

15.5. Homogeneous spaces. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let K be a closed subgroup of G, then by theorem 5.5 K is a closed Lie subgroup whose

Lie algebra will be denoted by $\mathfrak{k}$. By theorem 5.11 there is a unique structure of a smooth manifold on the quotient space G/K such that the projection $p : G \to G/K$ is a submersion, so by the implicit function theorem p admits local sections.

Theorem. *$(G, p, G/K, K)$ is a principal fiber bundle.*

Proof. The group multiplication of G restricts to a free right action $\mu : G \times K \to G$, whose orbits are exactly the fibers of p. By lemma 15.3 the result follows. $\square$

For the convenience of the reader we discuss now the best known homogeneous spaces.

The group $SO(n)$ acts transitively on $S^{n-1} \subset \mathbb{R}^n$. The isotropy group of the 'north pole' $(0, \dots, 0, 1)$ is the subgroup

$$\begin{pmatrix} 1 & 0 \\ 0 & SO(n-1) \end{pmatrix}$$

which we identify with $SO(n-1)$. So $S^{n-1} = SO(n)/SO(n-1)$ and we get a principal fiber bundle $(SO(n), p, S^{n-1}, SO(n-1))$. Likewise
$(O(n), p, S^{n-1}, O(n-1))$,
$(SU(n), p, S^{2n-1}, SU(n-1))$,
$(U(n), p, S^{2n-1}, U(n-1))$, and
$(Sp(n), p, S^{4n-1}, Sp(n-1))$ are principal fiber bundles.

The *Grassmann manifold* $G(k, n; \mathbb{R})$ is the space of all k-planes containing 0 in $\mathbb{R}^n$. The group $O(n)$ acts transitively on it and the isotropy group of the k-plane $\mathbb{R}^k \times \{0\}$ is the subgroup

$$\begin{pmatrix} O(k) & 0 \\ 0 & O(n-k) \end{pmatrix},$$

therefore $G(k, n; \mathbb{R}) = O(n)/O(k) \times O(n-k)$ is a compact manifold and we get the principal fiber bundle $(O(n), p, G(k, n; \mathbb{R}), O(k) \times O(n-k))$. Likewise
$(SO(n), p, G(k, n; \mathbb{R}), S(O(k) \times O(n-k)))$,
$(SO(n), p, \tilde{G}(k, n; \mathbb{R}), SO(k) \times SO(n-k))$,
$(U(n), p, G(k, n; \mathbb{C}), U(k) \times U(n-k))$, and
$(Sp(n), p, G(k, n; \mathbb{H}), Sp(k) \times Sp(n-k))$ are principal fiber bundles.

The *Stiefel manifold* $V(k, n; \mathbb{R})$ is the space of all orthonormal k-frames in $\mathbb{R}^n$. Clearly the group $O(n)$ acts transitively on $V(k, n; \mathbb{R})$ and the isotropy subgroup of $(e_1, \dots, e_k)$ is $\mathbb{I}_k \times O(n-k)$, so $V(k, n; \mathbb{R}) = O(n)/O(n-k)$ is a compact manifold, and $(O(n), p, V(k, n; \mathbb{R}), O(n-k))$ is a principal fiber bundle. But $O(k)$ also acts from the right on $V(k, n; \mathbb{R})$, its orbits are exactly the fibers of the projection $p : V(k,n;\mathbb{R}) \to G(k,n;\mathbb{R})$. So by lemma 15.3 we get a principal fiber bundle $(V(k,n,\mathbb{R}), p, G(k,n;\mathbb{R}), O(k))$. Indeed we have the following diagram where all arrows are projections of principal fiber bundles, and where the respective structure groups are written on the arrows:

(a)
$$\begin{tikzcd} O(n) \arrow[r, "O(n-k)"] \arrow[d, "O(k)"'] & V(k,n;\mathbb{R}) \arrow[d, "O(k)"] \\ V(n-k,n;\mathbb{R}) \arrow[r, "O(n-k)"'] & G(k,n;\mathbb{R}), \end{tikzcd}$$

$V(k,n)$ is also diffeomorphic to the space $\{ A \in L(\mathbb{R}^k,\mathbb{R}^n) : A^t.A = \mathbb{I}_k \}$, i.e. the space of all linear isometries $\mathbb{R}^k \to \mathbb{R}^n$. There are furthermore complex and quaternionic versions of the Stiefel manifolds.

15.6. Homomorphisms. Let $\chi : (P,p,M,G) \to (P',p',M',G)$ be a *principal fiber bundle homomorphism*, i.e. a smooth G-equivariant mapping $\chi : P \to P'$. Then obviously the diagram

(a)
$$\begin{tikzcd} P \arrow[r, "\chi"] \arrow[d, "p"'] & P' \arrow[d, "p'"] \\ M \arrow[r, "\bar\chi"'] & M' \end{tikzcd}$$

commutes for a uniquely determined smooth mapping $\bar\chi : M \to M'$. For each $x \in M$ the mapping $\chi_x := \chi|P_x : P_x \to P'_{\bar\chi(x)}$ is G-equivariant and therefore a diffeomorphism, so diagram (a) is a pullback diagram.

But the most general notion of a homomorphism of principal bundles is the following. Let $\Phi : G \to G'$ be a homomorphism of Lie groups. $\chi : (P,p,M,G) \to (P',p',M',G')$ is called a *homomorphism over* Φ *of principal bundles*, if $\chi : P \to P'$ is smooth and $\chi(u.g) = \chi(u).\Phi(g)$ holds in general. Then χ is fiber respecting, so diagram (a) makes again sense, but it is no longer a pullback diagram in general.

If χ covers the identity on the base, it is called a *reduction of the structure group* G' to G for the principal bundle (P',p',M',G') — the name comes from the case, when Φ is the embedding of a subgroup.

By the universal property of the pullback any general homomorphism χ of principal fiber bundles over a group homomorphism can be written as the composition of a reduction of structure groups and a pullback homomorphism as follows, where we also indicate the structure groups:

(b)
$$\begin{tikzcd}
(P,G) \arrow[r] \arrow[dr, "p"'] & (\bar\chi^* P', G') \arrow[r] \arrow[d] & (P', G') \arrow[d, "p'"] \\
 & M \arrow[r, "\bar\chi"] & M'.
\end{tikzcd}$$

15.7. Associated bundles. Let (P, p, M, G) be a principal bundle and let $\ell : G\times S \to S$ be a left action of the structure group G on a manifold S. We consider the right action $R : (P \times S) \times G \to P \times S$, given by $R((u, s), g) = (u.g, g^{-1}.s)$.

Theorem. *In this situation we have:*

(1) *The space $P \times_G S$ of orbits of the action R carries a unique smooth manifold structure such that the quotient map $q : P \times S \to P \times_G S$ is a submersion.*
(2) *$(P\times_G S, \bar p, M, S, G)$ is a G-bundle in a canonical way, where $\bar p : P\times_G S \to M$ is given by*

(a)
$$\begin{tikzcd}
P \times S \arrow[r, "q"] \arrow[d, "pr_1"] & P \times_G S \arrow[d, "\bar p"'] \\
P \arrow[r, "p"] & M.
\end{tikzcd}$$

In this diagram $q_u : \{u\}\times S \to (P\times_G S)_{p(u)}$ is a diffeomorphism for each $u \in P$.
(3) *$(P \times S, q, P \times_G S, G)$ is a principal fiber bundle with principal action R.*
(4) *If $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$ is a principal bundle atlas with cocycle of transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$, then together with the left action $\ell : G \times S \to S$ this cocycle is also one for the G-bundle $(P \times_G S, \bar p, M, S, G)$.*

Notation. $(P \times_G S, \bar p, M, S, G)$ is called the *associated bundle* for the action $\ell : G \times S \to S$. We will also denote it by $P[S, \ell]$ or simply $P[S]$ and we will write p for $\bar p$ if no confusion is possible. We also define the smooth mapping $\tau = \tau^S : P \times_M P[S, \ell] \to S$ by $\tau(u_x, v_x) := q_{u_x}^{-1}(v_x)$. It satisfies $\tau(u, q(u, s)) = s$, $q(u_x, \tau(u_x, v_x)) = v_x$, and $\tau(u_x.g, v_x) = g^{-1}.\tau(u_x, v_x)$. In the special situation, where $S = G$ and the action is left translation, so that $P[G] = P$, this mapping coincides with τ considered in 15.2.

Proof. In the setting of the diagram in (2) the mapping $p \circ pr_1$ is constant on the R-orbits, so $\bar p$ exists as a mapping. Let $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$ be a principal bundle atlas with transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$. We define $\psi_\alpha^{-1} : U_\alpha \times S \to \bar p^{-1}(U_\alpha) \subset P \times_G S$ by $\psi_\alpha^{-1}(x,s) = q(\varphi_\alpha^{-1}(x,e),s)$, which is fiber respecting. For each orbit in $\bar p^{-1}(x) \subset P \times_G S$ there is exactly one $s \in S$ such that this orbit passes through $(\varphi_\alpha^{-1}(x,e),s)$, namely $s = \tau^G(u_x, \varphi_\alpha^{-1}(x,e))^{-1}.s'$ if (u_x,s') is the orbit, since the principal right action is free. Thus $\psi_\alpha^{-1}(x, \quad) : S \to \bar p^{-1}(x)$ is bijective. Furthermore

$$\begin{aligned}
\psi_\beta^{-1}(x,s) &= q(\varphi_\beta^{-1}(x,e),s) \\
&= q(\varphi_\alpha^{-1}(x,\varphi_{\alpha\beta}(x).e),s) = q(\varphi_\alpha^{-1}(x,e).\varphi_{\alpha\beta}(x),s) \\
&= q(\varphi_\alpha^{-1}(x,e),\varphi_{\alpha\beta}(x).s) = \psi_\alpha^{-1}(x,\varphi_{\alpha\beta}(x).s),
\end{aligned}$$

so $\psi_\alpha\psi_\beta^{-1}(x,s) = (x, \varphi_{\alpha\beta}(x).s)$ So (U_α, ψ_α) is a G-atlas for $P \times_G S$ and makes it into a smooth manifold and a G-bundle. The defining equation for ψ_α shows that q is smooth and a submersion and consequently the smooth structure on $P \times_G S$ is uniquely defined, and $\bar p$ is smooth by the universal properties of a submersion.

By the definition of ψ_α the diagram

$$\text{(b)} \qquad \begin{tikzcd}
p^{-1}(U_\alpha) \times S \arrow[r, "\varphi_\alpha \times Id"] \arrow[d, "q"'] & U_\alpha \times G \times S \arrow[d, "Id \times \ell"] \\
\bar p^{-1}(U_\alpha) \arrow[r, "\psi_\alpha"] & U_\alpha \times S
\end{tikzcd}$$

commutes; since its lines are diffeomorphisms we conclude that $q_u : \{u\} \times S \to \bar p^{-1}(p(u))$ is a diffeomorphism. So (1), (2), and (4) are checked.
(3) follows directly from lemma 15.3. We give below an explicit chart construction. We rewrite the last diagram in the following form:

$$\text{(c)} \qquad \begin{tikzcd}
p^{-1}(U_\alpha) \times S \arrow[r, "="] & q^{-1}(V_\alpha) \arrow[r, "\lambda_\alpha"] \arrow[d, "q"'] & V_\alpha \times G \arrow[d, "pr_1"] \\
& \bar p^{-1}(U_\alpha) \arrow[r, "="] & V_\alpha
\end{tikzcd}$$

Here $V_\alpha := \bar p^{-1}(U_\alpha) \subset P \times_G S$ and the diffeomorphism λ_α is defined by $\lambda_\alpha^{-1}(\psi_\alpha^{-1}(x,s),g) := (\varphi_\alpha^{-1}(x,g), g^{-1}.s)$. Then we have

$$\begin{aligned}
\lambda_\beta^{-1}(\psi_\alpha^{-1}(x,s),g) &= \lambda_\beta^{-1}(\psi_\beta^{-1}(x,\varphi_{\beta\alpha}(x).s),g) \\
&= (\varphi_\beta^{-1}(x,g), g^{-1}.\varphi_{\beta\alpha}(x).s)
\end{aligned}$$

$$= (\varphi_\alpha^{-1}(x, \varphi_{\alpha\beta}(x).g), g^{-1}.\varphi_{\alpha\beta}(x)^{-1}.s)$$
$$= \lambda_\alpha^{-1}(\psi_\alpha^{-1}(x,s), \varphi_{\alpha\beta}(x).g),$$

so $\lambda_\alpha\lambda_\beta^{-1}(\psi_\alpha^{-1}(x,s),g) = (\psi_\alpha^{-1}(x,s), \varphi_{\alpha\beta}(x).g)$ and $(P\times S, q, P\times_G S, G)$ is a principal bundle with structure group G and the same cocycle $(\varphi_{\alpha\beta})$ we started with. □

15.8. Corollary. *Let (E,p,M,S,G) be a G-bundle, specified by a cocycle of transition functions $(\varphi_{\alpha\beta})$ with values in G and a left action ℓ of G on S. Then from the cocycle of transition functions we may glue a unique principal bundle (P,p,M,G) such that $E = P[S,\ell]$.* □

This is the usual way a differential geometer thinks of an associated bundle. He is given a bundle E, a principal bundle P, and the G-bundle structure then is described with the help of the mappings τ and q.

15.9. Equivariant mappings and associated bundles.

1. Let (P,p,M,G) be a principal fiber bundle and consider two left actions of G, $\ell : G\times S\to S$ and $\ell' : G\times S'\to S'$. Let furthermore $f : S\to S'$ be a G-equivariant smooth mapping, so $f(g.s) = g.f(s)$ or $f\circ\ell_g = \ell'_g\circ f$. Then $Id_P\times f : P\times S\to P\times S'$ is equivariant for the actions $R : (P\times S)\times G\to P\times S$ and $R' : (P\times S')\times G\to P\times S'$ and is thus a homomorphism of principal bundles, so there is an induced mapping

(a)
$$\begin{tikzcd} P\times S \arrow[r, "Id\times f"] \arrow[d, "q"'] & P\times S' \arrow[d, "q'"] \\ P\times_G S \arrow[r, "Id\times_G f"] & P\times_G S', \end{tikzcd}$$

which is fiber respecting over M, and a homomorphism of G-bundles in the sense of the definition 15.10 below.

2. Let $\chi : (P,p,M,G)\to(P',p',M',G)$ be a principal fiber bundle homomorphism as in 15.6. Furthermore we consider a smooth left action $\ell : G\times S\to S$. Then $\chi\times Id_S : P\times S\to P'\times S$ is G-equivariant and induces a mapping $\chi\times_G Id_S : P\times_G S\to P'\times_G S$, which is fiber respecting over M, fiber wise a diffeomorphism, and again a homomorphism of G-bundles in the sense of definition 15.10 below.

3. Now we consider the situation of 1 and 2 at the same time. We have two associated bundles $P[S,\ell]$ and $P'[S',\ell']$. Let $\chi : (P,p,M,G)\to(P',p',M',G)$ be a principal fiber bundle homomorphism and let $f : S\to S'$ be an G-equivariant

mapping. Then $\chi \times f : P \times S \to P' \times S'$ is clearly G-equivariant and therefore induces a mapping $\chi \times_G f : P[S,\ell] \to P'[S',\ell']$ which again is a homomorphism of G-bundles.

4. Let S be a point. Then $P[S] = P \times_G S = M$. Furthermore let $y \in S'$ be a fixpoint of the action $\ell' : G \times S' \to S'$, then the inclusion $i : \{y\} \hookrightarrow S'$ is G-equivariant, thus $Id_P \times i$ induces $Id_P \times_G i : M = P[\{y\}] \to P[S']$, which is a global section of the associated bundle $P[S']$.

If the action of G on S is trivial, so $g.s = s$ for all $s \in S$, then the associated bundle is trivial: $P[S] = M \times S$. For a trivial principal fiber bundle any associated bundle is trivial.

15.10. Definition. In the situation of 15.9, a smooth fiber respecting mapping $\gamma : P[S,\ell] \to P'[S',\ell']$ covering a smooth mapping $\bar\gamma : M \to M'$ of the bases is called a *homomorphism of G-bundles*, if the following conditions are satisfied: P is isomorphic to the pullback $\bar\gamma^* P'$, and the local representations of γ in pullback-related fiber bundle atlases belonging to the two G-bundles are fiber wise G-equivariant.

Let us describe this in more detail now. Let $(U'_\alpha, \psi'_\alpha)$ be a G-atlas for $P'[S',\ell']$ with cocycle of transition functions $(\varphi'_{\alpha\beta})$, belonging to the principal fiber bundle atlas $(U'_\alpha, \varphi'_\alpha)$ of (P', p', M', G). Then the pullback-related principal fiber bundle atlas $(U_\alpha = \bar\gamma^{-1}(U'_\alpha), \varphi_\alpha)$ for $P = \bar\gamma^* P'$ as described in the proof of 14.5 has the cocycle of transition functions $(\varphi_{\alpha\beta} = \varphi'_{\alpha\beta} \circ \bar\gamma)$; it induces the G-atlas (U_α, ψ_α) for $P[S,\ell]$. Then $(\psi'_\alpha \circ \gamma \circ \psi_\alpha^{-1})(x,s) = (\bar\gamma(x), \gamma_\alpha(x,s))$ and $\gamma_\alpha(x, \quad) : S \to S'$ is required to be G-equivariant for all α and all $x \in U_\alpha$.

Lemma. *Let $\gamma : P[S,\ell] \to P'[S',\ell']$ be a homomorphism of G-bundles as defined above. Then there is a homomorphism $\chi : (P,p,M,G) \to (P',p',M',G)$ of principal bundles and a G-equivariant mapping $f : S \to S'$ such that $\gamma = \chi \times_G f : P[S,\ell] \to P'[S',\ell']$.*

Proof. The homomorphism $\chi : (P,p,M,G) \to (P',p',M',G)$ of principal fiber bundles is already determined by the requirement that $P = \bar\gamma^* P'$, and we have $\bar\gamma = \bar\chi$. The G-equivariant mapping $f : S \to S'$ can be read off the following diagram

$$\text{(a)} \qquad \begin{array}{ccc} P \times_M P[S] & \xrightarrow{\tau^S} & S \\ {\scriptstyle \chi \times_M \gamma}\downarrow & & \downarrow{\scriptstyle f} \\ P' \times_{M'} P'[S'] & \xrightarrow{\tau^{S'}} & S', \end{array}$$

which by the assumptions is seen to be well defined in the right column. □

So a homomorphism of G-bundles is described by the whole triple ($\chi : P \to P', f : S \to S'$ (G-equivariant), $\gamma : P[S] \to P'[S']$), such that diagram (a) commutes.

15.11. Associated vector bundles. Let (P, p, M, G) be a principal fiber bundle, and consider a representation $\rho : G \to GL(V)$ of G on a finite dimensional vector space V. Then $P[V, \rho]$ is an associated fiber bundle with structure group G, but also with structure group $GL(V)$, for in the canonically associated fiber bundle atlas the transition functions have also values in $GL(V)$. So by section 6 $P[V, \rho]$ is a vector bundle.

Now let $\mathcal{F}$ be a covariant smooth functor from the category of finite dimensional vector spaces and linear mappings into itself, as considered in section 6.7. Then clearly $\mathcal{F} \circ \rho : G \to GL(V) \to GL(\mathcal{F}(V))$ is another representation of G and the associated bundle $P[\mathcal{F}(V), \mathcal{F} \circ \rho]$ coincides with the vector bundle $\mathcal{F}(P[V, \rho])$ constructed with the method of 6.7, but now it has an extra G-bundle structure. For contravariant functors $\mathcal{F}$ we have to consider the representation $\mathcal{F} \circ \rho \circ \nu$, similarly for bifunctors. In particular the bifunctor $L(V, W)$ may be applied to two different representations of two structure groups of two principal bundles over the same base M to construct a vector bundle $L(P[V, \rho], P'[V', \rho']) = (P \times_M P')[L(V, V'), L \circ ((\rho \circ \nu) \times \rho')]$.

If (E, p, M) is a vector bundle with n-dimensional fibers we may consider the open subset $GL(\mathbb{R}^n, E) \subset L(M \times \mathbb{R}^n, E)$, a fiber bundle over the base M, whose fiber over $x \in M$ is the space $GL(\mathbb{R}^n, E_x)$ of all invertible linear mappings. Composition from the right by elements of $GL(n)$ gives a free right action on $GL(\mathbb{R}^n, E)$ whose orbits are exactly the fibers, so by lemma 15.3 we have a principal fiber bundle $(GL(\mathbb{R}^n, E), p, M, GL(n))$. The associated bundle $GL(\mathbb{R}^n, E)[\mathbb{R}^n]$ for the banal representation of $GL(n)$ on $\mathbb{R}^n$ is isomorphic to the vector bundle (E, p, M) we started with, for the evaluation mapping $ev : GL(\mathbb{R}^n, E) \times \mathbb{R}^n \to E$ is invariant under the right action R of $GL(n)$, and locally in the image there are smooth sections to it, so it factors to a fiber linear diffeomorphism $GL(\mathbb{R}^n, E)[\mathbb{R}^n] = GL(\mathbb{R}^n, E) \times_{GL(n)} \mathbb{R}^n \to E$. The principal bundle $GL(\mathbb{R}^n, E)$ is called the *linear frame bundle of* E. Note that local sections of $GL(\mathbb{R}^n, E)$ are exactly the local frame fields of the vector bundle E as discussed in 6.4.

To illustrate the notion of reduction of structure group, we consider now a vector bundle $(E, p, M, \mathbb{R}^n)$ equipped with a *Riemannian metric* g, that is a section $g \in C^\infty(S^2E^*)$ such that g_x is a positive definite inner product on E_x for each $x \in M$. Any vector bundle admits Riemannian metrics: local existence is clear and we may glue with the help of a partition of unity on M, since the positive definite sections form an open convex subset. Now let

$s' = (s'_1, \dots, s'_n) \in C^\infty(GL(\mathbb{R}^n, E)|U)$ be a local frame field of the bundle E over $U \subset M$. Now we may apply the Gram-Schmidt orthonormalization procedure to the basis $(s_1(x), \dots, s_n(x))$ of E_x for each $x \in U$. Since this procedure is smooth (even real analytic), we obtain a frame field $s = (s_1, \dots, s_n)$ of E over U which is orthonormal with respect to g. We call it an *orthonormal frame field*. Now let (U_α) be an open cover of M with orthonormal frame fields $s^\alpha = (s^\alpha_1, \dots, s^\alpha_n)$, where s^α is defined on U_α. We consider the vector bundle charts $(U_\alpha, \psi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^n)$ given by the orthonormal frame fields: $\psi_\alpha^{-1}(x, v^1, \dots, v^n) = \sum s^\alpha_i(x).v^i =: s^\alpha(x).v$. For $x \in U_{\alpha\beta}$ we have $s^\alpha_i(x) = \sum s^\beta_j(x).g_{\beta\alpha}{}^j_i(x)$ for C^∞-functions $g_{\alpha\beta}{}^j_i : U_{\alpha\beta} \to \mathbb{R}$. Since $s^\alpha(x)$ and $s^\beta(x)$ are both orthonormal bases of E_x, the matrix $g_{\alpha\beta}(x) = (g_{\alpha\beta}{}^j_i(x))$ is an element of $O(n,\mathbb{R})$. We write $s^\alpha = s^\beta.g_{\beta\alpha}$ for short. Then we have $\psi_\beta^{-1}(x,v) = s^\beta(x).v = s^\alpha(x).g_{\alpha\beta}(x).v = \psi_\alpha^{-1}(x, g_{\alpha\beta}(x).v)$ and consequently $\psi_\alpha\psi_\beta^{-1}(x,v) = (x, g_{\alpha\beta}(x).v)$. So the $(g_{\alpha\beta} : U_{\alpha\beta} \to O(n,\mathbb{R}))$ are the cocycle of transition functions for the vector bundle atlas (U_α, ψ_α). So we have constructed an $O(n,\mathbb{R})$-structure on E. The corresponding principal fiber bundle will be denoted by $O(\mathbb{R}^n, (E,g))$; it is usually called the *orthonormal frame bundle* of E. It is derived from the linear frame bundle $GL(\mathbb{R}^n, E)$ by reduction of the structure group from $GL(n)$ to $O(n)$. The phenomenon discussed here plays a prominent role in the theory of *classifying spaces*.

15.12. Sections of associated bundles. Let (P, p, M, G) be a principal fiber bundle and $\ell : G \times S \to S$ a left action. Let $C^\infty(P,S)^G$ denote the space of all smooth mappings $f : P \to S$ which are G-equivariant in the sense that $f(u.g) = g^{-1}.f(u)$ holds for $g \in G$ and $u \in P$.

Theorem. *The sections of the associated bundle $P[S,\ell]$ correspond exactly to the G-equivariant mappings $P \to S$; we have a bijection $C^\infty(P,S)^G \cong C^\infty(P[S])$.* ∎

This result follows from 15.9 and 15.10. Since it is very important we include a direct proof.

Proof. If $f \in C^\infty(P,S)^G$ we construct $s_f \in C^\infty(P[S])$ in the following way: $\operatorname{graph}(f) = (Id, f) : P \to P \times S$ is G-equivariant, since we have $(Id,f)(u.g) = (u.g, f(u.g)) = (u.g, g^{-1}.f(u)) = ((Id,f)(u)).g$. So it induces a smooth section $s_f \in C^\infty(P[S])$ as seen from 15.9 and the diagram:

$$\text{(a)}\qquad \begin{array}{ccc} P \times \{Pt\} \cong P & \xrightarrow{(Id,f)} & P \times S \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ M & \xrightarrow{s_f} & P[S] \end{array}$$

If conversely $s \in C^\infty(P[S])$ we define $f_s \in C^\infty(P,S)^G$ by $f_s := \tau^S \circ (Id_P \times_M s) : P = P \times_M M \to P \times_M P[S] \to S$. This is G-equivariant since $f_s(u_x.g) = \tau^S(u_x.g, s(x)) = g^{-1}.\tau^S(u_x, s(x)) = g^{-1}.f_s(u_x)$ by 15.7. The two constructions are inverse to each other since we have $f_{s(f)}(u) = \tau^S(u, s_f(p(u))) = \tau^S(u, q(u, f(u))) = f(u)$ and $s_{f(s)}(p(u)) = q(u, f_s(u)) = q(u, \tau^S(u, s(p(u))) = s(p(u))$. $\square$

15.13. Induced representations. Let K be a closed subgroup of a Lie group G. Let $\rho : K \to GL(V)$ be a representation in a vector space V, which we assume to be finite dimensional for the beginning. Then we consider the principal fiber bundle $(G, p, G/K, K)$ and the associated vector bundle $(G[V], p, G/K)$. The smooth (or even continuous) sections of $G[V]$ correspond exactly to the K-equivariant mappings $f : G \to V$, those satisfying $f(gk) = \rho(k^{-1})f(g)$, by lemma 15.12. Each $g \in G$ acts as a principal bundle homomorphism by left translation

$$\begin{tikzcd} G \arrow[r, "\mu_g"] \arrow[d] & G \arrow[d] \\ G/K \arrow[r, "\bar\mu_g"] & G/K. \end{tikzcd}$$

So by 15.9 we have an induced isomorphism of vector bundles

$$\begin{tikzcd} G \arrow[r, "\mu_g"] \arrow[d] & G \arrow[d] \\ G[V] \arrow[r, "\mu_g \times_K V"] \arrow[d] & G[V] \arrow[d] \\ G/K \arrow[r, "\bar\mu_g"] & G/K \end{tikzcd}$$

which gives rise to the representation $\widetilde{\operatorname{ind}}_K^G \rho$ of G in the space $C^\infty(G[V])$, defined by

$$(\widetilde{\operatorname{ind}}_K^G \rho)(g)(s) := (\mu_g \times_K V) \circ s \circ \bar\mu_{g^{-1}} = (\mu_g \times_K V)_*(s).$$

Now let us assume that the original representation ρ is unitary, $\rho : K \to U(V)$ for a complex vector space V with inner product $\langle \quad , \quad \rangle_V$. Then $v \mapsto \|v\|^2 = \langle v, v\rangle$ is an invariant symmetric homogeneous polynomial $V \to \mathbb{R}$ of degree 2, so it is equivariant where K acts trivial on $\mathbb{R}$. By 15.9 again we get an induced mapping $G[V] \to G[\mathbb{R}] = G/K \times \mathbb{R}$, which we can polarize to a smooth fiberwise hermitian form $\langle\quad,\quad\rangle_{G[V]}$ on the vector bundle $G[V]$. We may also express this by

$$\langle v_x, w_x\rangle_{G[V]} = \langle \tau^V(u_x, v_x), \tau^V(u_x, w_x)\rangle_V$$

for some $u_x \in G_x$, using the mapping $\tau^V : G \times_{G/M} G[V] \to V$ from 15.7; it can be checked easily that it does not depend on the choice of u_x. Still another way to describe the fiberwise hermitian form is

$$\begin{tikzcd}
(G\times V)\times_{G/K}(G\times V) \arrow[d] \arrow[drr, "f"] & & \\
G[V]\times_{G/K} G[V] \arrow[rr, "\langle\ ,\ \rangle_{G[V]}"] \arrow[d] & & \mathbb{C} \\
G/K, & &
\end{tikzcd}$$

where $f((g_1, v_1), (g_2, v_2)) := \langle v_1, \rho(\tau^K(g_1, g_2))v_2\rangle_V$ for $\tau^K : G \times_K G \to K$, $\tau^K(g_1, g_2) = g_1^{-1}g_2$ from 15.2. From this last description it is also clear that each $g \in G$ acts as an isometric vector bundle homomorphism.

Now we consider the natural line bundle $\mathrm{Vol}^{1/2}(G/K)$ of all $\frac{1}{2}$-densities on the manifold G/K from 8.4. Then for $\frac{1}{2}$-densities $\mu_i \in C^\infty(\mathrm{Vol}^{1/2}(G/M))$ and any diffeomorphism $f : G/K \to G/K$ the push forward $f_*\mu_i$ is defined and for those with compact support we have $\int_{G/K}(f_*\mu_1.f_*\mu_2) = \int_{G/K} f_*(\mu_1.\mu_2) = \int_{G/K}\mu_1.\mu_2$. The hermitian inner product on $G[V]$ now defines a fiberwise hermitian mapping

$$(G[V]\otimes \mathrm{Vol}^{1/2}(G/K)) \times_{G/K} (G[V]\otimes \mathrm{Vol}^1/2(G/K)) \xrightarrow{\langle\ ,\ \rangle_{G[V]}} \mathrm{Vol}^{1/2}(G/M)$$

and on the space $C_c^\infty(G[V]\otimes \mathrm{Vol}^{1/2}(G/K))$ of all smooth sections with compact support we have the following hermitian inner product

$$\langle s_1, s_2\rangle := \int_{G/K}\langle s_1, s_2\rangle_{G[V]}.$$

Obviously the resulting action of the group G on $C^\infty(G[V]\otimes \mathrm{Vol}^{1/2}(G/K))$ is unitary with respect to the hermitian inner product, and it can be extended to the Hilbert space completion of this space of sections. The resulting unitary representation is called the *induced representation* and is denoted by $\mathrm{ind}_K^G\,\rho$.

If the original unitary representation $\rho : K \to U(V)$ is in an infinte dimensional Hilbert space V, one can first restrict the representation ρ to the subspace of smooth vectors, on which it is differentiable, and repeat the above construction with some modifications. See [Michor, 1990] for more details on this infinite dimensional construction.

15.14. Theorem. *Consider a principal fiber bundle (P, p, M, G) and a closed subgroup K of G. Then the reductions of structure group from G to K correspond bijectively to the global sections of the associated bundle $P[G/K, \bar\lambda]$ in a canonical way, where $\bar\lambda : G \times G/K \to G/K$ is the left action on the homogeneous space from 5.11.*

Proof. By theorem 15.12 the section $s \in C^\infty(P[G/K])$ corresponds to $f_s \in C^\infty(P, G/K)^G$, which is a surjective submersion since the action $\bar\lambda : G \times G/K \to G/K$ is transitive. Thus $P_s := f_s^{-1}(\bar e)$ is a submanifold of P which is stable under the right action of K on P. Furthermore the K-orbits are exactly the fibers of the mapping $p : P_s \to M$, so by lemma 15.3 we get a principal fiber bundle (P_s, p, M, K). The embedding $P_s \hookrightarrow P$ is then a reduction of structure groups as required.

If conversely we have a principal fiber bundle (P', p', M, K) and a reduction of structure groups $\chi : P' \to P$, then χ is an embedding covering the identity of M and is K-equivariant, so we may view P' as a sub fiber bundle of P which is stable under the right action of K. Now we consider the mapping $\tau : P \times_M P \to G$ from 15.2 and restrict it to $P \times_M P'$. Since we have $\tau(u_x, v_x.k) = \tau(u_x, v_x).k$ for $k \in K$ this restriction induces $f : P \to G/K$ by

$$\begin{tikzcd} P \times_M P' \arrow[r, "\tau"] \arrow[d] & G \arrow[d, "p"] \\ P = P \times_M P'/K \arrow[r, "f"] & G/K, \end{tikzcd}$$

since $P'/K = M$; and from $\tau(u_x.g, v_x) = g^{-1}.\tau(u_x, v_x)$ it follows that f is G-equivariant as required. Finally $f^{-1}(\bar e) = \{u \in P : \tau(u, P'_{p(u)}) \subseteq K\} = P'$, so the two constructions are inverse to each other. $\square$

15.15. The bundle of gauges. If (P, p, M, G) is a principal fiber bundle we denote by $\operatorname{Aut}(P)$ the group of all G-equivariant diffeomorphisms $\chi : P \to P$. Then $p \circ \chi = \bar\chi \circ p$ for a unique diffeomorphism $\bar\chi$ of M, so there is a group homomorphism from $\operatorname{Aut}(P)$ into the group $\operatorname{Diff}(M)$ of all diffeomorphisms of M. The kernel of this homomorphism is called $\operatorname{Gau}(P)$, the group of *gauge transformations*. So $\operatorname{Gau}(P)$ is the space of all $\chi : P \to P$ which satisfy $p \circ \chi = p$ and $\chi(u.g) = \chi(u).g$.

Theorem. *The group $Gau(P)$ of gauge transformations is equal to the space $C^\infty(P,(G,\mathrm{conj}))^G \cong C^\infty(P[G,\mathrm{conj}])$.*

Proof. We use again the mapping $\tau : P \times_M P \to G$ from 15.2. For $\chi \in \mathrm{Gau}(P)$ we define $f_\chi \in C^\infty(P,(G,\mathrm{conj}))^G$ by $f_\chi := \tau \circ (Id, \chi)$. Then $f_\chi(u.g) = \tau(u.g, \chi(u.g)) = g^{-1}.\tau(u,\chi(u)).g = \mathrm{conj}_{g^{-1}} f_\chi(u)$, so f_χ is indeed G-equivariant.

If conversely $f \in C^\infty(P,(G,\mathrm{conj}))^G$ is given, we define $\chi_f : P \to P$ by $\chi_f(u) := u.f(u)$. It is easy to check that χ_f is indeed in $\mathrm{Gau}(P)$ and that the two constructions are inverse to each other, namely

$$\begin{aligned}
&\chi_f(ug) = ugf(ug) = ugg^{-1}f(u)g = \chi_f(u)g,\\
&f_{\chi_f}(u) = \tau^G(u,\chi_f(u)) = \tau^G(u,u.f(u)) = \tau^G(u,u)f(u) = f(u),\\
&\chi_{f_\chi}(u) = uf_\chi(u) = u\tau^G(u,\chi(u)) = \chi(u). \quad \square
\end{aligned}$$

15.16. The tangent bundles of homogeneous spaces. Let G be a Lie group and K a closed subgroup, with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, respectively. We recall the mapping $\mathrm{Ad}_G : G \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ from 4.24 and put $\mathrm{Ad}_{G,K} := \mathrm{Ad}_G |K : K \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$. For $X \in \mathfrak{k}$ and $k \in K$ we have $\mathrm{Ad}_{G,K}(k)X = \mathrm{Ad}_G(k)X = \mathrm{Ad}_K(k)X \in \mathfrak{k}$, so $\mathfrak{k}$ is an invariant subspace for the representation $\mathrm{Ad}_{G,K}$ of K in $\mathfrak{g}$, and we have the factor representation $\mathrm{Ad}^\perp : K \to GL(\mathfrak{g}/\mathfrak{k})$. Then

$$0 \to \mathfrak{k} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{k} \to 0 \tag{a}$$

is short exact and K-equivariant.

Now we consider the principal fiber bundle $(G, p, G/K, K)$ and the associated vector bundles $G[\mathfrak{g}/\mathfrak{k}, \mathrm{Ad}^\perp]$ and $G[\mathfrak{k}, \mathrm{Ad}_K]$.

Theorem. *In these circumstances we have*

$$T(G/K) = G[\mathfrak{g}/\mathfrak{k}, \mathrm{Ad}^\perp] = (G \times_K \mathfrak{g}/\mathfrak{k}, p, G/K, \mathfrak{g}/\mathfrak{k}).$$

The left action $g \mapsto T(\bar\mu_g)$ of G on $T(G/K)$ corresponds to the canonical left action of G on $G \times_K \mathfrak{g}/\mathfrak{k}$. Furthermore $G[\mathfrak{g}/\mathfrak{k}, \mathrm{Ad}^\perp] \oplus G[\mathfrak{k}, \mathrm{Ad}_K]$ is a trivial vector bundle.

Proof. For $p : G \to G/K$ we consider the tangent mapping $T_e p : \mathfrak{g} \to T_{\bar e}(G/K)$ which is linear and surjective and induces a linear isomorphism $\overline{T_e p} : \mathfrak{g}/\mathfrak{k} \to T_{\bar e}(G/K)$. For $k \in K$ we have $p \circ \mathrm{conj}_k = p \circ \mu_k \circ \rho_{k^{-1}} = \bar\mu_k \circ p$ and consequently $T_e p \circ \mathrm{Ad}_{G,K}(k) = T_e p \circ T_e(\mathrm{conj}_k) = T_{\bar e}\bar\mu_k \circ T_e p$. Thus the isomorphism $\overline{T_e p} : \mathfrak{g}/\mathfrak{k} \to T_{\bar e}(G/K)$ is K-equivariant for the representations $\mathrm{Ad}^\perp$ and $T_{\bar e}\bar\lambda : k \mapsto T_{\bar e}\bar\mu_k$.

Let us now consider the associated vector bundle

$$G[T_{\bar e}(G/K), T_{\bar e}\bar\lambda] = (G \times_K T_{\bar e}(G/K), p, G/K, T_{\bar e}(G/K)),$$

which is isomorphic to the vector bundle $G[\mathfrak{g}/\mathfrak{k}, \mathrm{Ad}^\perp]$, since the representation spaces are isomorphic. The mapping $T_2\bar\lambda : G \times T_{\bar e}(G/K) \to T(G/K)$ (where T_2 is the second partial tangent functor) is K-invariant, since $T\bar\lambda((g,X)k) = T\bar\lambda(gk, T_{\bar e}\bar\mu_{k^{-1}}.X) = T\bar\mu_{gk}.T\bar\mu_{k^{-1}}.X = T\bar\mu_g.X$. Therefore it induces a mapping ψ as in the following diagram:

$$\text{(b)}\qquad \begin{tikzcd} & G \times T_{\bar e}(G/K) \arrow[dl, "q"'] \arrow[dr, "T\bar\lambda"] & \\ G \times_K T_{\bar e}(G/K) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & T(G/K) \arrow[dl, "\pi_{G/K}"] \\ & G/K & \end{tikzcd}$$

This mapping ψ is an isomorphism of vector bundles.

It remains to show the last assertion. The short exact sequence (a) induces a sequence of vector bundles over G/K:

$$G/K \times 0 \to G[\mathfrak{k}, \mathrm{Ad}_K] \to G[\mathfrak{g}, \mathrm{Ad}_{G,K}] \to G[\mathfrak{g}/\mathfrak{k}, \mathrm{Ad}^\perp] \to G/K \times 0$$

This sequence splits fiber wise thus also locally over G/K, so we get $G[\mathfrak{g}/\mathfrak{k}, \mathrm{Ad}^\perp] \oplus G[\mathfrak{k}, \mathrm{Ad}_K] \cong G[\mathfrak{g}, \mathrm{Ad}_{G,K}]$. We have to show that $G[\mathfrak{g}, \mathrm{Ad}_{G,K}]$ is a trivial vector bundle. Let $\varphi : G \times \mathfrak{g} \to G \times \mathfrak{g}$ be given by $\varphi(g,X) = (g, \mathrm{Ad}_G(g)X)$. Then for $k \in K$ we have

$$\begin{aligned} \varphi((g,X).k) &= \varphi(gk, \mathrm{Ad}_{G,K}(k^{-1})X) \\ &= (gk, \mathrm{Ad}_G(g.k.k^{-1})X) = (gk, \mathrm{Ad}_G(g)X). \end{aligned}$$

So φ is K-equivariant for the 'joint' K-action to the 'on the left' K-action and therefore induces a mapping $\bar\varphi$ as in the diagram:

$$\text{(c)}\qquad \begin{tikzcd} G \times \mathfrak{g} \arrow[rr, "\varphi"] \arrow[d, "q"'] & & G \times \mathfrak{g} \arrow[d] \\ G \times_K \mathfrak{g} \arrow[rr, "\bar\varphi"] \arrow[dr, "p"'] & & G/K \times \mathfrak{g} \arrow[dl, "pr_1"] \\ & G/K & \end{tikzcd}$$

The map $\bar\varphi$ is a vector bundle isomorphism. $\square$

15.17. Tangent bundles of Grassmann manifolds. From 15.5 we know that $(V(k,n) = O(n)/O(n-k), p, G(k,n), O(k))$ is a principal fiber bundle. Using the banal representation of $O(k)$ we consider the associated vector bundle $(E_k := V(k,n)[\mathbb{R}^k], p, G(k,n))$. It is called the *universal vector bundle* over $G(k,n)$ for reasons we will discuss below in section 16. Recall from 15.5 the description of $V(k,n)$ as the space of all linear isometries $\mathbb{R}^k \to \mathbb{R}^n$; we get from it the evaluation mapping $ev : V(k,n)\times\mathbb{R}^k \to \mathbb{R}^n$. The mapping (p, ev) in the diagram

(a)
$$\begin{tikzcd}
V(k,n)\times\mathbb{R}^k \arrow[d, "q"'] \arrow[dr, "{(p,ev)}"] & \\
V(k,n)\times_{O(k)}\mathbb{R}^k \arrow[r, "\psi"'] & G(k,n)\times\mathbb{R}^n
\end{tikzcd}$$

is $O(k)$-invariant for the action R and factors therefore to an embedding of vector bundles $\psi : E_k \to G(k,n)\times\mathbb{R}^n$. So the fiber $(E_k)_W$ over the k-plane W in $\mathbb{R}^n$ is just the linear subspace W. Note finally that the fiber wise orthogonal complement ${E_k}^\perp$ of E_k in the trivial vector bundle $G(k,n)\times\mathbb{R}^n$ with its standard Riemannian metric is isomorphic to the universal vector bundle E_{n-k} over $G(n-k,n)$, where the isomorphism covers the diffeomorphism $G(k,n)\to G(n-k,n)$ given also by the orthogonal complement mapping.

Corollary. *The tangent bundle of the Grassmann manifold is*

$$TG(k,n) \cong L(E_k, {E_k}^\perp).$$

Proof. We have $G(k,n) = O(n)/(O(k)\times O(n-k))$, so by theorem 15.16 we get

$$TG(k,n) = O(n) \underset{O(k)\times O(n-k)}{\times} (\mathfrak{so}(n)/(\mathfrak{so}(k)\times\mathfrak{so}(n-k))).$$

On the other hand we have $V(k,n) = O(n)/O(n-k)$ and the right action of $O(k)$ commutes with the right action of $O(n-k)$ on $O(n)$, therefore

$$V(k,n)[\mathbb{R}^k] = (O(n)/O(n-k)) \underset{O(k)}{\times} \mathbb{R}^k = O(n) \underset{O(k)\times O(n-k)}{\times} \mathbb{R}^k,$$

where $O(n-k)$ acts trivially on $\mathbb{R}^k$. Finally

$$\begin{aligned}
L(E_k, {E_k}^\perp) &= L\left(O(n) \underset{O(k)\times O(n-k)}{\times} \mathbb{R}^k, O(n) \underset{O(k)\times O(n-k)}{\times} \mathbb{R}^{n-k}\right)\\
&= O(n) \underset{O(k)\times O(n-k)}{\times} L(\mathbb{R}^k, \mathbb{R}^{n-k}),
\end{aligned}$$

where $O(k) \times O(n-k)$ acts on $L(\mathbb{R}^k, \mathbb{R}^{n-k})$ by $(A,B)(C) = B.C.A^{-1}$. Finally we have an $O(k) \times O(n-k)$ - equivariant linear isomorphism $L(\mathbb{R}^k, \mathbb{R}^{n-k}) \to \mathfrak{so}(n)/(\mathfrak{so}(k) \times \mathfrak{so}(n-k))$, as follows:

$$\mathfrak{so}(n)/(\mathfrak{so}(k) \times \mathfrak{so}(n-k)) = \frac{(\text{skew})}{\begin{pmatrix} \text{skew} & 0 \\ 0 & \text{skew} \end{pmatrix}} = \left\{ \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} : \quad A \in L(\mathbb{R}^k, \mathbb{R}^{n-k}) \right\} \quad \square$$

15.18. Tangent bundles and vertical bundles. Let (E, p, M, S) be a fiber bundle. The sub vector bundle $VE = \{ \xi \in TE : \quad Tp.\xi = 0 \}$ of TE is called the *vertical bundle* and is denoted by (VE, π_E, E).

Theorem. *Let (P, p, M, G) be a principal fiber bundle with principal right action $r : P \times G \to P$. Let $\ell : G \times S \to S$ be a left action. Then the following assertions hold:*

(1) *(TP, Tp, TM, TG) is again a principal fiber bundle with principal right action $Tr : TP \times TG \to TP$, where the structure group TG is the tangent group of G, see 5.16.*
(2) *The vertical bundle $(VP, \pi, P, \mathfrak{g})$ of the principal bundle is trivial as a vector bundle over P: $VP \cong P \times \mathfrak{g}$.*
(3) *The vertical bundle of the principal bundle as bundle over M is again a principal bundle: $(VP, p \circ \pi, M, TG)$.*
(4) *The tangent bundle of the associated bundle $P[S, \ell]$ is given by $T(P[S, \ell]) = TP[TS, T\ell]$.*
(5) *The vertical bundle of the associated bundle $P[S, \ell]$ is given by $V(P[S, \ell]) = P[TS, T_2\ell] = P \times_G TS$.*

Proof. Let $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$ be a principal fiber bundle atlas with cocycle of transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$. Since T is a functor which respects products, $(TU_\alpha, T\varphi_\alpha : TP|TU_\alpha \to TU_\alpha \times TG)$ is again a principal fiber bundle atlas with cocycle of transition functions $(T\varphi_{\alpha\beta} : TU_{\alpha\beta} \to TG)$, describing the principal fiber bundle (TP, Tp, TM, TG). The assertion about the principal action is obvious. So (1) follows. For completeness sake we include here the transition formula for this atlas in the right trivialization of TG:

$$T(\varphi_\alpha \circ \varphi_\beta^{-1})(\xi_x, T_e(\rho_g).X) = (\xi_x, T_e(\rho_{\varphi_{\alpha\beta}(x).g}).(\delta\varphi_{\alpha\beta}(\xi_x) + \mathrm{Ad}(\varphi_{\alpha\beta}(x))X)),$$

where $\delta\varphi_{\alpha\beta} \in \Omega^1(U_{\alpha\beta}; \mathfrak{g})$ is the right logarithmic derivative of $\varphi_{\alpha\beta}$, see 4.26.

(2) The mapping $(u,X)\mapsto T_e(r_u).X = T_{(u,e)}r.(0_u,X)$ is a vector bundle isomorphism $P\times\mathfrak{g}\to VP$ over P.
(3) Obviously $Tr : TP\times TG \to TP$ is a free right action which acts transitive on the fibers of $Tp : TP\to TM$. Since $VP = (Tp)^{-1}(0_M)$, the bundle $VP\to M$ is isomorphic to $TP|0_M$ and Tr restricts to a free right action, which is transitive on the fibers, so by lemma 15.3 the result follows.
(4) The transition functions of the fiber bundle $P[S,\ell]$ are given by the expression $\ell\circ(\varphi_{\alpha\beta}\times Id_S) : U_{\alpha\beta}\times S\to G\times S\to S$. Then the transition functions of $T(P[S,\ell])$ are $T(\ell\circ(\varphi_{\alpha\beta}\times Id_S)) = T\ell\circ(T\varphi_{\alpha\beta}\times Id_{TS}) : TU_{\alpha\beta}\times TS\to TG\times TS\to TS$, from which the result follows.
(5) Vertical vectors in $T(P[S,\ell])$ have local representations $(0_x,\eta_s)\in TU_{\alpha\beta}\times TS$. Under the transition functions of $T(P[S,\ell])$ they transform as $T(\ell\circ(\varphi_{\alpha\beta}\times Id_S)).(0_x,\eta_s) = T\ell.(0_{\varphi_{\alpha\beta}(x)},\eta_s) = T(\ell_{\varphi_{\alpha\beta}(x)}).\eta_s = T_2\ell.(\varphi_{\alpha\beta}(x),\eta_s)$ and this implies the result $\square$

16. Principal and Induced Connections

16.1. Principal connections. Let (P, p, M, G) be a principal fiber bundle. Recall from 14.3 that a (general) connection on P is a fiber projection $\Phi : TP \to VP$, viewed as a 1-form in $\Omega^1(P; TP)$. Such a connection Φ is called a *principal connection* if it is G-equivariant for the principal right action $r : P \times G \to P$, so that $T(r^g).\Phi = \Phi.T(r^g)$ and Φ is r^g-related to itself, or $(r^g)^*\Phi = \Phi$ in the sense of 13.16, for all $g \in G$. By theorem 13.15.6 the curvature $R = \frac{1}{2}.[\Phi, \Phi]$ is then also r^g-related to itself for all $g \in G$.

Recall from 15.18.2 that the vertical bundle of P is trivialized as a vector bundle over P by the principal action. So $\omega(X_u) := T_e(r_u)^{-1}.\Phi(X_u) \in \mathfrak{g}$ and in this way we get a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P; \mathfrak{g})$, which is called the *(Lie algebra valued) connection form* of the connection Φ. Recall from 5.13. the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(P)$ for the principal right action. The defining equation for ω can be written also as $\Phi(X_u) = \zeta_{\omega(X_u)}(u)$.

Lemma. *If $\Phi \in \Omega^1(P; VP)$ is a principal connection on the principal fiber bundle (P, p, M, G) then the connection form has the following two properties:*

(1) *ω reproduces the generators of fundamental vector fields, so we have $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$.*
(2) *ω is G-equivariant, $((r^g)^*\omega)(X_u) = \omega(T_u(r^g).X_u) = \operatorname{Ad}(g^{-1}).\omega(X_u)$ for all $g \in G$ and $X_u \in T_uP$. Consequently we have for the Lie derivative $\mathcal{L}_{\zeta_X}\omega = -\operatorname{ad}(X).\omega$.*

Conversely a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying (1) *defines a connection Φ on P by $\Phi(X_u) = T_e(r_u).\omega(X_u)$, which is a principal connection if and only if* (2) *is satisfied.*

Proof. (1). $T_e(r_u).\omega(\zeta_X(u)) = \Phi(\zeta_X(u)) = \zeta_X(u) = T_e(r_u).X$. Since $T_e(r_u) : \mathfrak{g} \to V_uP$ is an isomorphism, the result follows.

(2). Both directions follow from

$$\begin{aligned}
T_e(r_{ug}).\omega(T_u(r^g).X_u) &= \zeta_{\omega(T_u(r^g).X_u)}(ug) = \Phi(T_u(r^g).X_u)\\
T_e(r_{ug}).\operatorname{Ad}(g^{-1}).\omega(X_u) &= \zeta_{\operatorname{Ad}(g^{-1}).\omega(X_u)}(ug) = T_u(r^g).\zeta_{\omega(X_u)}(u)\\
&= T_u(r^g).\Phi(X_u) \quad \square
\end{aligned}$$

16.2. Curvature. Let Φ be a principal connection on the principal fiber bundle (P, p, M, G) with connection form $\omega \in \Omega^1(P; \mathfrak{g})$. We already noted in 16.1 that the curvature $R = \frac{1}{2}[\Phi, \Phi]$ is then also G-equivariant, $(r^g)^*R = R$ for all $g \in G$. Since R has vertical values we may again define a $\mathfrak{g}$-valued 2-form $\Omega \in \Omega^2(P; \mathfrak{g})$

by $\Omega(X_u, Y_u) := -T_e(r_u)^{-1}.R(X_u, Y_u)$, which is called the *(Lie algebra-valued) curvature form* of the connection. We also have $R(X_u, Y_u) = -\zeta_{\Omega(X_u,Y_u)}(u)$. We take the negative sign here to get the usual curvature form as in [Kobayashi-Nomizu I, 1963].

We equip the space $\Omega(P;\mathfrak{g})$ of all $\mathfrak{g}$-valued forms on P in a canonical way with the structure of a graded Lie algebra by

$$\begin{aligned}
&[\Psi,\Theta]_\wedge(X_1,\dots,X_{p+q}) = \\
&\qquad = \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}\sigma\,[\Psi(X_{\sigma 1},\dots,X_{\sigma p}),\Theta(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]_{\mathfrak{g}}
\end{aligned}$$

or equivalently by $[\psi\otimes X,\theta\otimes Y]_\wedge := \psi\wedge\theta\otimes[X,Y]_{\mathfrak{g}}$. From the latter description it is clear that $d[\Psi,\Theta]_\wedge = [d\Psi,\Theta]_\wedge + (-1)^{\deg\Psi}[\Psi, d\Theta]_\wedge$. In particular for $\omega\in\Omega^1(P;\mathfrak{g})$ we have $[\omega,\omega]_\wedge(X,Y) = 2[\omega(X),\omega(Y)]_{\mathfrak{g}}$.

Theorem. *The curvature form Ω of a principal connection with connection form ω has the following properties:*

(1) *Ω is horizontal, i.e. it kills vertical vectors.*
(2) *Ω is G-equivariant in the following sense: $(r^g)^*\Omega = \operatorname{Ad}(g^{-1}).\Omega$. Consequently $\mathcal{L}_{\zeta_X}\Omega = -\operatorname{ad}(X).\Omega$.*
(3) *The Maurer-Cartan formula holds: $\Omega = d\omega + \frac{1}{2}[\omega,\omega]_\wedge$.*

Proof. (1) is true for R by 14.4. For (2) we compute as follows:

$$\begin{aligned}
T_e(r_{ug}).((r^g)^*\Omega)(X_u,Y_u) &= T_e(r_{ug}).\Omega(T_u(r^g).X_u, T_u(r^g).Y_u) = \\
&= -R_{ug}(T_u(r^g).X_u, T_u(r^g).Y_u) = -T_u(r^g).((r^g)^*R)(X_u,Y_u) = \\
&= -T_u(r^g).R(X_u,Y_u) = T_u(r^g).\zeta_{\Omega(X_u,Y_u)}(u) = \\
&= \zeta_{\operatorname{Ad}(g^{-1}).\Omega(X_u,Y_u)}(ug) = T_e(r_{ug}).\operatorname{Ad}(g^{-1}).\Omega(X_u,Y_u), \quad \text{by 5.13.}
\end{aligned}$$

(3). For $X\in\mathfrak{g}$ we have $i_{\zeta_X}R = 0$ by (1), and using 16.1.(3) we get

$$\begin{aligned}
i_{\zeta_X}(d\omega + \frac{1}{2}[\omega,\omega]_\wedge) &= i_{\zeta_X}d\omega + \frac{1}{2}[i_{\zeta_X}\omega,\omega]_\wedge - \frac{1}{2}[\omega, i_{\zeta_X}\omega]_\wedge = \\
&= \mathcal{L}_{\zeta_X}\omega + [X,\omega]_\wedge = -\operatorname{ad}(X)\omega + \operatorname{ad}(X)\omega = 0
\end{aligned}$$

So the formula holds for vertical vectors, and for horizontal vector fields $X, Y\in C^\infty(H(P))$ we have

$$R(X,Y) = \Phi[X-\Phi X, Y-\Phi Y] = \Phi[X,Y] = \zeta_{\omega([X,Y])}$$

$$(d\omega + \frac{1}{2}[\omega,\omega])(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = -\omega([X,Y]) \quad \square$$

16.3. Lemma. *Any principal fiber bundle (P,p,M,G) (with paracompact basis) admits principal connections.*

Proof. Let $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)_\alpha$ be a principal fiber bundle atlas. Let us define $\gamma_\alpha(T\varphi_\alpha^{-1}(\xi_x, T_e\mu_g.X)) := X$ for $\xi_x \in T_xU_\alpha$ and $X \in \mathfrak{g}$. An easy computation involving lemma 5.13 shows that $\gamma_\alpha \in \Omega^1(P|U_\alpha;\mathfrak{g})$ satisfies the requirements of lemma 16.1 and thus is a principal connection on $P|U_\alpha$. Now let (f_α) be a smooth partition of unity on M which is subordinated to the open cover (U_α), and let $\omega := \sum_\alpha (f_\alpha \circ p)\gamma_\alpha$. Since both requirements of lemma 16.1 are invariant under convex linear combinations, ω is a principal connection on P. $\square$

16.4. Local descriptions of principal connections. We consider a principal fiber bundle (P,p,M,G) with some principal fiber bundle atlas $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$ and corresponding cocycle $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$ of transition functions. We consider the sections $s_\alpha \in C^\infty(P|U_\alpha)$ which are given by $\varphi_\alpha(s_\alpha(x)) = (x,e)$ and satisfy $s_\alpha.\varphi_{\alpha\beta} = s_\beta$, since we have in turn:

$$\begin{aligned}\varphi_\alpha(s_\beta(x)) &= \varphi_\alpha\varphi_\beta^{-1}(x,e) = (x, \varphi_{\alpha\beta}(x))\\ s_\beta(x) &= \varphi_\alpha^{-1}(x, e\varphi_{\alpha\beta}(e)), = \varphi_\alpha^{-1}(x,e)\varphi_{\alpha\beta}(x) = s_\alpha(x)\varphi_{\alpha\beta}(x).\end{aligned}$$

(1) Let $\Theta \in \Omega^1(G,\mathfrak{g})$ be the left logarithmic derivative of the identity, i.e. $\Theta(\eta_g) := T_g(\mu_{g^{-1}}).\eta_g$. We will use the forms $\Theta_{\alpha\beta} := {\varphi_{\alpha\beta}}^*\Theta \in \Omega^1(U_{\alpha\beta};\mathfrak{g})$.

Let $\Phi = \zeta \circ \omega \in \Omega^1(P;VP)$ be a principal connection with connection form $\omega \in \Omega^1(P;\mathfrak{g})$. We may associate the following local data to the connection:

(2) $\omega_\alpha := {s_\alpha}^*\omega \in \Omega^1(U_\alpha;\mathfrak{g})$, the physicists version of the connection.
(3) The Christoffel forms $\Gamma^\alpha \in \Omega^1(U_\alpha;\mathfrak{X}(G))$ from 14.7, which are given by $(0_x, \Gamma^\alpha(\xi_x,g)) = -T(\varphi_\alpha).\Phi.T(\varphi_\alpha)^{-1}(\xi_x,0_g)$.
(4) $\gamma_\alpha := (\varphi_\alpha^{-1})^*\omega \in \Omega^1(U_\alpha \times G;\mathfrak{g})$, the local expressions of ω.

Lemma. *These local data have the following properties and are related by the following formulas.*

(5) *The forms $\omega_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ satisfy the transition formulas*

$$\omega_\alpha = \operatorname{Ad}(\varphi_{\beta\alpha}^{-1})\omega_\beta + \Theta_{\beta\alpha},$$

and any set of forms like that with this transition behavior determines a unique principal connection.
(6) *We have $\gamma_\alpha(\xi_x, T\mu_g.X) = \gamma_\alpha(\xi_x,0_g) + X = \operatorname{Ad}(g^{-1})\omega_\alpha(\xi_x) + X$.*
(7) *We have $\Gamma^\alpha(\xi_x,g) = -T_e(\mu_g).\gamma_\alpha(\xi_x,0_g) = -T_e(\mu_g).\operatorname{Ad}(g^{-1})\omega_\alpha(\xi_x) = -T(\mu^g)\omega_\alpha(\xi_x)$, so $\Gamma^\alpha(\xi_x) = R_{\omega_\alpha(\xi_x)}$, a right invariant vector field.*

Proof. From the definition of the Christoffel forms we have

$$\begin{aligned}
(0_x, \Gamma^\alpha(\xi_x, g)) &= -T(\varphi_\alpha).\Phi.T(\varphi_\alpha)^{-1}(\xi_x, 0_g)\\
&= -T(\varphi_\alpha).T_e(r_{\varphi_\alpha^{-1}(x,g)})\omega.T(\varphi_\alpha)^{-1}(\xi_x, 0_g)\\
&= -T_e(\varphi_\alpha \circ r_{\varphi_\alpha^{-1}(x,g)})\omega.T(\varphi_\alpha)^{-1}(\xi_x, 0_g)\\
&= -(0_x, T_e(\mu_g)\omega.T(\varphi_\alpha)^{-1}(\xi_x, 0_g)) = -(0_x, T_e(\mu_g)\gamma_\alpha(\xi_x, 0_g)).
\end{aligned}$$

This is the first part of (7). The second part follows from (6).

$$\begin{aligned}
\gamma_\alpha(\xi_x, T\mu_g.X) &= \gamma_\alpha(\xi_x, 0_g) + \gamma_\alpha(0_x, T\mu_g.X)\\
&= \gamma_\alpha(\xi_x, 0_g) + \omega(T(\varphi_\alpha)^{-1}(0_x, T\mu_g.X))\\
&= \gamma_\alpha(\xi_x, 0_g) + \omega(\zeta_X(\varphi_\alpha^{-1}(x, g)))\\
&= \gamma_\alpha(\xi_x, 0_g) + X.
\end{aligned}$$

So the first part of (6) holds. The second part is seen from

$$\begin{aligned}
\gamma_\alpha(\xi_x, 0_g) &= \gamma_\alpha(\xi_x, T_e(\mu^g)0_e) = (\omega \circ T(\varphi_\alpha)^{-1} \circ T(Id_X \times \mu^g))(\xi_x, 0_e) =\\
&= (\omega \circ T(r^g \circ \varphi_\alpha^{-1}))(\xi_x, 0_e) = \mathrm{Ad}(g^{-1})\omega(T(\varphi_\alpha^{-1})(\xi_x, 0_e))\\
&= \mathrm{Ad}(g^{-1})({s_\alpha}^*\omega)(\xi_x) = \mathrm{Ad}(g^{-1})\omega_\alpha(\xi_x).
\end{aligned}$$

Via (7) the transition formulas for the ω_α are easily seen to be equivalent to the transition formulas for the Christoffel forms in lemma 14.7. A direct proof goes as follows: We have $s_\alpha(x) = s_\beta(x)\varphi_{\beta\alpha}(x) = r(s_\beta(x), \varphi_{\beta\alpha}(x))$ and thus

$$\begin{aligned}
\omega_\alpha(\xi_x) &= \omega(T_x(s_\alpha).\xi_x)\\
&= (\omega \circ T_{(s_\beta(x), \varphi_{\beta\alpha}(x))} r)((T_x s_\beta.\xi_x, 0_{\varphi_{\beta\alpha}(x)}) - (0_{s_\beta}(x), T_x\varphi_{\beta\alpha}.\xi_x))\\
&= \omega(T(r^{\varphi_{\beta\alpha}(x)}).T_x(s_\beta).\xi_x) + \omega(T_{\varphi_{\beta\alpha}(x)}(r_{s_\beta(x)}).T_x(\varphi_{\beta\alpha}).\xi_x)\\
&= \mathrm{Ad}(\varphi_{\beta\alpha}(x)^{-1})\omega(T_x(s_\beta).\xi_x)\\
&\qquad + \omega(T_{\varphi_{\beta\alpha}(x)}(r_{s_\beta(x)}).T(\mu_{\varphi_{\beta\alpha}(x)} \circ \mu_{\varphi_{\beta\alpha}(x)^{-1}})T_x(\varphi_{\beta\alpha}).\xi_x)\\
&= \mathrm{Ad}(\varphi_{\beta\alpha}(x)^{-1})\omega_\beta(\xi_x)\\
&\qquad + \omega(T_e(r_{s_\beta(x)\varphi_{\beta\alpha}(x)}).\Theta_{\beta\alpha}.\xi_x)\\
&= \mathrm{Ad}(\varphi_{\beta\alpha}(x)^{-1})\omega_\beta(\xi_x) + \Theta_{\beta\alpha}(\xi_x). \quad \square
\end{aligned}$$

16.5. The covariant derivative. Let (P,p,M,G) be a principal fiber bundle with principal connection $\Phi = \zeta \circ \omega$. We consider the horizontal projection $\chi = Id_{TP} - \Phi : TP \to HP$, cf. 14.3, which satisfies $\chi \circ \chi = \chi$, $\operatorname{im} \chi = HP$, $\ker \chi = VP$, and $\chi \circ T(r^g) = T(r^g) \circ \chi$ for all $g \in G$.

If W is a finite dimensional vector space, we consider the mapping $\chi^* : \Omega(P;W) \to \Omega(P;W)$ which is given by

$$(\chi^*\varphi)_u(X_1,\dots,X_k) = \varphi_u(\chi(X_1),\dots,\chi(X_k)).$$

The mapping χ^* is a projection onto the subspace of *horizontal differential forms*, i.e. the space $\Omega_{hor}(P;W) := \{\psi \in \Omega(P;W) : i_X\psi = 0 \text{ for } X \in VP\}$. The notion of horizontal form is independent of the choice of a connection.

The projection χ^* has the following properties: $\chi^*(\varphi \wedge \psi) = \chi^*\varphi \wedge \chi^*\psi$ if one of the two forms has values in $\mathbb{R}$; $\chi^* \circ \chi^* = \chi^*$; $\chi^* \circ (r^g)^* = (r^g)^* \circ \chi^*$ for all $g \in G$; $\chi^*\omega = 0$; and $\chi^* \circ \mathcal{L}(\zeta_X) = \mathcal{L}(\zeta_X) \circ \chi^*$. They follow easily from the corresponding properties of χ, the last property uses that $\mathrm{Fl}_t^{\zeta(X)} = r^{\exp tX}$.

We define the *covariant exterior derivative* $d_\omega : \Omega^k(P;W) \to \Omega^{k+1}(P;W)$ by the prescription $d_\omega := \chi^* \circ d$.

Theorem. *The covariant exterior derivative d_ω has the following properties.*

(1) $d_\omega(\varphi \wedge \psi) = d_\omega(\varphi) \wedge \chi^*\psi + (-1)^{\deg \varphi}\chi^*\varphi \wedge d_\omega(\psi)$ *if φ or ψ is real valued.*
(2) $\mathcal{L}(\zeta_X) \circ d_\omega = d_\omega \circ \mathcal{L}(\zeta_X)$ *for each $X \in \mathfrak{g}$.*
(3) $(r^g)^* \circ d_\omega = d_\omega \circ (r^g)^*$ *for each $g \in G$.*
(4) $d_\omega \circ p^* = d \circ p^* = p^* \circ d : \Omega(M;W) \to \Omega_{hor}(P;W)$.
(5) $d_\omega\omega = \Omega$, *the curvature form.*
(6) $d_\omega\Omega = 0$, *the Bianchi identity.*
(7) $d_\omega \circ \chi^* - d_\omega = \chi^* \circ i(R)$, *where R is the curvature.*
(8) $d_\omega \circ d_\omega = \chi^* \circ i(R) \circ d$.
(9) *Let $\Omega_{\mathrm{hor}}(P,\mathfrak{g})^G$ be the algebra of all horizontal G-equivariant $\mathfrak{g}$-valued forms, i.e. $(r^g)^*\psi = Ad(g^{-1})\psi$. Then for any $\psi \in \Omega_{\mathrm{hor}}(P,\mathfrak{g})^G$ we have $d_\omega\psi = d\psi + [\omega,\psi]_\wedge$.*
(10) *The mapping $\psi \mapsto \zeta_\psi$, where $\zeta_\psi(X_1,\dots,X_k)(u) = \zeta_{\psi(X_1,\dots,X_k)(u)}(u)$, is an isomorphism between $\Omega_{\mathrm{hor}}(P,\mathfrak{g})^G$ and the algebra $\Omega_{\mathrm{hor}}(P,VP)^G$ of all horizontal G-equivariant forms with values in the vertical bundle VP. Then we have $\zeta_{d_\omega\psi} = -[\Phi,\zeta_\psi]$.*

Proof. (1) through (4) follow from the properties of χ^*.

(5) We have

$$\begin{aligned}
(d_\omega\omega)(\xi,\eta) &= (\chi^* d\omega)(\xi,\eta) = d\omega(\chi\xi,\chi\eta)\\
&= (\chi\xi)\omega(\chi\eta) - (\chi\eta)\omega(\chi\xi) - \omega([\chi\xi,\chi\eta])\\
&= -\omega([\chi\xi,\chi\eta]) \text{ and}\\
-\zeta(\Omega(\xi,\eta)) &= R(\xi,\eta) = \Phi[\chi\xi,\chi\eta] = \zeta_{\omega([\chi\xi,\chi\eta])}.
\end{aligned}$$

(6) Using 16.2 we have

$$\begin{aligned}
d_\omega\Omega &= d_\omega(d\omega + \tfrac12[\omega,\omega]_\wedge)\\
&= \chi^* dd\omega + \tfrac12\chi^* d[\omega,\omega]_\wedge\\
&= \tfrac12\chi^*([d\omega,\omega]_\wedge - [\omega,d\omega]_\wedge) = \chi^*[d\omega,\omega]_\wedge\\
&= [\chi^* d\omega,\chi^*\omega]_\wedge = 0, \text{ since } \chi^*\omega = 0.
\end{aligned}$$

(7) For $\varphi \in \Omega(P;W)$ we have

$$\begin{aligned}
(d_\omega\chi^*\varphi)&(X_0,\dots,X_k) = (d\chi^*\varphi)(\chi(X_0),\dots,\chi(X_k))\\
&= \sum_{0\le i\le k}(-1)^i\chi(X_i)((\chi^*\varphi)(\chi(X_0),\dots,\widehat{\chi(X_i)},\dots,\chi(X_k)))\\
&\quad+\sum_{i<j}(-1)^{i+j}(\chi^*\varphi)([\chi(X_i),\chi(X_j)],\chi(X_0),\dots\\
&\qquad\qquad\dots,\widehat{\chi(X_i)},\dots,\widehat{\chi(X_j)},\dots)\\
&= \sum_{0\le i\le k}(-1)^i\chi(X_i)(\varphi(\chi(X_0),\dots,\widehat{\chi(X_i)},\dots,\chi(X_k)))\\
&\quad+\sum_{i<j}(-1)^{i+j}\varphi([\chi(X_i),\chi(X_j)]-\Phi[\chi(X_i),\chi(X_j)],\chi(X_0),\dots\\
&\qquad\qquad\dots,\widehat{\chi(X_i)},\dots,\widehat{\chi(X_j)},\dots)\\
&= (d\varphi)(\chi(X_0),\dots,\chi(X_k)) + (i_R\varphi)(\chi(X_0),\dots,\chi(X_k))\\
&= (d_\omega + \chi^* i_R)(\varphi)(X_0,\dots,X_k).
\end{aligned}$$

(8) $d_\omega d_\omega = \chi^* d\chi^* d = (\chi^* i_R + \chi^* d)d = \chi^* i_R d$ holds by (7).

(9) If we insert one vertical vector field, say ζ_X for $X \in \mathfrak{g}$, into $d_\omega\psi$, we get 0 by definition. For the right hand side we use $i_{\zeta_X}\psi = 0$ and $\mathcal{L}_{\zeta_X}\psi = \frac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^{\zeta_X}_t)^*\psi = \frac{\partial}{\partial t}\big|_0 (r^{\exp tX}) * \psi = \frac{\partial}{\partial t}\big|_0 \mathrm{Ad}(\exp(-tX))\psi = -ad(X)\psi$ to get

$$\begin{aligned}
i_{\zeta_X}(d\psi + [\omega,\psi]_\wedge) &= i_{\zeta_X}d\psi + di_{\zeta_X}\psi + [i_{\zeta_X}\omega,\psi] - [\omega, i_{\zeta_X}\psi]\\
&= \mathcal{L}_{\zeta_X}\psi + [X,\psi] = -ad(X)\psi + [X,\psi] = 0.
\end{aligned}$$

Let now all vector fields ξ_i be horizontal, then we get

$$\begin{aligned}(d_\omega\psi)(\xi_0,\dots,\xi_k) = (\chi^* d\psi)(\xi_0,\dots,\xi_k) = d\psi(\xi_0,\dots,\xi_k),\\ (d\psi + [\omega,\psi]_\wedge)(\xi_0,\dots,\xi_k) = d\psi(\xi_0,\dots,\xi_k).\end{aligned}$$

So the first formula holds.

(10) We proceed in a similar manner. Let Ψ be in the space $\Omega^\ell_{\text{hor}}(P,VP)^G$ of all horizontal G-equivariant forms with vertical values. Then for each $X \in \mathfrak{g}$ we have $i_{\zeta_X}\Psi = 0$; furthermore the G-equivariance $(r^g)^*\Psi = \Psi$ implies that $\mathcal{L}_{\zeta_X}\Psi = [\zeta_X,\Psi] = 0$ by 13.16.(5). Using formula 13.11.(2) we have

$$\begin{aligned}i_{\zeta_X}[\Phi,\Psi] &= [i_{\zeta_X}\Phi,\Psi] - [\Phi, i_{\zeta_X}\Psi] + i([\Phi,\zeta_X])\Psi + i([\Psi,\zeta_X])\Phi\\ &= [\zeta_X,\Psi] - 0 + 0 + 0 = 0.\end{aligned}$$

Let now all vector fields ξ_i again be horizontal, then from the huge formula 13.9 for the Frölicher-Nijenhuis bracket only the following terms in the third and fifth line survive:

$$\begin{aligned}&[\Phi,\Psi](\xi_1,\dots,\xi_{\ell+1}) =\\ &\quad= \tfrac{(-1)^\ell}{\ell!}\sum_\sigma \operatorname{sign}\sigma\ \Phi([\Psi(\xi_{\sigma 1},\dots,\xi_{\sigma\ell}),\xi_{\sigma(\ell+1)}])\\ &\qquad + \tfrac{1}{(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\ \Phi(\Psi([\xi_{\sigma 1},\xi_{\sigma 2}],\xi_{\sigma 3},\dots,\xi_{\sigma(\ell+1)}).\end{aligned}$$

For $f : P \to \mathfrak{g}$ and horizontal ξ we have $\Phi[\xi,\zeta_f] = \zeta_{\xi(f)} = \zeta_{df(\xi)}$: It is $C^\infty(P,\mathbb{R})$-linear in ξ; or imagine it in local coordinates. So the last expression becomes

$$-\zeta(d\psi(\xi_0,\dots,\xi_k)) = -\zeta(d\psi(\xi_0,\dots,\xi_k)) = -\zeta((d\psi + [\omega,\psi]_\wedge)(\xi_0,\dots,\xi_k))$$

as required. $\square$

16.6. Theorem. *Let (P,p,M,G) be a principal fiber bundle with principal connection ω. Then the parallel transport for the principal connection is globally defined and G-equivariant.*

In detail: For each smooth curve $c : \mathbb{R} \to M$ there is a smooth mapping $\operatorname{Pt}_c : \mathbb{R} \times P_{c(0)} \to P$ such that the following holds:

1. $\operatorname{Pt}(c,t,u) \in P_{c(t)}$, $\operatorname{Pt}(c,0) = Id_{P_{c(0)}}$, *and* $\omega(\frac{d}{dt}\operatorname{Pt}(c,t,u)) = 0$.
2. $\operatorname{Pt}(c,t) : P_{c(0)} \to P_{c(t)}$ *is G-equivariant, i.e.* $\operatorname{Pt}(c,t,u.g) = \operatorname{Pt}(c,t,u).g$ *holds for all $g \in G$ and $u \in P$. Moreover we have* $\operatorname{Pt}(c,t)^*(\zeta_X|P_{c(t)}) = \zeta_X|P_{c(0)}$ *for all $X \in \mathfrak{g}$.*
3. *For any smooth function $f : \mathbb{R} \to \mathbb{R}$ we have*
$\operatorname{Pt}(c,f(t),u) = \operatorname{Pt}(c\circ f,t,\operatorname{Pt}(c,f(0),u))$.

Proof. By 16.4 the Christoffel forms $\Gamma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(G))$ of the connection ω with respect to a principal fiber bundle atlas $(U_\alpha, \varphi_\alpha)$ are given by $\Gamma^\alpha(\xi_x) = R_{\omega_\alpha(\xi_x)}$, so they take values in the Lie subalgebra $\mathfrak{X}_R(G)$ of all right invariant vector fields on G, which are bounded with respect to any right invariant Riemannian metric on G. Each right invariant metric on a Lie group is complete. So the connection is complete by the remark in 14.9.

Properties (1) and (3) follow from theorem 14.8, and (2) is seen as follows: $\omega(\frac{d}{dt}\operatorname{Pt}(c,t,u).g) = \operatorname{Ad}(g^{-1})\omega(\frac{d}{dt}\operatorname{Pt}(c,t,u)) = 0$ implies that $\operatorname{Pt}(c,t,u).g = \operatorname{Pt}(c,t,u.g)$. For the second assertion we compute for $u \in P_{c(0)}$:

$$\begin{aligned}\operatorname{Pt}(c,t)^*(\zeta_X|P_{c(t)})(u) &= T\operatorname{Pt}(c,t)^{-1}\zeta_X(\operatorname{Pt}(c,t,u))\\ &= T\operatorname{Pt}(c,t)^{-1}\tfrac{d}{ds}|_0\operatorname{Pt}(c,t,u).\exp(sX)\\ &= T\operatorname{Pt}(c,t)^{-1}\tfrac{d}{ds}|_0\operatorname{Pt}(c,t,u.\exp(sX))\\ &= \tfrac{d}{ds}|_0\operatorname{Pt}(c,t)^{-1}\operatorname{Pt}(c,t,u.\exp(sX))\\ &= \tfrac{d}{ds}|_0 u.\exp(sX) = \zeta_X(u). \quad\square\end{aligned}$$

16.7. Holonomy groups. Let (P,p,M,G) be a principal fiber bundle with principal connection $\Phi = \zeta\circ\omega$. We assume that M is connected and we fix $x_0 \in M$.

In 14.10 we defined the holonomy group $\operatorname{Hol}(\Phi,x_0) \subset \operatorname{Diff}(P_{x_0})$ as the group of all $\operatorname{Pt}(c,1) : P_{x_0} \to P_{x_0}$ for c any piecewise smooth closed loop through x_0. (Reparametrizing c by a function which is flat at each corner of c we may assume that any c is smooth.) If we consider only those curves c which are nullhomotopic, we obtain the restricted holonomy group $\operatorname{Hol}_0(\Phi,x_0)$, a normal subgroup.

Now let us fix $u_0 \in P_{x_0}$. The elements $\tau(u_0,\operatorname{Pt}(c,t,u_0)) \in G$ form a subgroup of the structure group G which is isomorphic to $\operatorname{Hol}(\Phi,x_0)$; we denote it by $\operatorname{Hol}(\omega,u_0)$ and we call it also the *holonomy group* of the connection. Considering only nullhomotopic curves we get the *restricted holonomy group* $\operatorname{Hol}_0(\omega,u_0)$ a normal subgroup of $\operatorname{Hol}(\omega,u_0)$.

Theorem. *1. We have* $\operatorname{Hol}(\omega,u_0.g) = \operatorname{conj}(g^{-1})\operatorname{Hol}(\omega,u_0)$ *and* $\operatorname{Hol}_0(\omega,u_0.g) = \operatorname{conj}(g^{-1})\operatorname{Hol}_0(\omega,u_0)$.

2. For each curve c in M with $c(0) = x_0$ we have $\operatorname{Hol}(\omega,\operatorname{Pt}(c,t,u_0)) = \operatorname{Hol}(\omega,u_0)$ *and* $\operatorname{Hol}_0(\omega,\operatorname{Pt}(c,t,u_0)) = \operatorname{Hol}_0(\omega,u_0)$.

3. $\operatorname{Hol}_0(\omega,u_0)$ *is a connected Lie subgroup of G and the quotient group* $\operatorname{Hol}(\omega,u_0)/\operatorname{Hol}_0(\omega,u_0)$ *is at most countable, so* $\operatorname{Hol}(\omega,u_0)$ *is also a Lie subgroup of G.*

4. The Lie algebra $\operatorname{hol}(\omega,u_0) \subset \mathfrak{g}$ *of* $\operatorname{Hol}(\omega,u_0)$ *is linearly generated by* $\{\Omega(X_u,Y_u) : X_u,Y_u \in T_uP\}$. *It is isomorphic to the Lie algebra* $\operatorname{hol}(\Phi,x_0)$ *we considered in 14.10.*

5. For $u_0 \in P_{x_0}$ let $P(\omega, u_0)$ be the set of all $\operatorname{Pt}(c, t, u_0)$ for c any (piecewise) smooth curve in M with $c(0) = x_0$ and for $t \in \mathbb{R}$. Then $P(\omega, u_0)$ is a sub fiber bundle of P which is invariant under the right action of $\operatorname{Hol}(\omega, u_0)$; so it is itself a principal fiber bundle over M with structure group $\operatorname{Hol}(\omega, u_0)$ and we have a reduction of structure group, cf. 15.6 and 15.14. The pullback of ω to $P(\omega, u_0)$ is then again a principal connection form $i^*\omega \in \Omega^1(P(\omega, u_0); \operatorname{hol}(\omega, u_0))$.

6. P is foliated by the leaves $P(\omega, u)$, $u \in P_{x_0}$.

7. If the curvature $\Omega = 0$ then $\operatorname{Hol}_0(\omega, u_0) = \{e\}$ and each $P(\omega, u)$ is a covering of M. They are all isomorphic and are associated to the universal covering of M, which is a principal fiber bundle with structure group the fundamental group $\pi_1(M)$.

In view of assertion 5 a principal connection ω is called *irreducible *-principle connection* if $\operatorname{Hol}(\omega, u_0)$ equals the structure group G for some (equivalently any) $u_0 \in P_{x_0}$.

Proof. 1. This follows from the properties of the mapping τ from 15.2 and from the from the G-equivariance of the parallel transport:

$$\tau(u_0.g, \operatorname{Pt}(c, 1, u_0.g)) = \tau(u_0, \operatorname{Pt}(c, 1, u_0).g) = g^{-1}.\tau(u_0, \operatorname{Pt}(c, 1, u_0)).g.$$

Note that we have an isomorphism

$$\begin{aligned}
\operatorname{Hol}(\omega, u_0) &\to \operatorname{Hol}(\Phi, x_0) \\
g &\mapsto (u \mapsto f_g(u) = u_0.g.\tau(u_0, u)) \\
g_f := \tau(u_0, f(u_0)) &\leftarrow f.
\end{aligned}$$

So via the diffeomorphism $\tau(u_0, \quad) : P_{x_0} \to G$ the action of the holonomy group $\operatorname{Hol}(\Phi, u_0)$ on P_{x_0} is conjugate to the left translation of $\operatorname{Hol}(\omega, u_0)$ on G.

2. By reparameterizing the curve c we may assume that $t = 1$, and we put $\operatorname{Pt}(c, 1, u_0) =: u_1$. Then by definition for an element $g \in G$ we have $g \in \operatorname{Hol}(\omega, u_1)$ if and only if $g = \tau(u_1, \operatorname{Pt}(e, 1, u_1))$ for some closed smooth loop e through $x_1 := c(1) = p(u_1)$, i. e.

$$\begin{aligned}
\operatorname{Pt}(c, 1)(r^g(u_0)) = r^g(\operatorname{Pt}(c, 1)(u_0)) = u_1 g = \operatorname{Pt}(e, 1)(\operatorname{Pt}(c, 1)(u_0)) \\
u_0 g = \operatorname{Pt}(c, 1)^{-1} \operatorname{Pt}(e, 1) \operatorname{Pt}(c, 1)(u_0) = \operatorname{Pt}(c.e.c^{-1}, 3)(u_0),
\end{aligned}$$

where $c.e.c^{-1}$ is the curve travelling along $c(t)$ for $0 \le t \le 1$, along $e(t-1)$ for $1 \le t \le 3$, and along $c(3-t)$ for $2 \le t \le 3$. This is equivalent to $g \in \operatorname{Hol}(\omega, u_0)$. Furthermore e is nullhomotopic if and only if $c.e.c^{-1}$ is nullhomotopic, so we also have $\operatorname{Hol}_0(\omega, u_1) = \operatorname{Hol}_0(\omega, u_0)$.

3. Let $c:[0,1]\to M$ be a nullhomotopic curve through x_0 and let $h:\mathbb{R}^2\to M$ be a smooth homotopy with $h_1|[0,1]=c$ and $h(0,s)=h(t,0)=h(t,1)=x_0$. We consider the pullback bundle

$$\begin{tikzcd} h^*P \arrow[r, "p^*h"] \arrow[d, "h^*p"'] & P \arrow[d, "p"] \\ \mathbb{R}^2 \arrow[r, "h"] & M. \end{tikzcd}$$

Then for the parallel transport Pt^Φ on P and for the parallel transport $\operatorname{Pt}^{h^*\Phi}$ of the pulled back connection we have

$$\operatorname{Pt}^\Phi(h_t,1,u_0)=(p^*h)\operatorname{Pt}^{h^*\Phi}((t,\quad),1,u_0)=(p^*h)\operatorname{Fl}_1^{C^{h^*\Phi}\partial_s}(t,u_0).$$

So $t\mapsto\tau(u_0,\operatorname{Pt}^\Phi(h_t,1,u_0))$ is a smooth curve in the Lie group G starting from e, so $\operatorname{Hol}_0(\omega,u_0)$ is an arcwise connected subgroup of G. By the theorem of Yamabe (which we mentioned without proof in 5.6) the subgroup $\operatorname{Hol}_0(\omega,u_0)$ is a Lie subgroup of G. The quotient group $\operatorname{Hol}(\omega,u_0)/\operatorname{Hol}_0(\omega,u_0)$ is a countable group, since by Morse theory M is homotopy equivalent to a countable CW-complex, so the fundamental group $\pi_1(M)$ is countably generated, thus countable.

4. Note first that for $g\in G$ and $X\in\mathfrak{X}(M)$ we have for the horizontal lift $(r^g)^*CX=CX$, since $(r^g)^*\Phi=\Phi$ implies $T_u(r^g).H_uP=H_{u.g}P$ and thus

$$\begin{aligned}T_u(r^g).C(X,u)&=T_u(r^g).(T_up|H_uP)^{-1}(X(p(u)))\\&=(T_{u.g}p|H_{u.g}P)^{-1}(X(p(u)))=C(X,u.g).\end{aligned}$$

Thus $\operatorname{hol}(\omega)$ is an ideal in the Lie algebra $\mathfrak{g}$, since

$$\begin{aligned}\operatorname{Ad}(g^{-1})\Omega(C(X,u),C(Y,u))&=\Omega(T_u(r^g).C(X,u),T_u(r^g).C(Y,u))\\&=\Omega(C(X,u.g),C(Y,u.g))\in\operatorname{hol}(\omega).\end{aligned}$$

We consider now the mapping

$$\begin{aligned}\xi^{u_0}&:\operatorname{hol}(\omega)\to\mathfrak{X}(P_{x_0})\\\xi_X^{u_0}(u)&=\zeta_{\operatorname{Ad}(\tau(u_0,u)^{-1})X}(u).\end{aligned}$$

It turns out that $\xi_X^{u_0}$ is related to the right invariant vector field R_X on G under the diffeomorphism $\tau(u_0,\quad)=(r_{u_0})^{-1}:P_{x_0}\to G$, since we have

$$\begin{aligned}T_g(r_{u_0}).R_X(g)&=T_g(r_{u_0}).T_e(\mu^g).X=T_{u_0}(r^g).T_e(r_{u_0}).X\\&=T_{u_0}(r^g)\zeta_X(u_0)=\zeta_{\operatorname{Ad}(g^{-1})X}(u_0.g)=\xi_X^{u_0}(u_0.g).\end{aligned}$$

Thus ξ^{u_0} is a Lie algebra anti homomorphism, and each vector field $\xi^{u_0}_X$ on P_{x_0} is complete. The dependence of ξ^{u_0} on u_0 is explained by

$$\begin{aligned}\xi^{u_0 g}_X(u) &= \zeta_{\mathrm{Ad}(\tau(u_0 g,u)^{-1})X}(u) = \zeta_{\mathrm{Ad}(\tau(u_0,u)^{-1})\,\mathrm{Ad}(g)X}(u)\\ &= \xi^{u_0}_{\mathrm{Ad}(g)X}(u).\end{aligned}$$

Recall now that the holonomy Lie algebra $\mathrm{hol}(\Phi, x_0)$ is the closed linear span of all vector fields of the form $\mathrm{Pt}(c,1)^* R(CX,CY)$, where $X,Y \in T_xM$ and c is a curve from x_0 to x. Then we have for $u = \mathrm{Pt}(c,1,u_0)$

$$\begin{aligned}R(C(X,u),C(Y,u)) &= \zeta_{\Omega(C(X,u),C(Y,u))}(u)\\ R(CX,CY)(ug) &= T(r^g)R(CX,CY)(u) = T(r^g)\zeta_{\Omega(C(X,u),C(Y,u))}(u)\\ &= \zeta_{\mathrm{Ad}(g^{-1})\Omega(C(X,u),C(Y,u))}(ug) = \xi^u_{\Omega(C(X,u),C(Y,u))}(ug)\\ (\mathrm{Pt}(c,1)^*R(CX,CY))(u_0.g) &=\\ &= T(\mathrm{Pt}(c,1)^{-1})\zeta_{\mathrm{Ad}(g^{-1})\Omega(C(X,u),C(Y,u))}(\mathrm{Pt}(c,1,u_0.g))\\ &= (\mathrm{Pt}(c,1)^*\zeta_{\mathrm{Ad}(g^{-1})\Omega(C(X,u),C(Y,u))})(u_0.g)\\ &= \zeta_{\mathrm{Ad}(g^{-1})\Omega(C(X,u),C(Y,u))}(u_0.g) \qquad \text{by 16.6.(2)}\\ &= \xi^{u_0}_{\Omega(C(X,u),C(Y,u))}(u_0.g).\end{aligned}$$

So $\xi^{u_0} : \mathrm{hol}(\omega) \to \mathrm{hol}(\Phi,x_0)$ is a Lie algebra anti isomorphism. Moreover $\mathrm{hol}(\Phi,x_0)$ consists of complete vector fields and we may apply theorem 14.11 (only claim 3) which tells us that the Lie algebra of the Lie group $\mathrm{Hol}(\Phi,x_0)$ is $\mathrm{hol}(\Phi,x_0)$. The diffeomorphism $\tau(u_0,\quad) : P_{x_0} \to G$ intertwines the actions and the infinitesimal actions in the right way.

5. We define the sub vector bundle $E \subset TP$ by $E_u := H_uP + T_e(r_u).\,\mathrm{hol}(\omega)$. From the proof of 4 it follows that $\xi^{u_0}_X$ are sections of E for each $X \in \mathrm{hol}(\omega)$, thus E is a vector bundle. Any vector field $\eta \in \mathfrak{X}(P)$ with values in E is a linear combination with coefficients in $C^\infty(P,\mathbb{R})$ of horizontal vector fields CX for $X \in \mathfrak{X}(M)$ and of ζ_Z for $Z \in \mathrm{hol}(\omega)$. Their Lie brackets are in turn

$$\begin{aligned}[CX,CY](u) &= C[X,Y](u) + R(CX,CY)(u)\\ &= C[X,Y](u) + \zeta_{\Omega(C(X,u),C(Y,u))}(u) \in C^\infty(E)\\ [\zeta_Z, CX] &= \mathcal{L}_{\zeta_Z}CX = \tfrac{d}{dt}|_0(\mathrm{Fl}^{\zeta_Z}_t)^*CX = 0,\end{aligned}$$

since $(r^g)^*CX = CX$, see step 4 above. So E is an integrable subbundle and induces a foliation by 3.25.2. Let $L(u_0)$ be the leaf of the foliation through u_0. Since for a curve c in M the parallel transport $\mathrm{Pt}(c,t,u_0)$ is tangent to the leaf, we have $P(\omega, u_0) \subseteq L(u_0)$. By definition the holonomy group $\mathrm{Hol}(\Phi, x_0)$ acts transitively and freely on $P(\omega, u_0) \cap P_{x_0}$, and by 4 the restricted holonomy group $\mathrm{Hol}_0(\Phi, x_0)$ acts transitively on each connected component of $L(u_0) \cap P_{x_0}$, since the vertical part of E is spanned by the generating vector fields of this action. This is true for any fiber since we may conjugate the holonomy groups by a suitable parallel transport to each fiber. Thus $P(\omega, u_0) = L(u_0)$ and by lemma 15.2 the sub fiber bundle $P(\omega, x_0)$ is a principal fiber bundle with structure group $\mathrm{Hol}(\omega, u_0)$. Since all horizontal spaces $H_u P$ with $u \in P(\omega, x_0)$ are tangential to $P(\omega, x_0)$, the connection Φ restricts to a principal connection on $P(\omega, x_0)$ and we obtain the looked for reduction of the structure group.

6. This is obvious from the proof of 5.

7. If the curvature Ω is everywhere 0, the holonomy Lie algebra is zero, so $P(\omega, u)$ is a principal fiber bundle with discrete structure group, $p|P(\omega, u) : P(\omega, u) \to M$ is a local diffeomorphism, since $T_u P(\omega, u) = H_u P$ and Tp is invertible on it. By the right action of the structure group we may translate each local section of p to any point of the fiber, so p is a covering map. Parallel transport defines a group homomorphism $\varphi : \pi_1(M, x_0) \to \mathrm{Hol}(\Phi, x_0)$ (see the proof of 3). Let $\tilde M$ be the universal covering space of M, then from topology one knows that $\tilde M \to M$ is a principal fiber bundle with discrete structure group $\pi_1(M, x_0)$. Let $\pi_1(M)$ act on $\mathrm{Hol}(\Phi, x_0)$ by left translation via φ, then the mapping $f : \tilde M \times \mathrm{Hol}(\Phi, x_0) \to P(\omega, u_0)$ which is given by $f([c], g) = \mathrm{Pt}(c, 1, u_0).g$ is $\pi_1(M)$-invariant and thus factors to a mapping $\tilde M[\mathrm{Hol}(\Phi, x_0)] \to P(\omega, u_0)$ which is an isomorphism of $\mathrm{Hol}(\Phi, x_0)$-bundles since the upper mapping admits local sections by the curve lifting property of the universal cover. $\square$

16.8. Inducing principal connections on associated bundles.
Let (P, p, M, G) be a principal bundle with principal right action $r : P \times G \to P$ and let $\ell : G \times S \to S$ be a left action of the structure group G on some manifold S. Then we consider the associated bundle $P[S] = P[S, \ell] = P \times_G S$, constructed in 15.7. Recall from 15.18 that its tangent and vertical bundle are given by $T(P[S, \ell]) = TP[TS, T\ell] = TP \times_{TG} TS$ and $V(P[S, \ell]) = P[TS, T_2\ell] = P \times_G TS$.

Let $\Phi = \zeta \circ \omega \in \Omega^1(P; TP)$ be a principal connection on the principal bundle P. We construct the *induced connection* $\bar\Phi \in \Omega^1(P[S], T(P[S]))$ by factorizing as in the following diagram:

$$\begin{tikzcd}
TP \times TS \arrow[r, "\Phi \times Id"] \arrow[d, "Tq = q'"'] & TP \times TS \arrow[r, "="] \arrow[d, "q'"'] & T(P \times S) \arrow[d, "Tq"'] \\
TP \times_{TG} TS \arrow[r, "\bar\Phi"] & TP \times_{TG} TS \arrow[r, "="] & T(P \times_G S).
\end{tikzcd}$$

Let us first check that the top mapping $\Phi\times Id$ is TG-equivariant. For $g\in G$ and $X\in\mathfrak g$ the inverse of $T_e(\mu_g)X$ in the Lie group TG is denoted by $(T_e(\mu_g)X)^{-1}$, see lemma 5.16. Furthermore by 5.13 we have

$$\begin{aligned}Tr(\xi_u, T_e(\mu_g)X) &= T_u(r^g)\xi_u + Tr((0_P\times L_X)(u,g))\\ &= T_u(r^g)\xi_u + T_g(r_u)(T_e(\mu_g)X)\\ &= T_u(r^g)\xi_u + \zeta_X(ug).\end{aligned}$$

We may compute

$$\begin{aligned}&(\Phi\times Id)(Tr(\xi_u, T_e(\mu_g)X), T\ell((T_e(\mu_g)X)^{-1},\eta_s))\\ &\quad= (\Phi(T_u(r^g)\xi_u + \zeta_X(ug)), T\ell((T_e(\mu_g)X)^{-1},\eta_s))\\ &\quad= (\Phi(T_u(r^g)\xi_u) + \Phi(\zeta_X(ug)), T\ell((T_e(\mu_g)X)^{-1},\eta_s))\\ &\quad= ((T_u(r^g)\Phi\xi_u) + \zeta_X(ug), T\ell((T_e(\mu_g)X)^{-1},\eta_s))\\ &\quad= (Tr(\Phi(\xi_u), T_e(\mu_g)X), T\ell((T_e(\mu_g)X)^{-1},\eta_s)).\end{aligned}$$

So the mapping $\Phi\times Id$ factors to $\bar\Phi$ as indicated in the diagram, and we have $\bar\Phi\circ\bar\Phi=\bar\Phi$ from $(\Phi\times Id)\circ(\Phi\times Id)=\Phi\times Id$. The mapping $\bar\Phi$ is fiberwise linear, since $\Phi\times Id$ and $q'=Tq$ are. The image of $\bar\Phi$ is

$$\begin{aligned}q'(VP\times TS) &= q'(\ker(Tp: TP\times TS\to TM))\\ &= \ker(Tp: TP\times_{TG} TS\to TM) = V(P[S,\ell]).\end{aligned}$$

Thus $\bar\Phi$ is a connection on the associated bundle $P[S]$. We call it the *induced connection*.

From the diagram it also follows, that the vector valued forms $\Phi\times Id\in\Omega^1(P\times S; TP\times TS)$ and $\bar\Phi\in\Omega^1(P[S];T(P[S]))$ are $(q: P\times S\to P[S])$-related. So by 13.15 we have for the curvatures

$$\begin{aligned}R_{\Phi\times Id} &= \tfrac12[\Phi\times Id, \Phi\times Id] = \tfrac12[\Phi,\Phi]\times 0 = R_\Phi\times 0,\\ R_{\bar\Phi} &= \tfrac12[\bar\Phi,\bar\Phi],\end{aligned}$$

that they are also q-related, i.e. $Tq\circ(R_\Phi\times 0) = R_{\bar\Phi}\circ(Tq\times_M Tq)$.

By uniqueness of the solutions of the defining differential equation we also get that

$$\mathrm{Pt}_{\bar\Phi}(c,t,q(u,s)) = q(\mathrm{Pt}_\Phi(c,t,u),s).$$

16.9. Recognizing induced connections. We consider again a principal fiber bundle (P,p,M,G) and a left action $\ell : G\times S\to S$. Suppose that $\Psi\in\Omega^1(P[S];T(P[S]))$ is a connection on the associated bundle $P[S]=P[S,\ell]$. Then the following question arises: When is the connection Ψ induced from a principal connection on P? If this is the case, we say that Ψ is compatible with the G-structure on $P[S]$. The answer is given in the following

Theorem. *Let Ψ be a (general) connection on the associated bundle $P[S]$. Let us suppose that the action ℓ is infinitesimally effective, i.e. the fundamental vector field mapping $\zeta:\mathfrak{g}\to\mathfrak{X}(S)$ is injective.*

Then the connection Ψ is induced from a principal connection ω on P if and only if the following condition is satisfied:

> *In some (equivalently any) fiber bundle atlas (U_α,ψ_α) of $P[S]$ belonging to the G-structure of the associated bundle the Christoffel forms $\Gamma^\alpha\in\Omega^1(U_\alpha;\mathfrak{X}(S))$ have values in the sub Lie algebra $\mathfrak{X}_{fund}(S)$ of fundamental vector fields for the action ℓ.*

Proof. Let $(U_\alpha,\varphi_\alpha : P|U_\alpha\to U_\alpha\times G)$ be a principal fiber bundle atlas for P. Then by the proof of theorem 15.7 the induced fiber bundle atlas $(U_\alpha,\psi_\alpha : P[S]|U_\alpha\to U_\alpha\times S)$ is given by

$$\psi_\alpha^{-1}(x,s)=q(\varphi_\alpha^{-1}(x,e),s), \tag{1}$$

$$(\psi_\alpha\circ q)(\varphi_\alpha^{-1}(x,g),s)=(x,g.s). \tag{2}$$

Let $\Phi=\zeta\circ\omega$ be a principal connection on P and let $\bar\Phi$ be the induced connection on the associated bundle $P[S]$. By 14.7 its Christoffel symbols are given by

$$\begin{aligned}
(0_x,\Gamma^\alpha_{\bar\Phi}(\xi_x,s)) &= -(T(\psi_\alpha)\circ\bar\Phi\circ T(\psi_\alpha^{-1}))(\xi_x,0_s)\\
&= -(T(\psi_\alpha)\circ\bar\Phi\circ Tq\circ(T(\varphi_\alpha^{-1})\times Id))(\xi_x,0_e,0_s) && \text{by (1)}\\
&= -(T(\psi_\alpha)\circ Tq\circ(\Phi\times Id))(T(\varphi_\alpha^{-1})(\xi_x,0_e),0_s) && \text{by 16.8}\\
&= -(T(\psi_\alpha)\circ Tq)(\Phi(T(\varphi_\alpha^{-1})(\xi_x,0_e)),0_s)\\
&= (T(\psi_\alpha)\circ Tq)(T(\varphi_\alpha^{-1})(0_x,\Gamma^\alpha_\Phi(\xi_x,e)),0_s) && \text{by 16.4.(3)}\\
&= -T(\psi_\alpha\circ q\circ(\varphi_\alpha^{-1}\times Id))(0_x,\omega_\alpha(\xi_x),0_s) && \text{by 16.4.(7)}\\
&= -T_e(\ell^s)\omega_\alpha(\xi_x) && \text{by (2)}\\
&= -\zeta_{\omega_\alpha(\xi_x)}(s).
\end{aligned}$$

So the condition is necessary. Now let us conversely suppose that a connection Ψ on $P[S]$ is given such that the Christoffel forms Γ^α_Ψ with respect to a fiber bundle atlas of the G-structure have values in $\mathfrak{X}_{fund}(S)$. Then unique $\mathfrak{g}$-valued forms $\omega_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})$ are given by the equation

$$\Gamma^\alpha_\Psi(\xi_x) = \zeta(\omega_\alpha(\xi_x)),$$

since the action is infinitesimally effective. From the transition formulas 14.7 for the Γ^α_Ψ follow the transition formulas 16.4.(5) for the ω^α, so that they give a unique principal connection on P, which by the first part of the proof induces the given connection Ψ on $P[S]$. $\square$

16.10. Inducing principal connections on associated vector bundles. Let (P, p, M, G) be a principal fiber bundle and let $\rho : G \to GL(W)$ be a representation of the structure group G on a finite dimensional vector space W. We consider the associated vector bundle $(E := P[W, \rho], p, M, W)$, which was treated in some detail in 15.11.

Recall from 6.11 that $T(E) = TP \times_{TG} TW$ has two vector bundle structures with the projections

$$\begin{aligned} \pi_E : T(E) = TP \times_{TG} TW &\to P \times_G W = E, \\ Tp \circ pr_1 : T(E) = TP \times_{TG} TW &\to TM. \end{aligned}$$

Now let $\Phi = \zeta \circ \omega \in \Omega^1(P; TP)$ be a principal connection on P. We consider the induced connection $\bar\Phi \in \Omega^1(E; T(E))$ from 16.8. A look at the diagram below shows that the induced connection is linear in both vector bundle structures. We say that it is a *linear connection* on the associated bundle.

$$\begin{tikzcd}
TP \times TW \arrow[rr, "\Phi \times Id"] \arrow[dr, "\pi"'] \arrow[ddd, "Tq"'] & & TP \times TW \arrow[dl, "\pi"] \arrow[ddd, "Tq"] & TP \times W \times W \\
& P \times W \arrow[d, "q"] & & \\
& P \times_G W = E & & \\
TP \times_{TG} TW \arrow[ur, "\pi_E"] \arrow[rr, "\bar\Phi"'] \arrow[dr, "Tp"'] & & TP \times_{TG} TW \arrow[ul, "\pi_E"'] \arrow[dl, "Tp"] \arrow[r, equal] & TE \\
& TM & &
\end{tikzcd}$$

Recall now from 6.11 the vertical lift $vl_E : E \times_M E \to VE$, which is an isomorphism, pr_1–π_E–fiberwise linear and also p–Tp–fiberwise linear.

Now we define the *connector* K of the linear connection $\bar{\Phi}$ by

$$K := pr_2 \circ (vl_E)^{-1} \circ \bar{\Phi} : TE \to VE \to E \times_M E \to E.$$

Lemma. *The connector $K : TE \to E$ is a vector bundle homomorphism for both vector bundle structures on TE and satisfies $K \circ vl_E = pr_2 : E \times_M E \to TE \to E$.*

So K is π_E–p–fiberwise linear and Tp–p–fiberwise linear.

Proof. This follows from the fiberwise linearity of the composants of K and from its definition. $\square$

16.11. Linear connections. If (E, p, M) is a vector bundle, a connection $\Psi \in \Omega^1(E; TE)$ such that $\Psi : TE \to VE \to TE$ is also Tp–Tp–fiberwise linear is called a *linear connection.* An easy check with 16.9 or a direct construction shows that Ψ is then induced from a unique principal connection on the linear frame bundle $GL(\mathbb{R}^n, E)$ of E (where n is the fiber dimension of E).

Equivalently a linear connection may be specified by a connector $K : TE \to E$ with the three properties of lemma 16.10. For then $HE := \{\xi_u : K(\xi_u) = 0_{p(u)}\}$ is a complement to VE in TE which is Tp–fiberwise linearly chosen.

16.12. Covariant derivative on vector bundles. Let (E, p, M) be a vector bundle with a linear connection, given by a connector $K : TE \to E$ with the properties in lemma 16.10.

For any manifold N, smooth mapping $s : N \to E$, and vector field $X \in \mathfrak{X}(N)$ we define the *covariant derivative* of s along X by

$$\nabla_X s := K \circ Ts \circ X : N \to TN \to TE \to E. \tag{1}$$

If $f : N \to M$ is a fixed smooth mapping, let us denote by $C^\infty_f(N, E)$ the vector space of all smooth mappings $s : N \to E$ with $p \circ s = f$ – they are called sections of E along f. From the universal property of the pullback it follows that the vector space $C^\infty_f(N, E)$ is canonically linearly isomorphic to the space $C^\infty(f^*E)$ of sections of the pullback bundle. Then the covariant derivative may be viewed as a bilinear mapping

$$\nabla : \mathfrak{X}(N) \times C^\infty_f(N, E) \to C^\infty_f(N, E). \tag{2}$$

In particular for $f = Id_M$ we have

$$\nabla : \mathfrak{X}(M) \times C^\infty(E) \to C^\infty(E).$$

Lemma. *This covariant derivative has the following properties:*

(3) $\nabla_X s$ *is* $C^\infty(N,\mathbb{R})$*-linear in* $X \in \mathfrak{X}(N)$*. So for a tangent vector* $X_x \in T_xN$ *the mapping* $\nabla_{X_x} : C^\infty_f(N,E) \to E_{f(x)}$ *makes sense and we have* $(\nabla_X s)(x) = \nabla_{X(x)} s$.

(4) $\nabla_X s$ *is* $\mathbb{R}$*-linear in* $s \in C^\infty_f(N,E)$.

(5) $\nabla_X(h.s) = dh(X).s + h.\nabla_X s$ *for* $h \in C^\infty(N,\mathbb{R})$*, the derivation property of* ∇_X.

(6) *For any manifold* Q *and smooth mapping* $g : Q \to N$ *and* $Y_y \in T_yQ$ *we have* $\nabla_{Tg.Y_y} s = \nabla_{Y_y}(s \circ g)$*. If* $Y \in \mathfrak{X}(Q)$ *and* $X \in \mathfrak{X}(N)$ *are* g*-related, then we have* $\nabla_Y(s \circ g) = (\nabla_X s) \circ g$.

Proof. All these properties follow easily from the definition (1). $\square$

Remark. Property (6) is not well understood in some differential geometric literature. See e.g. the clumsy and unclear treatment of it in [Eells-Lemaire, 1983].

For vector fields $X, Y \in \mathfrak{X}(M)$ and a section $s \in C^\infty(E)$ an easy computation shows that

$$\begin{aligned} R^E(X,Y)s :&= \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]} s \\ &= ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})s \end{aligned}$$

is $C^\infty(M,\mathbb{R})$-linear in X, Y, and s. By the method of 7.3 it follows that R^E is a 2-form on M with values in the vector bundle $L(E,E)$, i.e. $R^E \in \Omega^2(M; L(E,E))$. It is called the *curvature* of the covariant derivative.

For $f : N \to M$, vector fields X, $Y \in \mathfrak{X}(N)$ and a section $s \in C^\infty_f(N,E)$ along f one may prove that

$$\nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]} s = (f^*R^E)(X,Y)s := R^E(Tf.X, Tf.Y)s.$$

16.13. Covariant exterior derivative. Let (E,p,M) be a vector bundle with a linear connection, given by a connector $K : TE \to E$.

For a smooth mapping $f : N \to M$ let $\Omega(N; f^*E)$ be the vector space of all forms on N with values in the vector bundle f^*E. We can also view them as forms on N with values along f in E, but we do not introduce an extra notation for this.

The graded space $\Omega(N; f^*E)$ is a graded $\Omega(N)$-module via

$$\begin{aligned} &(\varphi \wedge \Phi)(X_1, \dots, X_{p+q}) = \\ &\qquad = \tfrac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma)\ \varphi(X_{\sigma 1}, \dots, X_{\sigma p})\Phi(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}). \end{aligned}$$

It is easily seen that the graded module homomorphisms $H : \Omega(N; f^*E) \to \Omega(N; f^*E)$ (so that $H(\varphi \wedge \Phi) = (-1)^{\deg H.\deg\varphi}\varphi \wedge H(\Phi))$ are exactly the mappings $\mu(A)$ for $A \in \Omega^p(N; f^*L(E,E))$, which are given by

$$(\mu(A)\Phi)(X_1, \dots, X_{p+q}) = \\ = \tfrac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma)\; A(X_{\sigma 1}, \dots, X_{\sigma p})(\Phi(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})).$$

The *covariant exterior derivative* $d_\nabla : \Omega^p(N; f^*E) \to \Omega^{p+1}(N; f^*E)$ is defined by (where the X_i are vector fields on N)

$$\begin{aligned}(d_\nabla\Phi)(X_0, \dots, X_p) &= \sum_{i=0}^p (-1)^i \nabla_{X_i}\Phi(X_0, \dots, \widehat{X_i}, \dots, X_p)\\ &\quad + \sum_{0\le i<j\le p} (-1)^{i+j}\Phi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p).\end{aligned}$$

Lemma. *The covariant exterior derivative is well defined and has the following properties.*

(1) *For $s \in C^\infty(f^*E) = \Omega^0(N; f^*E)$ we have $(d_\nabla s)(X) = \nabla_X s$.*
(2) $d_\nabla(\varphi \wedge \Phi) = d\varphi \wedge \Phi + (-1)^{\deg\varphi}\varphi \wedge d_\nabla\Phi$.
(3) *For smooth $g : Q \to N$ and $\Phi \in \Omega(N; f^*E)$ we have $d_\nabla(g^*\Phi) = g^*(d_\nabla\Phi)$.*
(4) $d_\nabla d_\nabla\Phi = \mu(f^*R^E)\Phi$.

Proof. It suffices to investigate decomposable forms $\Phi = \varphi \otimes s$ for $\varphi \in \Omega^p(N)$ and $s \in C^\infty(f^*E)$. Then from the definition we have $d_\nabla(\varphi \otimes s) = d\varphi \otimes s + (-1)^p\varphi \wedge d_\nabla s$. Since by 16.12.(3) $d_\nabla s \in \Omega^1(N; f^*E)$, the mapping d_∇ is well defined. This formula also implies (2) immediately. (3) follows from 16.12.(6). (4) is checked as follows:

$$\begin{aligned}d_\nabla d_\nabla(\varphi \otimes s) &= d_\nabla(d\varphi \otimes s + (-1)^p\varphi \wedge d_\nabla s) \text{ by (2)}\\ &= 0 + (-1)^{2p}\varphi \wedge d_\nabla d_\nabla s\\ &= \varphi \wedge \mu(f^*R^E)s \text{ by the definition of } R^E\\ &= \mu(f^*R^E)(\varphi \otimes s). \quad \square\end{aligned}$$

16.14. Let (P, p, M, G) be a principal fiber bundle and let $\rho : G \to GL(W)$ be a representation of the structure group G on a finite dimensional vector space W.

Theorem. *There is a canonical isomorphism from the space of $P[W,\rho]$-valued differential forms on M onto the space of horizontal G-equivariant W-valued differential forms on P:*

$$\begin{aligned} q^\sharp : \Omega(M;P[W,\rho]) \to \Omega_{hor}(P;W)^G = \{\varphi \in \Omega(P;W) : i_X\varphi = 0 \\ \text{for all } X \in VP, (r^g)^*\varphi = \rho(g^{-1})\circ\varphi \text{ for all } g\in G\}. \end{aligned}$$

In particular for $W=\mathbb{R}$ with trivial representation we see that

$$p^* : \Omega(M) \to \Omega_{hor}(P)^G = \{\varphi\in\Omega_{hor}(P) : (r^g)^*\varphi = \varphi\}$$

is also an isomorphism. The isomorphism

$$q^\sharp : \Omega^0(M;P[W]) = C^\infty(P[W]) \to \Omega^0_{hor}(P;W)^G = C^\infty(P,W)^G$$

is a special case of the one from 15.12.

Proof. Recall the smooth mapping $\tau^G : P\times_M P\to G$ from 15.2, which satisfies $r(u_x,\tau^G(u_x,v_x)) = v_x$, $\tau^G(u_x.g,u'_x.g') = g^{-1}.\tau^G(u_x,u'_x).g'$, and $\tau^G(u_x,u_x)=e$.

Let $\varphi\in\Omega^k_{hor}(P;W)^G$, $X_1,\dots,X_k\in T_uP$, and $X'_1,\dots,X'_k\in T_{u'}P$ such that $T_up.X_i = T_{u'}p.X'_i$ for each i. Then we have for $g=\tau^G(u,u')$, so that $ug=u'$:

$$\begin{aligned} q(u,\varphi_u(X_1,\dots,X_k)) &= q(ug,\rho(g^{-1})\varphi_u(X_1,\dots,X_k)) \\ &= q(u',((r^g)^*\varphi)_u(X_1,\dots,X_k)) \\ &= q(u',\varphi_{ug}(T_u(r^g).X_1,\dots,T_u(r^g).X_k)) \\ &= q(u',\varphi_{u'}(X'_1,\dots,X'_k)), \text{ since } T_u(r^g)X_i - X'_i \in V_{u'}P. \end{aligned}$$

By this a vector bundle valued form $\Phi\in\Omega^k(M;P[W])$ is uniquely determined.

For the converse recall the smooth mapping $\tau^W : P\times_M P[W,\rho]\to W$ from 15.7, which satisfies $\tau^W(u,q(u,w)) = w$, $q(u_x,\tau^W(u_x,v_x)) = v_x$, and $\tau^W(u_xg,v_x) = \rho(g^{-1})\tau^W(u_x,v_x)$.

For $\Phi\in\Omega^k(M;P[W])$ we define $q^\sharp\Phi\in\Omega^k(P;W)$ as follows. For $X_i\in T_uP$ we put

$$(q^\sharp\Phi)_u(X_1,\dots,X_k) := \tau^W(u,\Phi_{p(u)}(T_up.X_1,\dots,T_up.X_k)).$$

Then $q^\sharp\Phi$ is smooth and horizontal. For $g\in G$ we have

$$\begin{aligned} ((r^g)^*(q^\sharp\Phi))_u(X_1,\dots,X_k) &= (q^\sharp\Phi)_{ug}(T_u(r^g).X_1,\dots,T_u(r^g).X_k) \\ &= \tau^W(ug,\Phi_{p(ug)}(T_{ug}p.T_u(r^g).X_1,\dots,T_{ug}p.T_u(r^g).X_k)) \\ &= \rho(g^{-1})\tau^W(u,\Phi_{p(u)}(T_up.X_1,\dots,T_up.X_k)) \\ &= \rho(g^{-1})(q^\sharp\Phi)_u(X_1,\dots,X_k). \end{aligned}$$

Clearly the two constructions are inverse to each other. $\square$

16.15. Let (P,p,M,G) be a principal fiber bundle with a principal connection $\Phi = \zeta \circ \omega$, and let $\rho : G \to GL(W)$ be a representation of the structure group G on a finite dimensional vector space W. We consider the associated vector bundle $(E := P[W,\rho], p, M, W)$, the induced connection $\bar{\Phi}$ on it and the corresponding covariant derivative.

Theorem. *The covariant exterior derivative d_ω from 16.5 on P and the covariant exterior derivative for $P[W]$-valued forms on M are connected by the mapping $q^\sharp$ from 16.14, as follows:*

$$q^\sharp \circ d_\nabla = d_\omega \circ q^\sharp : \Omega(M;P[W]) \to \Omega_{hor}(P;W)^G.$$

Proof. Let us consider first $f \in \Omega^0_{hor}(P;W)^G = C^\infty(P,W)^G$, then $f = q^\sharp s$ for $s \in C^\infty(P[W])$ and we have $f(u) = \tau^W(u, s(p(u)))$ and $s(p(u)) = q(u, f(u))$ by 16.14 and 15.12. Therefore we have $Ts.Tp.X_u = Tq(X_u, Tf.X_u)$, where $Tf.X_u = (f(u), df(X_u)) \in TW = W \times W$. If $\chi : TP \to HP$ is the horizontal projection as in 16.5, we have $Ts.Tp.X_u = Ts.Tp.\chi.X_u = Tq(\chi.X_u, Tf.\chi.X_u)$. So we get

$$\begin{aligned}
(q^\sharp d_\nabla s)(X_u) &= \tau^W(u, (d_\nabla s)(Tp.X_u)) \\
&= \tau^W(u, \nabla_{Tp.X_u} s) && \text{by 16.13.(1)} \\
&= \tau^W(u, K.Ts.Tp.X_u) && \text{by 16.12.(1)} \\
&= \tau^W(u, K.Tq(\chi.X_u, Tf.\chi.X_u)) && \text{from above} \\
&= \tau^W(u, pr_2.vl^{-1}_{P[W]}.\bar{\Phi}.Tq(\chi.X_u, Tf.\chi.X_u)) && \text{by 16.10} \\
&= \tau^W(u, pr_2.vl^{-1}_{P[W]}.Tq.(\Phi \times Id)(\chi.X_u, Tf.\chi.X_u))) && \text{by 16.8} \\
&= \tau^W(u, pr_2.vl^{-1}_{P[W]}.Tq(0_u, Tf.\chi.X_u))) && \text{since } \Phi.\chi = 0 \\
&= \tau^W(u, q.pr_2.vl^{-1}_{P\times W}.(0_u, Tf.\chi.X_u))) && \text{since } q \text{ is fiber linear} \\
&= \tau^W(u, q(u, df.\chi.X_u)) = (\chi^* df)(X_u) \\
&= (d_\omega q^\sharp s)(X_u).
\end{aligned}$$

Now we turn to the general case. It suffices to check the formula for a decomposable $P[W]$-valued form $\Psi = \psi \otimes s \in \Omega^k(M, P[W])$, where $\psi \in \Omega^k(M)$ and

$s \in C^\infty(P[W])$. Then we have

$$\begin{aligned}
d_\omega q^\sharp(\psi \otimes s) &= d_\omega(p^*\psi \cdot q^\sharp s)\\
&= d_\omega(p^*\psi)\cdot q^\sharp s + (-1)^k \chi^* p^*\psi \wedge d_\omega q^\sharp s \qquad \text{by 16.5.(1)}\\
&= \chi^* p^* d\psi \cdot q^\sharp s + (-1)^k p^*\psi \wedge q^\sharp d_\nabla s \qquad \text{from above and 16.5.(4)}\\
&= p^* d\psi \cdot q^\sharp s + (-1)^k p^*\psi \wedge q^\sharp d_\nabla s\\
&= q^\sharp(d\psi \otimes s + (-1)^k \psi \wedge d_\nabla s)\\
&= q^\sharp d_\nabla(\psi \otimes s). \quad \square
\end{aligned}$$

16.16. Corollary. *In the situation of theorem 16.15 above we have for the Lie algebra valued curvature form $\Omega \in \Omega^2_{hor}(P;\mathfrak{g})$ and the curvature $R^{P[W]} \in \Omega^2(M; L(P[W],P[W]))$ the relation*

$$q^\sharp_{L(P[W],P[W])} R^{P[W]} = \rho' \circ \Omega,$$

where $\rho' = T_e\rho : \mathfrak{g} \to L(W,W)$ is the derivative of the representation ρ.

Proof. We use the notation of the proof of theorem 16.15. By this theorem we have for X, $Y \in T_uP$

$$\begin{aligned}
(d_\omega d_\omega q^\sharp_{P[W]} s)_u(X,Y) &= (q^\sharp d_\nabla d_\nabla s)_u(X,Y)\\
&= (q^\sharp R^{P[W]} s)_u(X,Y)\\
&= \tau^W(u, R^{P[W]}(T_up.X, T_up.Y)s(p(u)))\\
&= (q^\sharp_{L(P[W],P[W])} R^{P[W]})_u(X,Y)(q^\sharp_{P[W]} s)(u).
\end{aligned}$$

On the other hand we have by theorem 16.5.(8)

$$\begin{aligned}
(d_\omega d_\omega q^\sharp s)_u(X,Y) &= (\chi^* i_R d q^\sharp s)_u(X,Y)\\
&= (dq^\sharp s)_u(R(X,Y)) \qquad \text{since } R \text{ is horizontal}\\
&= (dq^\sharp s)(-\zeta_{\Omega(X,Y)}(u)) \qquad \text{by 16.2}\\
&= \tfrac{\partial}{\partial t}\big|_0 (q^\sharp s)(\mathrm{Fl}^{\zeta_{\Omega(X,Y)}}_{-t}(u))\\
&= \tfrac{\partial}{\partial t}\big|_0 \tau^W(u.\exp(-t\Omega(X,Y)), s(p(u.\exp(-t\Omega(X,Y)))))\\
&= \tfrac{\partial}{\partial t}\big|_0 \tau^W(u.\exp(-t\Omega(X,Y)), s(p(u)))\\
&= \tfrac{\partial}{\partial t}\big|_0 \rho(\exp t\Omega(X,Y))\tau^W(u, s(p(u))) \qquad \text{by 15.7}\\
&= \rho'(\Omega(X,Y))(q^\sharp s)(u). \quad \square
\end{aligned}$$

17. Characteristic classes

17.1. Invariants of Lie algebras. Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $\bigotimes \mathfrak{g}^*$ be the tensor algebra over the dual space $\mathfrak{g}^*$, the graded space of all multilinear real (or complex) functionals on $\mathfrak{g}$. Let $S(\mathfrak{g}^*)$ be the symmetric algebra over $\mathfrak{g}^*$ which corresponds to the algebra of polynomial functions on $\mathfrak{g}$. The adjoint representation $\operatorname{Ad} : G \to L(\mathfrak{g}, \mathfrak{g})$ induces representations $\operatorname{Ad}^* : G \to L(\bigotimes \mathfrak{g}^*, \bigotimes \mathfrak{g}^*)$ and also $\operatorname{Ad}^* : G \to L(S(\mathfrak{g}^*), S(\mathfrak{g}^*))$, which are both given by $\operatorname{Ad}^*(g)f = f \circ (\operatorname{Ad}(g^{-1}) \otimes \cdots \otimes \operatorname{Ad}(g^{-1}))$. A tensor $f \in \bigotimes \mathfrak{g}^*$ (or a polynomial $f \in S(\mathfrak{g}^*)$) is called an *invariant of the Lie algebra* if $\operatorname{Ad}^*(g)f = f$ for all $g \in G$. If the Lie group G is connected, f is an invariant if and only if $\mathcal{L}_X f = 0$ for all $X \in \mathfrak{g}$, where $\mathcal{L}_X$ is the restriction of the Lie derivative to left invariant tensor fields on G, which coincides with the unique extension of $\operatorname{ad}(X)^* : \mathfrak{g}^* \to \mathfrak{g}^*$ to a derivation on $\bigotimes \mathfrak{g}^*$ or $S(\mathfrak{g}^*)$, respectively. Compare this with the proof of 12.16.(2). Obvious the space of all invariants is a graded subalgebra of $\bigotimes \mathfrak{g}^*$ or $S(\mathfrak{g}^*)$, respectively. The usual notation for the algebra of invariant polynomials is $I(G) := \bigoplus_{k \geq 0} I^k(G)$, where $I^k(G)$ is the invariant subspace of $S^k(\mathfrak{g}^*)$.

We will later determine the generating systems of the algebra of invariant polynomials for the most important Lie algebras.

17.2. The Chern-Weil forms. Let (P, p, M, G) be a principal fiber bundle with principal connection $\Phi = \zeta \circ \omega$ and curvature $R = \zeta \circ \Omega$. For $\psi_i \in \Omega^{p_i}(P; \mathfrak{g})$ and $f \in S^k(\mathfrak{g}^*) \subset \bigotimes^k \mathfrak{g}^*$ we have the differential forms

$$\begin{aligned} \psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k &\in \Omega^{p_1 + \cdots + p_k}(P; \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}), \\ f \circ (\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k) &\in \Omega^{p_1 + \cdots + p_k}(P). \end{aligned}$$

The exterior derivative of the latter one is clearly given by

$$\begin{aligned} d(f \circ (\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k)) &= f \circ d(\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k) \\ &= f \circ \Big(\textstyle\sum_{i=1}^k (-1)^{p_1 + \cdots + p_{i-1}} \psi_1 \otimes_\wedge \cdots \otimes_\wedge d\psi_i \otimes_\wedge \cdots \otimes_\wedge \psi_k \Big) \end{aligned}$$

Let us now consider an invariant polynomial $f \in I^k(G)$ and the curvature form $\Omega \in \Omega^2_{\text{hor}}(P, \mathfrak{g})^G$. Then the $2k$-form $f \circ (\Omega \otimes_\wedge \cdots \otimes_\wedge \Omega)$ is horizontal since by 16.2.(2) Ω is horizontal. It is also G-invariant since by 16.2.(2) we have

$$\begin{aligned} (r^g)^*(f \circ (\Omega \otimes_\wedge \cdots \otimes_\wedge \Omega)) &= f \circ ((r^g)^* \Omega \otimes_\wedge \cdots \otimes_\wedge (r^g)^* \Omega) \\ &= f \circ (\operatorname{Ad}(g^{-1}) \Omega \otimes_\wedge \cdots \otimes_\wedge \operatorname{Ad}(g^{-1}) \Omega) \\ &= f \circ (\Omega \otimes_\wedge \cdots \otimes_\wedge \Omega). \end{aligned}$$

So by theorem 16.14 there is a uniquely defined $2k$-form $\operatorname{cw}(f,P,\omega)\in\Omega^{2k}(M)$ with $p^*\operatorname{cw}(f,P,\omega)=f\circ(\Omega\otimes_\wedge\dots\otimes_\wedge\Omega)$, which we will call the *Chern-Weil form* of f.

If $g:N\to M$ is a smooth mapping, then for the pullback bundle g^*P the Chern-Weil form is given by $\operatorname{cw}(f,g^*P,g^*\omega)=g^*\operatorname{cw}(f,P,\omega)$, which is easily seen by applying p^*.

17.3. Theorem. The Chern-Weil homomorphism. *In the setting of 17.2 we have:*

1. For $f\in I^k(G)$ the Chern Weil form $\operatorname{cw}(f,P,\omega)$ is closed: $d\operatorname{cw}(f,P,\omega)=0$. So there is a well defined cohomology class $\operatorname{Cw}(f,P)=[\operatorname{cw}(f,P,\omega)]\in H^{2k}(M)$, called the characteristic class of the invariant polynomial f.

2. The characteristic class $\operatorname{Cw}(f,P)$ does not depend on the choice of the principal connection ω.

3. The mapping $\operatorname{Cw}_P:I^*(G)\to H^{2*}(M)$ is a homomorphism of commutative algebras, and it is called the Chern-Weil homomorphism.

4. If $g:N\to M$ is a smooth mapping, then the Chern-Weil homomorphism for the pullback bundle g^*P is given by

$$\operatorname{Cw}_{g^*P}=g^*\circ\operatorname{Cw}_P:I^*(G)\to H^{2*}(N).$$

Proof. 1. Since $f\in I^k(G)$ is invariant we have for any $X\in\mathfrak g$

$$\begin{aligned}0&=\tfrac{d}{dt}|_0\operatorname{Ad}(\exp(-tX_0))^*f(X_1,\dots,X_k)=\operatorname{ad}(X_0)^*f(X_1,\dots,X_k)\\&=\textstyle\sum_{i=1}^k f(X_1,\dots,[X_0,X_i],\dots,X_k)=\sum_{i=1}^k f([X_0,X_i],X_1,\dots,\widehat{X_i}\dots,X_k).\end{aligned}$$

This implies that

$$\begin{aligned}d(f\circ(\Omega\otimes_\wedge\dots\otimes_\wedge\Omega))&=f\circ\left(\textstyle\sum_{i=1}^k\Omega\otimes_\wedge\dots\otimes_\wedge d\Omega\otimes_\wedge\dots\otimes_\wedge\Omega\right)\\&=k\,f\circ(d\Omega\otimes_\wedge\dots\otimes_\wedge\Omega)+k\,f\circ([\omega,\Omega]_{\otimes_\wedge}\otimes_\wedge\dots\otimes_\wedge\Omega)\\&=k\,f\circ(d_\omega\Omega\otimes_\wedge\Omega\otimes_\wedge\dots\otimes_\wedge\Omega)=0,\qquad\text{by 16.5.6.}\\p^*d\operatorname{cw}(f,P,\omega)&=d\,p^*\operatorname{cw}(f,P,\omega)\\&=d\,(f\circ(\Omega\otimes_\wedge\dots\otimes_\wedge\Omega))=0,\end{aligned}$$

and thus $d\operatorname{cw}(f,P,\omega)=0$ since p^* is injective.

2. Let ω_0, $\omega_1\in\Omega^1(P,\mathfrak g)^G$ be two principal connections. Then we consider the principal bundle $(P\times\mathbb R,p\times Id,M\times\mathbb R,G)$ and the principal connection $\tilde\omega=(1-t)\omega_0+t\omega_1=(1-t)(pr_1)^*\omega_0+t(pr_1)^*\omega_1$ on it, where t is the coordinate

function on $\mathbb{R}$. Let $\tilde{\Omega}$ be the curvature form of $\tilde{\omega}$. Let $\operatorname{ins}_s : P \to P \times \mathbb{R}$ be the embedding at level s, $\operatorname{ins}_s(u) = (u, s)$. Then we have in turn by 16.2.(3) for $s = 0, 1$

$$\begin{aligned}
\omega_s &= (\operatorname{ins}_s)^* \tilde{\omega} \\
\Omega_s &= d\omega_s + \tfrac{1}{2}[\omega_s, \omega_s]_\wedge \\
&= d(\operatorname{ins}_s)^* \tilde{\omega} + \tfrac{1}{2}[(\operatorname{ins}_s)^* \tilde{\omega}, (\operatorname{ins}_s)^* \tilde{\omega}]_\wedge \\
&= (\operatorname{ins}_s)^* (d\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}]_\wedge) \\
&= (\operatorname{ins}_s)^* \tilde{\Omega}.
\end{aligned}$$

So we get for $s = 0, 1$

$$\begin{aligned}
p^*(\operatorname{ins}_s)^* \operatorname{cw}(f, P \times \mathbb{R}, \tilde{\omega}) &= (\operatorname{ins}_s)^*(p \times Id_{\mathbb{R}})^* \operatorname{cw}(f, P \times \mathbb{R}, \tilde{\omega}) \\
&= (\operatorname{ins}_s)^*(f \circ (\tilde{\Omega} \otimes_\wedge \dots \otimes_\wedge \tilde{\Omega})) \\
&= f \circ ((\operatorname{ins}_s)^* \tilde{\Omega} \otimes_\wedge \dots \otimes_\wedge (\operatorname{ins}_s)^* \tilde{\Omega}) \\
&= f \circ (\Omega_s \otimes_\wedge \dots \otimes_\wedge \Omega_s) \\
&= p^* \operatorname{cw}(f, P, \omega_s).
\end{aligned}$$

Since p^* is injective we get $(ins_s)^* \operatorname{cw}(f, P \times \mathbb{R}, \tilde{\omega}) = \operatorname{cw}(f, P, \omega_s)$ for $s = 0, 1$, and since ins_0 and ins_1 are smoothly homotopic, the cohomology classes coincide.

3. and 4. are obvious. □

17.4. Local description of characteristic classes. Let (P, p, M, G) be a principal fiber bundle with a principal connection $\omega \in \Omega^1(P, \mathfrak{g})^G$. Let $s_\alpha \in C^\infty(P|U_\alpha)$ be a collection of local smooth sections of the bundle such that (U_α) is an open cover of M. Recall (from the proof of 15.3 for example) that then $\varphi_\alpha = (p, \tau^G(s_\alpha \circ p, \quad)) : P|U_\alpha \to U_\alpha \times G$ is a principal fiber bundle atlas with transition functions $\varphi_{\alpha\beta}(x) = \tau^G(s_\alpha(x), s_\beta(x))$.

Then we consider the physicists version from 16.4 of the connection ω which is descibed by the forms $\omega_\alpha := s_\alpha^* \omega \in \Omega^1(U_\alpha; \mathfrak{g})$. They transform according to $\omega_\alpha = \operatorname{Ad}(\varphi_{\alpha\beta})\omega_\beta + \Theta_{\alpha\beta}$, where $\Theta_{\alpha\beta} = \varphi_{\alpha\beta} d\varphi_{\alpha\beta}$ if G is a matrix group, see lemma 16.4. This affine transformation law is due to the fact that ω is not horizontal.

Let $\Omega = d\omega + \frac{1}{2}[\omega, \omega]_\wedge \in \Omega^2_{\text{hor}}(P, \mathfrak{g})^G$ be the curvature of ω, then we consider again the local forms of the curvature:

$$\begin{aligned}
\Omega_\alpha : &= s_\alpha^* \Omega = s^*(d\omega + \tfrac{1}{2}[\omega, \omega]_\wedge) \\
&= d(s_\alpha^* \omega) + \tfrac{1}{2}[s_\alpha^* \omega, s_\alpha^*]_\wedge \\
&= d\omega_\alpha + \tfrac{1}{2}[\omega_\alpha, \omega_\alpha]_\wedge
\end{aligned}$$

Recall from theorem 16.14 that we have an isomorphism $q^\sharp : \Omega(M, P[\mathfrak{g}, \mathrm{Ad}]) \to \Omega_{\mathrm{hor}}(P, \mathfrak{g})^G$. Then $\Omega_\alpha = s_\alpha^*\Omega$ is the local expression of $(q^\sharp)^{-1}(\Omega)$ for the induced chart $P[\mathfrak{g}]|U_\alpha \to U_\alpha \times \mathfrak{g}$, thus we have the the simple transformation formula $\Omega_\alpha = \mathrm{Ad}(\varphi_{\alpha\beta})\Omega_\beta$.

If now $f \in I^k(G)$ is an invariant of G, for the Chern-Weil form $\mathrm{cw}(f, P, \omega)$ we have

$$\begin{aligned}
\mathrm{cw}(f, P, \omega)|U_\alpha :&= s_\alpha^*(q^\sharp \,\mathrm{cw}(f, P, \omega)) = s_\alpha^*(f \circ (\Omega \otimes_\wedge \cdots \otimes_\wedge \Omega)) \\
&= f \circ (s_\alpha^*\Omega \otimes_\wedge \cdots \otimes_\wedge s_\alpha^*\Omega) \\
&= f \circ (\Omega_\alpha \otimes_\wedge \cdots \otimes_\wedge \Omega_\alpha),
\end{aligned}$$

where $\Omega_\alpha \otimes_\wedge \cdots \otimes_\wedge \Omega_\alpha \in \Omega^{2k}(U_\alpha; \mathfrak{g} \otimes \cdots \otimes \mathfrak{g})$.

17.5. Characteristic classes for vector bundles. For a real vector bundle $(E, p, M, \mathbb{R}^n)$ the characteristic classes are by definition the characteristic classes of the linear frame bundle $(GL(\mathbb{R}^n, E), p, M, GL(n, \mathbb{R}))$. We write $\mathrm{Cw}(f, E) := \mathrm{Cw}(f, GL(\mathbb{R}^n, E))$ for short. Likewise for complex vector bundles.

Let (P, p, M, G) be a principal bundle and let $\rho : G \to GL(V)$ be a representation in a finite dimensional vector space. If ω is a principal connection form on P with curvature form Ω, then for the induced covariant derivative ∇ on the associated vector bundle $P[V]$ and its curvature $R^{P[V]}$ we have $q^\sharp R^{P[V]} = \rho' \circ \Omega$ by corollary 16.16. So if the representation ρ is infinitesimally effective, i. e. if $\rho' : \mathfrak{g} \to L(V, V)$ is injective, then we see that actually $R^{P[V]} \in \Omega^2(M; P[\mathfrak{g}])$. If $f \in I^k(G)$ is an invariant, then we have the induced mapping

$$\begin{tikzcd}
P \times (\bigotimes^k \mathfrak{g}) \arrow[r, "Id_P \times f"] \arrow[d, "q"'] & P \times \mathbb{R} \arrow[d, "q"] \\
P[\bigotimes^k \mathfrak{g}] \arrow[r, "P[f]"] & M \times \mathbb{R}.
\end{tikzcd}$$

So the Chern-Weil form can also be written as

$$\mathrm{cw}(f, P, \omega) = P[f] \circ (R^{P[V]} \otimes_\wedge \cdots \otimes_\wedge R^{P[V]}).$$

Sometimes we will make use of this expression.

All characteristic classes for a trivial vector bundle are zero, since the frame bundle is then trivial and admits a principal connection with curvature 0.

We will determine the classical bases for the algebra of invariants for the matrix groups $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $U(n)$, and discuss the resulting characteristic classes for vector bundles.

17.6. The characteristic coefficients. . For a matrix $A \in \mathfrak{gl}(n,\mathbb{R}) = L(\mathbb{R}^n,\mathbb{R}^n)$ we consider the characteristic coefficients $c^n_k(A)$ which are given by the implicit equation

$$\det(A + t\mathbb{I}) = \sum_{k=0}^{n} c^n_k(A).t^{n-k}. \tag{1}$$

From lemma 12.19 we have $c^n_k(A) = \operatorname{Trace}(\Lambda^k A : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n)$. The characteristic coefficient c^n_k is a homogeneous invariant polynomial of degree k, since we have $\det(\operatorname{Ad}(g)A + t\mathbb{I}) = \det(gAg^{-1} + t\mathbb{I}) = \det(g(A + t\mathbb{I})g^{-1}) = \det(A + t\mathbb{I})$.

Lemma. *We have*

$$c^{n+m}_k\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \sum_{j=0}^{k} c^n_j(A)c^m_{k-j}(B).$$

Proof. We have

$$\begin{aligned}
\det\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + t\mathbb{I}_{n+m}\right) &= \det(A + t\mathbb{I}_n)\det(B + t\mathbb{I}_m) \\
&= \left(\sum_{k=0}^{n} c^n_k(A)t^{n-k}\right)\left(\sum_{j=0}^{m} c^m_j(A)t^{m-l}\right) \\
&= \sum_{k=0}^{n+m}\left(\sum_{j=0}^{k} c^n_j(A)c^m_{k-j}(B)\right)t^{n+m-k}. \qquad \square
\end{aligned}$$

17.7. Pontryagin classes. Let (E,p,M) be a real vector bundle. Then the *Pontryagin classes* are given by

$$p_k(E) := \left(\frac{-1}{2\pi\sqrt{-1}}\right)^{2k} \operatorname{Cw}(c^{\dim E}_{2k}, E) \in H^{4k}(M;\mathbb{R}).$$

The factor $\frac{-1}{2\pi\sqrt{-1}}$ makes this class to be an integer class (in $H^{4k}(M,\mathbb{Z})$) and makes several integral formulas (like the Gauss-Bonnet-Chern formula) more beautiful. In principle one should always replace the curvature Ω by $\frac{-1}{2\pi\sqrt{-1}}\Omega$. The inhomogeneous cohomology class

$$p(E) := \sum_{k\geq 0} p_k(E) \in H^{4*}(M,\mathbb{R})$$

is called the *total Pontryagin class.*

Theorem. *For the Pontryagin classes we have:*

1. If E_1 and E_2 are two real vector bundles over a manifold M, then for the fiberwise direct sum we have

$$p(E_1 \oplus E_2) = p(E_1) \wedge p(E_2) \in H^{4*}(M, \mathbb{R}).$$

2. For the pullback of a vector bundle along $f : N \to M$ we have

$$p(f^*E) = f^*p(E).$$

3. For a real vector bundle and an invariant $f \in I^k(GL(n, \mathbb{R}))$ for odd k we have $\operatorname{Cw}(f, E) = 0$. Thus the Pontryagin classes exist only in dimension $0, 4, 8, 12, \dots$.

Proof. 1. If $\omega^i \in \Omega^1(GL(\mathbb{R}^{n_i}, E_i), \mathfrak{gl}(n_i))^{GL(n_i)}$ are principal connection forms for the frame bundles of the two vector bundles, then for local frames of the two bundles $s^i_\alpha \in C^\infty(GL(\mathbb{R}^{n_i}, E_i|U_\alpha)$ the forms

$$\omega_\alpha := \begin{pmatrix} \omega^1_\alpha & 0 \\ 0 & \omega^2_\alpha \end{pmatrix} \in \Omega^1(U_\alpha, \mathfrak{gl}(n_1 + n_2))$$

are exactly the local expressions of the direct sum connection, and from lemma 17.6 we see that $p_k(E_1 \oplus E_2) = \sum_{j=0}^k p_j(E_1)p_{k-j}(E_2)$ holds which implies the desired result.

2. This follows from 17.3.4.

3. Choose a fiber Riemannian metric g on E, consider the corresponding orthonormal frame bundle $(O(\mathbb{R}^n, E), p, M, O(n, \mathbb{R}))$, and choose a principal connection ω for it. Then the local expression with respect to local orthonormal frame fields s_α are skew symmetric matrices of 1-forms. So the local curvature forms are also skew symmetric. Any real matrix is conjugate to its transposed (use Jordan's normal form), so there are invertible matrices g_α such that $g_\alpha \Omega_\alpha g_\alpha^{-1} = -\Omega_\alpha$. But then

$$\begin{aligned} f \circ (\Omega_\alpha \otimes_\wedge \dots \otimes_\wedge \Omega_\alpha) &= f \circ (g_\alpha \Omega_\alpha g_\alpha^{-1} \otimes_\wedge \dots \otimes_\wedge g_\alpha \Omega_\alpha g_\alpha^{-1}) \\ &= f \circ ((-\Omega_\alpha) \otimes_\wedge \dots \otimes_\wedge (-\Omega_\alpha)) \\ &= (-1)^k f \circ (\Omega_\alpha \otimes_\wedge \dots \otimes_\wedge \Omega_\alpha). \end{aligned}$$

This implies that $\operatorname{Cw}(f, E) = 0$ if k is odd. $\square$

17.8. Remarks. 1. If two vector bundles E and F are *stably equivalent*, i. e. $E\oplus(M\times\mathbb{R}^p)\cong F\oplus(M\times\mathbb{R}^q)$, then $p(E)=p(F)$. This follows from 17.7.1 and 2.

2. If for a vector bundle E for some k the bundle $\overbrace{E\oplus\cdots\oplus E}^{k}\oplus(M\times\mathbb{R}^l)$ is trivial, then $p(E)=1$ since $p(E)^k=1$.

3. Let (E,p,M) be a vector bundle over a compact oriented manifold M. For $j_i\in\mathbb{N}_0$ we put

$$\lambda_{j_1,\dots,j_r}(E):=\int_M p_1(E)^{j_1}\dots p_r(E)^{j_r}\in\mathbb{R},$$

where the integral is set to be 0 on each degree which is not equal to $\dim M$. Then these *Pontryagin numbers* are indeed integers, see [Milnor-Stasheff, ??]. For example we have

$$\lambda_{j_1,\dots,j_r}(T(\mathbb{C}P^n))=\binom{2n+1}{j_1}\dots\binom{2n+1}{j_r}.$$

17.9. The trace coefficients. For a matrix $A\in\mathfrak{gl}(n,\mathbb{R})=L(\mathbb{R}^n,\mathbb{R}^n)$ the *trace coefficients* are given by

$$\operatorname{tr}^n_k(A):=\operatorname{Trace}(A^k)=\operatorname{Trace}(\overbrace{A\circ\ldots\circ A}^{k}).$$

Obviously tr^n_k is an invariant polynomial, homogeneous of degree k. To a direct sum of two matrices $A\in\mathfrak{gl}(n)$ and $B\in\mathfrak{gl}(m)$ it reacts clearly by

$$\operatorname{tr}^{n+m}_k\begin{pmatrix}A&0\\0&B\end{pmatrix}=\operatorname{Trace}\begin{pmatrix}A^k&0\\0&B^k\end{pmatrix}=\operatorname{tr}^n_k(A)+\operatorname{tr}^m_k(B).$$

The tensor product (sometimes also called Kronecker product) of A and B is given by $A\otimes B=(A^i_jB^k_l)_{(i,k),(j,l)\in n\times m}$ in terms of the canonical bases. Since we have $\operatorname{Trace}(A\otimes B)=\sum_{i,k}A^i_iB^k_k=\operatorname{Trace}(A)\operatorname{Trace}(B)$, we also get

$$\begin{aligned}\operatorname{tr}^{nm}_k(A\otimes B)&=\operatorname{Trace}((A\otimes B)^k)=\operatorname{Trace}(A^k\otimes B^k)=\operatorname{Trace}(A^k)\operatorname{Trace}(B^k)\\&=\operatorname{tr}^n_k(A)\operatorname{tr}^m_k(B).\end{aligned}$$

Lemma. *The trace coefficients and the characteristic coefficients are connected by the following recursive equation:*

$$c_k^n(A) = \tfrac{1}{k}\sum_{j=0}^{k-1}(-1)^{k-j-1}c_j^n(A)\,\mathrm{tr}_{k-j}^n(A).$$

Proof. For a matrix $A \in \mathfrak{gl}(n)$ let us denote by $C(A)$ the matrix of the signed algebraic complements of A (also called the classical adjoint), i. e.

(1) $$C(A)^i_j = (-1)^{i+j}\det\left(A \begin{array}{l}\text{without } i\text{-th column,}\\ \text{without } j\text{-th row}\end{array}\right)$$

Then Cramer's rule reads

(2) $$A.C(A) = C(A).A = \det(A).\mathbb{I},$$

and the derivative of the determinant is given by

(3) $$d\det(A)X = \mathrm{Trace}(C(A)X).$$

Note that $C(A)$ is a homogeneous matrix valued polynomial of degree $n-1$ in A. We define now matrix valued polynomials $a_k(A)$ by

(4) $$C(A+t\mathbb{I}) = \sum_{k=0}^{n-1} a_k(A)t^{n-k-1}.$$

We claim that for $A \in \mathfrak{gl}(n)$ and $k = 0, 1, \dots, n-1$ we have

(5) $$a_k(A) = \sum_{j=0}^{k}(-1)^j c_{k-j}^n(A)A^j.$$

We prove this in the following way: from (2) we have

$$(A+t\mathbb{I})C(A+t\mathbb{I}) = \det(A+t\mathbb{I})\mathbb{I},$$

and we insert (4) and 17.6.(1) to get in turn

$$(A+t\mathbb{I})\sum_{k=0}^{n-1} a_k(A)t^{n-k-1} = \sum_{j=0}^{n} c_j^n(A)t^{n-j}\mathbb{I}$$

$$\sum_{k=0}^{n-1} A.a_k(A)t^{n-k-1} + \sum_{k=0}^{n-1} a_k(A)t^{n-k} = \sum_{j=0}^{n} c_j^n(A)t^{n-j}\mathbb{I}$$

We put $a_{-1}(A) := 0 =: a_n(A)$ and compare coefficients of t^{n-k} in the last equation to get the recursion formula

$$A.a_{k-1}(A) + a_k(A) = c_k^n(A)\mathbb{I}$$

which immediately leads to to the desired formula (5), even for $k = 0, 1, \dots, n$. If we start this computation with the two factors in (2) reversed we get $A.a_k(A) = a_k(A).A$. Note that (5) for $k = n$ is exactly the *Caley-Hamilton equation*

$$0 = a_n(A) = \sum_{j=0}^{n} c_{n-j}^n(A)A^j.$$

We claim that

$$\text{(6)} \qquad \operatorname{Trace}(a_k(A)) = (n-k)c_k^n(A).$$

We use (3) for the proof:

$$\begin{aligned}
\tfrac{\partial}{\partial t}\big|_0 (\det(A + t\mathbb{I})) &= d\det(A + t\mathbb{I})\ \tfrac{\partial}{\partial t}\big|_0 (A + t\mathbb{I}) = \operatorname{Trace}(C(A + t\mathbb{I})\mathbb{I}) \\
&= \operatorname{Trace}\left(\sum_{k=0}^{n-1} a_k(A)t^{n-k-1}\right) = \sum_{k=0}^{n-1} \operatorname{Trace}(a_k(A))t^{n-k-1}. \\
\tfrac{\partial}{\partial t}\big|_0 (\det(A + t\mathbb{I})) &= \tfrac{\partial}{\partial t}\big|_0 \left(\sum_{k=0}^{n} c_k^n(A)t^{n-k}\right) \\
&= \sum_{k=0}^{n} (n-k)c_k^n(A)t^{n-k-1}.
\end{aligned}$$

Comparing coefficients leads to the result (6).

Now we may prove the lemma itself by the following computation:

$$\begin{aligned}
(n-k)c_k^n(A) &= \operatorname{Trace}(a_k(A)) \qquad \text{by (6)} \\
&= \operatorname{Trace}\left(\sum_{j=0}^{k} (-1)^j c_{k-j}^n(A)A^j\right) \qquad \text{by (5)} \\
&= \sum_{j=0}^{k} (-1)^j c_{k-j}^n(A) \operatorname{Trace}(A^j)
\end{aligned}$$

$$= n\, c^n_k(A) + \sum_{j=1}^{k} (-1)^j c^n_{k-j}(A)\, \mathrm{tr}^n_j(A).$$

$$c^n_k(A) = -\tfrac{1}{k} \sum_{j=1}^{k} (-1)^j c^n_{k-j}(A)\, \mathrm{tr}^n_j(A)$$

$$= \tfrac{1}{k} \sum_{j=0}^{k-1} (-1)^{k-j-1} c^n_j(A)\, \mathrm{tr}^n_{k-j}(A). \quad \square$$

17.10. The trace classes. Let (E, p, M) be a real vector bundle. Then the trace classes are given by

$$\text{(1)} \qquad \mathrm{tr}_k(E) := \left(\frac{-1}{2\pi\sqrt{-1}}\right)^{2k} \mathrm{Cw}(\mathrm{tr}^{\dim E}_{2k}, E) \in H^{4k}(M, \mathbb{R}).$$

Between the trace classes and the Pontryagin classes there are the following relations for $k \geq 1$

$$\text{(2)} \qquad p_k(E) = \frac{-1}{2k} \sum_{j=0}^{k-1} p_j(E) \wedge \mathrm{tr}_{k-j}(E),$$

which follows directly from lemma 17.9 above.

The inhomogeneous cohomology class

$$\text{(3)} \qquad \mathrm{tr}(E) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \mathrm{tr}_k(E) = \mathrm{Cw}(\mathrm{Trace} \circ \exp, E)$$

is called the *Pontryagin character* of E. In the second expression we use the smooth invariant function $\mathrm{Trace} \circ \exp : \mathfrak{gl}(n) \to \mathbb{R}$ which is given by

$$\mathrm{Trace}(\exp(A)) = \mathrm{Trace}\left(\sum_{k \geq 0} \frac{A^k}{k!}\right) = \sum_{k \geq 0} \frac{1}{k!} \mathrm{Trace}(A^k).$$

Of course one should first take the Taylor series at 0 of it and then take the Chern-Weil class of each homogeneous part separately.

Theorem. *Let (E_i, p, M) be vector bundles over the same base manifold M. Then we have*

(1) $\operatorname{tr}(E_1 \oplus E_2) = \operatorname{tr}(E_1) + \operatorname{tr}(E_2)$.
(2) $\operatorname{tr}(E_1 \otimes E_2) = \operatorname{tr}(E_1) \wedge \operatorname{tr}(E_2)$.
(3) $\operatorname{tr}(g^* E) = g^* \operatorname{tr}(E)$ *for any smooth mapping* $g : N \to M$.

Clearly stably equivalent vector bundles have equal Pontryagin characters. Statements 1 and 2 say that one may view the Pontryagin character as a ring homomorphism from the real K-theory into cohomology,

$$\operatorname{tr} : K_{\mathbb{R}}(M) \to H^{4*}(M; \mathbb{R}).$$

Statement 3 says, that it is even a natural transformation.

Proof. 1. This can be proved in the same way as 17.7.1, but we indicate another method which will be used also in the proof of 2 below. Covariant derivatives for E_1 and E_2 induce a covariant derivative on $E_1 \oplus E_2$ by $\nabla_X^{E_1 \oplus E_2}(s_1, s_2) = (\nabla_X^{E_1} s_1, \nabla_X^{E_2}, s_2)$. For the curvature operators we clearly have

$$R^{E_1 \oplus E_2} = R^{E_1} \oplus R^{E_2} = \begin{pmatrix} R^{E_1} & 0 \\ 0 & R^{E_2} \end{pmatrix}$$

So the result follows from 17.9 with the help of 17.5.

2. We have an induced covariant derivative on $E_1 \otimes E_2$ given by $\nabla_X^{E_1 \otimes E_2} s_1 \otimes s_2 = (\nabla_X^{E_1} s_1) \otimes s_2 + s_1 \otimes (\nabla_X^{E_2} s_2)$. Then for the curvatures we get obviously $R^{E_1 \otimes E_2}(X, Y) = R^{E_1}(X, Y) \otimes Id_{E_2} + Id_{E_1} \otimes R^{E_2}(X, Y)$. The two summands of the last expression commute, so we get

$$(R^{E_1} \otimes Id_{E_2} + Id_{E_1} \otimes R^{E_2})^{\circ_\wedge, k} = \sum_{j=0}^{k} \binom{k}{j} (R^{E_1})^{\circ_\wedge, j} \otimes_\wedge (R^{E_2})^{\circ_\wedge, k-j},$$

where the product involved is given as in

$$(R^E \circ_\wedge R^E)(X_1, \dots, X_4) = \tfrac{1}{2!2!} \sum_\sigma \operatorname{sign}(\sigma) R^E(X_{\sigma 1}, X_{\sigma 2}) \circ R^E(X_{\sigma 3}, X_{\sigma 4}),$$

which makes $(\Omega(M; L(E, E)), \circ_\wedge)$ into a graded associative algebra. The next computation takes place in a commutative subalgebra of it:

$$\begin{aligned} \operatorname{tr}(E_1 \otimes E_2) &= [\operatorname{Trace} \exp(R^{E_1} \otimes Id_{E_2} + Id_{E_1} \otimes R^{E_2})]_{H(M)} \\ &= [\operatorname{Trace}(\exp(R^{E_1}) \otimes_\wedge \exp(R^{E_2}))]_{H(M)} \\ &= [\operatorname{Trace}(\exp(R^{E_1})) \wedge \operatorname{Trace}(\exp(R^{E_2}))]_{H(M)} \\ &= \operatorname{tr}(E_1) \wedge \operatorname{tr}(E_2). \end{aligned}$$

3. This is a general fact. □

17.11. The Pfaffian coefficient. Let (V, g) be a real Euclidian vector space of dimension n, with a positive definite inner product g. Then for each p we have an induced inner product on $\Lambda^p V$ which is given by

$$\langle x_1 \wedge \dots \wedge x_p, y_1 \wedge \dots \wedge y_p \rangle_g = \det(g(x_i, y_j)_{i,j}).$$

Moreover the inner product g, when viewed as a linear isomorphism $g : V \to V^*$, induces an isomorphism $\beta : \Lambda^2 V \to L_{g,\text{ skew}}(V, V)$ which is given on decomposable forms by $\beta(x \wedge y)(z) = g(x, z)y - g(y, z)x$. We also have

$$\beta^{-1}(A) = A \circ g^{-1} \in L_{\text{skew}}(V^*, V) = \{B \in L(V^*, V) : B^t = -B\} \cong \Lambda^* V, \text{ where}$$

$$B^t : V^* \xrightarrow{B^*} V^{**} \xrightarrow{\cong} V.$$

Now we assume that V is of even dimension n and is oriented. Then there is a unique element $e \in \Lambda^n V$ which is positive and normed: $\langle e, e \rangle_g = 1$. We define

$$\operatorname{Pf}^g(A) := \frac{1}{n!} \langle e, \overbrace{\beta^{-1}(A) \wedge \dots \wedge \beta^{-1}(A)}^{n/2} \rangle_g.$$

This is a homogeneous polynomial of degree $n/2$ on $\mathfrak{gl}(n)$. Its polarisation is the $n/2$-linear symmetric functional

$$\operatorname{Pf}^g(A_1, \dots, A_{n/2}) = \frac{1}{n!} \langle e, \beta^{-1}(A_1) \wedge \dots \wedge \beta^{-1}(A_{n/2}) \rangle_g.$$

Lemma. *1. If $U \in O(V, g)$ then $\operatorname{Pf}^g(U.A.U^{-1}) = \det(U) \operatorname{Pf}^g(A)$, so Pf^g is invariant under the adjoint action of $SO(V, g)$.*

2. If $X \in L_{g,\text{ skew}}(V, V) = \mathfrak{o}(V, g)$ then we have

$$\sum_{i=1}^{n/2} \operatorname{Pf}^g(A_1, \dots, [X, A_i], \dots, A_{n/2}) = 0.$$

Proof. We have $U \in O(V, g)$ if and only if $g(Ux, Uy) = g(x, y)$. For $g : V \to V^*$ this means $U^* g U = g$ and $U^{-1} g^{-1} (U^{-1})^* = g^{-1}$, so we get $\beta^{-1}(UAU^{-1}) = UAU^{-1} g^{-1} = UAg^{-1}U^* = \Lambda^2(U)\beta^{-1}(A)$ and in turn:

$$\begin{aligned}
\operatorname{Pf}^g(UAU^{-1}) &= \frac{1}{n!} \langle e, \Lambda^n(U)(\beta^{-1}(A) \wedge \dots \wedge \beta^{-1}(A)) \rangle_g \\
&= \frac{1}{n!} \det(U) \langle \Lambda^n(U) e, \Lambda^n(U)(\beta^{-1}(A) \wedge \dots \wedge \beta^{-1}(A)) \rangle_g \\
&= \frac{1}{n!} \det(U) \langle e, \beta^{-1}(A) \wedge \dots \wedge \beta^{-1}(A) \rangle_g \\
&= \det(U) \operatorname{Pf}^g(A).
\end{aligned}$$

2. This follows from 1. by differentiation, see the beginning of the proof of 17.3. $\square$

17.12. The Pfaffian class. Let (E, p, M, V) be a vector bundle which is fiber oriented and of even fiber dimension. If we choose a fiberwise Riemannian metric on E, we in fact reduce the linear frame bundle of E to the oriented orthonormal one $SO(\mathbb{R}^n, E)$. On the Lie algebra $\mathfrak{o}(n, \mathbb{R})$ of the structure group $SO(n, \mathbb{R})$ the Pfaffian form Pf of the standard inner product is an invariant, $\operatorname{Pf} \in I^{n/2}(SO(n, \mathbb{R}))$. We define the *Pfaffian class* of the oriented bundle E by

$$\operatorname{Pf}(E) := \left(\frac{-1}{2\pi\sqrt{-1}}\right)^{n/2} \frac{1}{(n/2)!} \operatorname{Cw}(\operatorname{Pf}, SO(\mathbb{R}^n, E)) \in H^n(M).$$

It does not depend on the choice of the Riemannian metric on E, since for any two fiberwise Riemannian metrics g_1 and g_2 on E there is an isometric vector bundle isomorphism $f : (E, g_1) \to (E, g_2)$ covering the identity of M, which pulls a $SO(n)$-connection for (E, g_2) to an $SO(n)$-connection for (E, g_1). So the two Pfaffian classes coincide since then $\operatorname{Pf}^1 \circ (f^*\Omega_2 \otimes_\wedge \cdots \otimes_\wedge f^*\Omega_2) = \operatorname{Pf}^2 \circ (\Omega_2 \otimes_\wedge \cdots \otimes_\wedge \Omega_2)$.

Theorem. *The Pfaffian class of oriented even dimensional vector bundles has the following properties:*

1. $\operatorname{Pf}(E)^2 = (-1)^{n/2} p_{n/2}(E)$ *where n is the fiber dimension of E.*
2. $\operatorname{Pf}(E_1 \oplus E_2) = \operatorname{Pf}(E_1) \wedge \operatorname{Pf}(E_2)$
3. $\operatorname{Pf}(g^*)(E) = g^* \operatorname{Pf}(E)$ *for smooth $g : N \to M$.*

Proof. This is left as an exercise for the reader. □

17.13. Chern classes. Let (E, p, M) be a complex vector bundle over the smooth manifold M. So the structure group is $GL(n, \mathbb{C})$ where n is the fiber dimension. Recall now the explanation of the characteristic coefficients c_k^n in 17.6 and insert complex numbers everywhere. Then we get the characteristic coefficients $c_k^n \in I^k(GL(n, \mathbb{C}))$, which are just the extensions of the real ones to the complexification.

We define then the *Chern classes* by

(1) $$c_k(E) := \left(\frac{-1}{2\pi\sqrt{-1}}\right)^k \operatorname{Cw}(c_k^{\dim E}, E) \in H^{2k}(M; \mathbb{R}).$$

The *total Chern class* is again the inhomogeneous cohomology class

(2) $$c(E) := \sum_{k=0}^{\dim_{\mathbb{C}} E} c_k(E) \in H^{2*(M;\mathbb{R})}.$$

It has the following properties:

$$c(\bar{E}) = (-1)^{\dim_{\mathbb{C}} E} c(E) \tag{3}$$

$$c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2) \tag{4}$$

$$c(g^* E) = g^* c(E) \quad \text{for smooth } g : N \to M \tag{5}$$

One can show (see [Milnor-Stasheff, 1974]) that (2), (4), (5), and the following normalisation determine the total Chern class already completely: The total Chern class of the canonical complex line bundle over S^2 (the square root of the tangent bundle with respect to the tensor product) is $1 + \omega_{S^2}$, where ω_{S^2} is the canonical volume form on S^2 with total volume 1.

Lemma. *Then Chern classes are real cohomology classes.*

Proof. We choose a hermitian metric on the complex vector bundle E, i. e. we reduce the structure group from $GL(n, \mathbb{C})$ to $U(n)$. Then the curvature Ω of a $U(n)$-principal connection has values in the Lie algebra $\mathfrak{u}(n)$ of skew hermitian matrices A with $A^* = -A$. But then we have $c_k^n(-\sqrt{-1}A) \in \mathbb{R}$ since $\overline{\det_{\mathbb{C}}(-\sqrt{-1}A + t\mathbb{I})} = \det_{\mathbb{C}}(\overline{-\sqrt{-1}A} + t\mathbb{I}) = \det_{\mathbb{C}}(-\sqrt{-1}A + t\mathbb{I})$. □

17.14. The Chern character. The *trace classes of a complex vector bundle* are given by

$$\operatorname{tr}_k(E) := \left(\frac{-1}{2\pi\sqrt{-1}}\right)^k \operatorname{Cw}(\operatorname{tr}_k^{\dim E}, E) \in H^{2k}(M, \mathbb{R}). \tag{1}$$

They are also real cohomology classes, and we have $\operatorname{tr}_0(E) = \dim_{\mathbb{C}} E$, the fiber dimension of E, and $\operatorname{tr}_1(E) = c_1(E)$. In general we have the follwoing recursive relation between the Chern classes and the trace classes:

$$c_k(E) = \frac{-1}{k} \sum_{j=0}^{k-1} c_j(E) \wedge \operatorname{tr}_{k-j}(E), \tag{2}$$

which follows directly from lemma 17.9. The inhomogeneous cohomology class

$$\operatorname{ch}(E) := \sum_{k \geq 0} \frac{1}{k!} \operatorname{tr}_k(E) \in H^{2*}(M, \mathbb{R}) \tag{3}$$

is called the *Chern character* of the complex vector bundle E. With the same methods as for the Pontryagin character one can show that the Chern character satisfies the following properties:

$$\operatorname{ch}(E_1 \oplus E_2) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2) \tag{4}$$

$$\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch}(E_1) \wedge \operatorname{ch}(E_2) \tag{5}$$

$$\operatorname{ch}(g^* E) = g^* \operatorname{ch}(E) \tag{6}$$

From these it clearly follows that the Chern character can be viewed as a ring homomorphism from complex K-theory into even cohomology,

$$\operatorname{ch} : K_{\mathbb{C}}(M) \to H^{2*}(M, \mathbb{R}),$$

which is natural.

Finally we remark that the Pfaffian class of the underlying real vector bundle of a complex vectorbundle E of complex fiber dimension n coincides with the Chern class $c_n(E)$. But there is a new class, the Todd class, see below.

17.15. The Todd class. On the vector space $\mathfrak{gl}(n, \mathbb{C})$ of all complex $(n \times n)$-matrices we consider the smooth function

$$f(A) := \det_{\mathbb{C}} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} A^k \right). \tag{1}$$

It is the unique smooth function which satisfies the functional equation

$$\det(A).f(A) = \det(\mathbb{I} - \exp(-A)).$$

Clearly f is invariant under $\operatorname{Ad}(GL(n, \mathbb{C}))$ and $f(0) = 1$, so we may consider the invariant smooth function, defined near 0, $\operatorname{Td} : \mathfrak{gl}(n, \mathbb{C}) \supset U \to \mathbb{C}$, which is given by $\operatorname{Td}(A) = 1/f(A)$. It is uniquely defined by the functional equation

$$\det(A) = \operatorname{Td}(A) \det(\mathbb{I} - \exp(-A))$$
$$\det(\tfrac{1}{2}A) \det(\exp(\tfrac{1}{2}A)) = \operatorname{Td}(A) \det(\sinh(\tfrac{1}{2}A)).$$

The *Todd class* of a complex vector bundle is then given by

$$\begin{aligned} \operatorname{Td}(E) &= \left[GL(\mathbb{C}^n, E)[\operatorname{Td}] \left(\sum_{k \geq 0} \left(\frac{-1}{2\pi\sqrt{-1}} R^E \right)^{\otimes_\wedge, k} \right) \right]_{H^{2*}(M, \mathbb{R})} \\ &= \operatorname{Cw}(\operatorname{Td}, E). \end{aligned} \tag{2}$$

The Todd class is a real cohomology class since for $A \in \mathfrak{u}(n)$ we have $\operatorname{Td}(-A) = \operatorname{Td}(A^*) = \overline{\operatorname{Td}(A)}$. Since $\operatorname{Td}(0) = 1$, the Todd class $\operatorname{Td}(E)$ is an invertible element of $H^{2*}(M, \mathbb{R})$.

17.16. The Atiyah-Singer index formula (roughly). Let E_i be complex vector bundles over a compact manifold M, and let $D : C^\infty(E_1) \to C^\infty(E_2)$ be an elliptic pseudodifferential operator of order p. Then for appropriate Sobolev completions D prolongs to a bounded Fredholm operator between Hilbert spaces $D : \mathcal{H}^{d+p}(E_1) \to \mathcal{H}^d(E_2)$. Its *index* $\operatorname{index}(D)$ is defined as the dimension of the kernel minus dimension of the cokernel, which does not depend on d if it is high enough. The Atiyah-Singer index formula says that

$$\operatorname{index}(D) = (-1)^{\dim M} \int_{TM} \operatorname{ch}(\sigma(D)) \operatorname{Td}(TM \otimes \mathbb{C}),$$

where $\sigma(D)$ is a virtual vector bundle (with compact support) on $TM \setminus 0$, a formal difference of two vector bundles, the so called symbol bundle of D.

See [Boos, 1977] for a rather unprecise introduction, [Shanahan, 1978] for a very short introduction, [Gilkey, 1984] for an analytical treatment using the heat kernel method, [Lawson, Michelsohn, 1989] for a recent treatment and the papers by Atiyah and Singer for the real thing.

Special cases are The Gauss-Bonnet-Chern formula, and the Riemann-Roch-Hizebruch formula.

18. Jets

Jet spaces or jet bundles consist of the invariant expressions of Taylor developments up to a certain order of smooth mappings between manifolds. Their invention goes back to Ehresmann [Ehresmann, 1951]. We could have treated them from the beginning and could have mixed them into every chapter; but it is also fine to have all results collected in one place.

18.1. Contact. Recall that smooth functions $f, g : \mathbb{R} \to \mathbb{R}$ are said to have *contact of order* k at 0 if all their values and all derivatives up to order k coincide.

Lemma. *Let $f, g : M \to N$ be smooth mappings between smooth manifolds and let $x \in M$. Then the following conditions are equivalent.*

(1) *For each smooth curve $c : \mathbb{R} \to M$ with $c(0) = x$ and for each smooth function $h \in C^\infty(M, \mathbb{R})$ the two functions $h \circ f \circ c$ and $h \circ g \circ c$ have contact of order k at 0.*
(2) *For each chart (U, u) of M centered at x and each chart (V, v) of N with $f(x) \in V$ the two mappings $v \circ f \circ u^{-1}$ and $v \circ g \circ u^{-1}$, defined near 0 in $\mathbb{R}^m$, with values in $\mathbb{R}^n$, have the same Taylor development up to order k at 0.*
(3) *For some charts (U, u) of M and (V, v) of N with $x \in U$ and $f(x) \in V$ we have*
$$\frac{\partial^{|\alpha|}}{\partial u^\alpha}\bigg|_x (v \circ f) = \frac{\partial^{|\alpha|}}{\partial u^\alpha}\bigg|_x (v \circ g)$$
for all multi indices $\alpha \in \mathbb{N}^m$ with $0 \le |\alpha| \le k$.
(4) *$T^k_x f = T^k_x g$, where T^k is the k-th iterated tangent bundle functor.*

Proof. This is an easy exercise in Analysis.

18.2. Definition. If the equivalent conditions of lemma 18.1 are satisfied, we say that f and g have the same *k-jet at x* and we write $j^k f(x)$ or $j^k_x f$ for the resulting equivalence class and call it the *k -jet at x* of f; x is called the *source* of the k-jet, $f(x)$ is its *target*.

The space of all k-jets of smooth mappings from M to N is denoted by $J^k(M, N)$. We have the *source mapping* $\alpha : J^k(M, N) \to M$ and the *target mapping* $\beta : J^k(M, N) \to N$, given by $\alpha(j^k f(x)) = x$ and $\beta(j^k f(x)) = f(x)$. We will also write $J^k_x(M, N) := \alpha^{-1}(x)$, $J^k(M, N)_y := \beta^{-1}(y)$, and $J^k_x(M, N)_y := J^k_x(M, N) \cap J^k(M, N)_y$ for the spaces of jets with source x, target y, and both, respectively. For $l < k$ we have a canonical surjective mapping $\pi^k_l : J^k(M, N) \to J^l(M, N)$, given by $\pi^k_l(j^k f(x)) := j^l f(x)$. This mapping respects the fibers of α and β and $\pi^k_0 = (\alpha, \beta) : J^k(M, N) \to M \times N$.

18.3. .. Now we look at the case $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$.

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth mapping. Then by 18.1.3 the k-jet $j^k f(x)$ of f ant x has a canonical representative, namely the Taylor polynomial of order k of f at x:

$$\begin{aligned} f(x+y) &= f(x) + df(x).y + \frac{1}{2!}d^2 f(x)y^2 + \dots + \frac{1}{k!}d^k f(x).y^k + o(|y|^k) \\ &=: f(x) + \operatorname{Tay}_x^k f(y) + o(|y|^k) \end{aligned}$$

Here y^k is short for $(y, y, \dots, y)$, k-times. The 'Taylor polynomial without constant'

$$\operatorname{Tay}_x^k f : y \mapsto \operatorname{Tay}_x^k(y) := df(x).y + \frac{1}{2!}d^2 f(x).y^2 + \dots + \frac{1}{k!}d^k f(x).y^k$$

is an element of the linear space

$$P^k(m,n) := \bigoplus_{j=1}^{k} L^j_{sym}(\mathbb{R}^m, \mathbb{R}^n),$$

where $L^j_{sym}(\mathbb{R}^m, \mathbb{R}^n)$ is the vector space of all j-linear symmetric mappings $\mathbb{R}^m \times \dots \times \mathbb{R}^m \to \mathbb{R}^n$, where we silently use the total polarization of polynomials. Conversely each polynomial $p \in P^k(m,n)$ defines a k-jet $j_0^k(y \mapsto z + p(x + y))$ with arbitrary source x and target z. So we get canonical identifications $J_x^k(\mathbb{R}^m, \mathbb{R}^n)_z \cong P^k(m,n)$ and

$$J^k(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^m \times \mathbb{R}^n \times P^k(m,n).$$

If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open subsets then clearly $J^k(U,V) \cong U \times V \times P^k(m,n)$ in the same canonical way.

For later uses we consider now the *truncated composition*

$$\bullet : P^k(m,n) \times P^k(p,m) \to P^k(p,n),$$

where $p \bullet q$ is just the polynomial $p \circ q$ without all terms of order $> k$. Obviously it is a polynomial, thus real analytic mapping. Now let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and $W \subset \mathbb{R}^p$ be open subsets and consider the fibered product

$$\begin{aligned} J^k(U,V) \times_U J^k(W,U) &= \{ (\sigma, \tau) \in J^k(U,V) \times J^k(W,U) : \alpha(\sigma) = \beta(\tau) \} \\ &= U \times V \times W \times P^k(m,n) \times P^k(p,m). \end{aligned}$$

Then the mapping

$$\gamma : J^k(U,V)\times_U J^k(W,U)\to J^k(W,V),$$
$$\gamma(\sigma,\tau)=\gamma((\alpha(\sigma),\beta(\sigma),\bar\sigma),(\alpha(\tau),\beta(\tau),\bar\tau))=(\alpha(\tau),\beta(\sigma),\bar\sigma\bullet\bar\tau),$$

is a real analytic mapping, called the *fibered composition of jets*.

Let U, $U'\subset\mathbb{R}^m$ and $V\subset\mathbb{R}^n$ be open subsets and let $g:U'\to U$ be a smooth diffeomorphism. We define a mapping $J^k(g,V):J^k(U,V)\to J^k(U,V')$ by $J^k(g,V)(j^kf(x))=j^k(f\circ g)(g^{-1}(x))$. Using the canonical representation of jets from above we get $J^k(g,V)(\sigma)=\gamma(\sigma,j^kg(g^{-1}(x)))$ or $J^k(g,V)(x,y,\bar\sigma)=(g^{-1}(x),y,\bar\sigma\bullet\operatorname{Tay}^k_{g^{-1}(x)}g)$. If g is a C^p diffeomorphism then $J^k(g,V)$ is a C^{p-k} diffeomorphism. If $g':U''\to U'$ is another diffeomorphism, then clearly $J^k(g',V)\circ J^k(g,V)=J^k(g\circ g',V)$ and $J^k(\quad,V)$ is a contravariant functor acting on diffeomorphisms between open subsets of $\mathbb{R}^m$. Since the truncated composition $\bar\sigma\mapsto\bar\sigma\bullet\operatorname{Tay}^k_{g^{-1}(x)}g$ is linear, the mapping $J^k_x(g,\mathbb{R}^n):=J^k(g,\mathbb{R}^n)|J^k_x(U,\mathbb{R}^n):J^k_x(U,\mathbb{R}^n)\to J^k_{g^{-1}(x)}(U',\mathbb{R}^n)$ is also linear.

If more generally $g:M'\to M$ is a diffeomorphism between manifolds the same formula as above defines a bijective mapping $J^k(g,N):J^k(M,N)\to J^k(M',N)$ and clearly $J^k(\quad,N)$ is a contravariant functor defined on the category of manifolds and diffeomorphisms.

Now let $U\subset\mathbb{R}^m$, $V\subset\mathbb{R}^n$, and $W\subset\mathbb{R}^p$ be open subsets and let $h:V\to W$ be a smooth mapping. Then we define $J^k(U,h):J^k(U,V)\to J^k(U,W)$ by $J^k(U,h)(j^kf(x))=j^k(h\circ f)(x)$ or equivalently by

$$J^k(U,h)(x,y,\bar\sigma)=(x,h(y),\operatorname{Tay}^k_y h\bullet\bar\sigma).$$

If h is C^p, then $J^k(U,h)$ is C^{p-k}. Clearly $J^k(U,\quad)$ is a covariant functor acting on smooth mappings between open subsets of finite dimensional vector spaces. The mapping $J^k_x(U,h)_y:J^k_x(U,V)_y\to J^k(U,W)_{h(y)}$ is linear if and only if the mapping $\bar\sigma\mapsto\operatorname{Tay}^k_y h\bullet\bar\sigma$ is linear, so if h is affine or if $k=1$.

If $h:N\to N'$ is a smooth mapping between manifolds we have by the same prescription a mapping $J^k(M,h):J^k(M,N)\to J^k(M,N')$ and $J^k(M,\quad)$ turns out to be a functor on the category of manifolds and smooth mappings.

18.4. The differential group G^k_m.. The k-jets at 0 of diffeomorphisms of $\mathbb{R}^m$ which map 0 to 0 form a group under truncated composition, which will be denoted by $GL^k(m,\mathbb{R})$ or G^k_m for short, and will be called the *differential group of order* k. Clearly an arbitrary 0-respecting k-jet $\sigma\in P^k(m,m)$ is in G^k_m if and only if its linear part is invertible, thus

$$G^k_m=GL^k(m,\mathbb{R})=GL(m)\oplus\bigoplus_{j=2}^k L^j_{\text{sym}}(\mathbb{R}^m,\mathbb{R}^m)=:GL(m)\times P^k_2(m),$$

where we put $P^k_2(m) = \bigoplus_{j=2}^k L^j_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^m)$ for the space of all polynomial mappings without constant and linear term of degree $\le k$. Since the truncated composition is even a polynomial mapping, G^k_m is a Lie group, and clearly the mapping $\pi^k_l : G^k_m \to G^l_m$ is a homomorphism of Lie groups, so $\ker(\pi^k_l) = \bigoplus_{j=l+1}^k L^j_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^m) =: P^k_{l+1}(m)$ is a normal subgroup for all l. The exact sequence of groups

$$\{e\} \to P^k_{l+1}(m) \to G^k_m \to G^l_m \to \{e\}$$

splits if and only if $l = 1$; only then we have a semidirect product.

18.5. Theorem. *If M and N are smooth manifolds, the following results hold.*

(1) *$J^k(M,N)$ is a smooth manifold (it is of class C^{r-k} if M and N are of class C^r); a canonical atlas is given by all charts $(J^k(U,V), J^k(u^{-1}, v))$, where (U,u) is a chart on M and (V,v) is a chart on N.*
(2) *$(J^k(M,N), (\alpha,\beta), M\times N, P^k(m,n), G^k_m\times G^k_n)$ is a fiber bundle with structure group, where $m = dimM$, $n = dimN$, and where $(\gamma,\chi) \in G^k_m \times G^k_n$ acts on $\sigma \in P^k(m,n)$ by $(\gamma,\chi).\sigma = \chi \bullet \sigma \bullet \gamma^{-1}$.*
(3) *If $f : M \to N$ is a smooth mapping then $j^k f : M \to J^k(M,N)$ is also smooth (it is C^{r-k} if f is C^r), sometimes called the k-jet extension of f. We have $\alpha \circ j^k f = Id_M$ and $\beta \circ j^k f = f$.*
(4) *If $g : M' \to M$ is a (C^r-) diffeomorphism then also the induced mapping $J^k(g,N) : J^k(M,N) \to J^k(M',N)$ is a (C^{r-k}-) diffeomorphism.*
(5) *If $h : N \to N'$ is a (C^r-) mapping then $J^k(M,h) : J^k(M,N) \to J^k(M,N')$ is a (C^{r-k}-) mapping. $J^k(M, \quad)$ is a covariant functor from the category of smooth manifolds and smooth mappings into itself which maps each of the following classes of mappings into itself: immersions, embeddings, closed embeddings, submersions, surjective submersions, fiber bundle projections. Furthermore $J^k(\quad,\quad)$ is a contra-covariant bifunctor.*
(6) *The projections $\pi^k_l : J^k(M,N) \to J^l(M,N)$ are smooth and natural, i.e. they commute with the mappings from (4) and (5).*
(7) *$(J^k(M,N), \pi^k_l, J^l(M,N), P^k_{l+1}(m,n))$ are fiber bundles for all l. The bundle $(J^k(M,N), \pi^k_{k-1}, J^{k-1}(M,N), L^k_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n))$ is an affine bundle. The first jet space $J^1(M,N)$ is a vector bundle, it is isomorphic to the bundle $(L(TM,TN), (\pi_M, \pi_N), M \times N)$. Moreover we have $J^1_0(\mathbb{R}, N) = TN$ and $J^1(M,\mathbb{R})_0 = T^*M$.*

Proof. We use 18.3 heavily. Let (U_γ, u_γ) be an atlas of M and let $(V_\varepsilon, v_\varepsilon)$ be an atlas of N. Then $J^k(u_\gamma^{-1}, v_\varepsilon) : (\alpha,\beta)^{-1}(U_\gamma \times V_\varepsilon) \to J^k(u_\gamma(U_\gamma), v_\varepsilon(V_\varepsilon))$ is a bijective mapping and the chart change looks like

$$J^k(u_\gamma^{-1}, v_\varepsilon) \circ J^k(u_\delta^{-1}, v_\nu)^{-1} = J^k(u_\delta \circ u_\gamma^{-1}, v_\varepsilon \circ v_\nu^{-1})$$

by the functorial properties of $J^k(\quad,\quad)$. With the identification topology $J^k(M,N)$ is Hausdorff, since it is a fiber bundle and the usual argument for gluing fiber bundles applies. So (1) follows.

Now we make this manifold atlas into a fiber bundle by using as charts $(U_\gamma \times V_\varepsilon), \psi_{(\gamma,\varepsilon)} : J^k(M,N)|U_\gamma \times V_\varepsilon \to U_\gamma \times V_\varepsilon \times P^k(m,n)$, where $\psi_{(\gamma,\varepsilon)}(\sigma) = (\alpha(\sigma), \beta(\sigma), J^k_{\alpha(\sigma)}(u_\gamma^{-1}, v_\varepsilon)_{\beta(\sigma)}$. We then get as transition functions

$$\begin{aligned}\psi_{(\gamma,\varepsilon)}\psi_{(\delta,\nu)}(x,y,\bar\sigma) &= (x, y, J^k_{u_\delta(x)}(u_\delta \circ u_\gamma^{-1}, v_\varepsilon \circ v_\nu^{-1})(\bar\sigma)) \\ &= \big(x, y, \operatorname{Tay}^k_{v_\nu(y)}(v_\varepsilon \circ v_\nu^{-1}) \bullet \bar\sigma \bullet \operatorname{Tay}^k_{u_\gamma(x)}(u_\delta \circ u_\gamma^{-1})\big),\end{aligned}$$

and (2) follows.

(3), (4), and (6) are obvious from 18.3, mainly by the functorial properties of $J^k(\quad,\quad)$.

(5). We will show later that these assertions hold in a much more general situation, see the chapter on product preserving functors. It is clear from 18.3 that $J^k(M,h)$ is a smooth mapping. The rest follows by looking at special chart representations of h and the induced chart representations for $J^k(M,h)$.

It remains to show (7) and here we concentrate on the affine bundle. Let $a_1 + a \in GL(n) \times P^k_2(n,n)$, $\sigma + \sigma_k \in P^{k-1}(m,n) \oplus L^k_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n)$, and $b_1 + b \in GL(m) \times P^k_2(m,m)$, then the only term of degree k containing σ_k in $(a + a_k) \bullet (\sigma + \sigma_k) \bullet (b + b_k)$ is $a_1 \circ \sigma_k \circ b_1^k$, which depends linearly on σ_k. To this the degree k-components of compositions of the lower order terms of σ with the higher order terms of a and b are added, and these may be quite arbitrary. So an affine bundle results.

We have $J^1(M,N) = L(TM,TN)$ since both bundles have the same transition functions. Finally we have $J^1_0(\mathbb{R},N) = L(T_0\mathbb{R},TN) = TN$, and $J^1(M,\mathbb{R})_0 = L(TM,T_0\mathbb{R}) = T^*M$ $\square$

18.6. Frame bundles and natural bundles.. Let M be a manifold of dimension m. We consider the jet bundle $J^1_0(\mathbb{R}^m, M) = L(T_0\mathbb{R}^m, TM)$ and the open subset $inv J^1_0(\mathbb{R}^m, M)$ of all invertible jets. This is visibly equal to the linear frame bundle of TM as treated in 15.11.

Note that a mapping $f : \mathbb{R}^m \to M$ is locally invertible near 0 if and only if $j^1 f(0)$ is invertible. A jet σ will be called *invertible* if its order 1-part $\pi^k_1(\sigma) \in J^1_0(\mathbb{R}^m, M)$ is invertible. Let us now consider the open subset $inv J^k_0(\mathbb{R}^m, M) \subset$

$J^1_0(\mathbb{R}^m, M)$ of all invertible jets and let us denote it by $P^k M$. Then by 12.5.2 we have a principal fiber bundle $(P^k M, \pi_M, M, G^k_m)$ which is called the *k-th order frame bundle* of the manifold M. Its principal right action r can be described in several ways. By the fiber composition of jets:

$$r = \gamma : inv J^k_0(\mathbb{R}^m, \mathbb{R}^m) \times inv J^k_0(\mathbb{R}^m, M) = G^k_m \times P^k M \to P^k M;$$

or by the functorial property of the jet bundle:

$$r^{j^k g(0)} = inv J^k_0(g, M)$$

for a local diffeomorphism $g : \mathbb{R}^m, 0 \to \mathbb{R}^m, 0$.

If $h : M \to M'$ is a local diffeomorphism, the induced mapping $J^k_0(\mathbb{R}^m, h)$ maps the open subset $P^k M$ into $P^k M'$. By the second description of the principal right action this induced mapping is a homomorphism of principal fiber bundles which we will denote by $P^k(h) : P^k M \to P^k M'$. Thus P^k becomes a covariant functor from the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms into the category of all principal fiber bundles with structure group G^k_m over m-dimensional manifolds and homomorphisms of principal fiber bundles covering local diffeomorphisms.

If we are given any smooth left action $\ell : G^k_m \times S \to S$ on some manifold S, the associated bundle construction from theorem 15.7 gives us a fiber bundle $P^k M[S, \ell] = P^k M \times_{G^k_m} S$ over M for each m-dimensional manifold M; by 15.9.2 this describes a functor $P^k(\quad)[S, \ell]$ from the category $\mathcal{M}f_m$ into the category of all fiber bundles over m-dimensional manifolds with standard fiber S and G^k_m-structure, and homomorphisms of fiber bundles covering local diffeomorphisms. These bundles are also called *natural bundles* or *geometric objects*.

It is one of the aims of this book to prove that under mild conditions all functors between the mentioned categories are of the form described above. This will involve some rather hard analytical results.

18.7. Theorem. *If (E, p, M, S) is a fiber bundle, let us denote by $J^k(E)$ the space of all k-jets of sections of E. Then we have:*

(1) *$J^k(E)$ is a closed submanifold of $J^k(M, E)$.*
(2) *The first jet bundle $J^1(E)$ is an affine subbundle of the vector bundle $J^1(M, E) = L(TM, TE)$, in fact we have $J^1(E) = \{\, \sigma \in L(TM, TE) : Tp \circ \sigma = Id_{TM} \,\}$.*
(3) *$(J^k(E), \pi^k_{k-1}, J^{k-1}(E))$ is an affine bundle.*
(4) *If (E, p, M) is a vector bundle, then $(J^k(E), \alpha, M)$ is also a vector bundle. If $\phi : E \to E'$ is a homomorphism of vector bundles covering the identity, then $J^k(\varphi)$ is of the same kind.*

Proof. (1). By 18.6.5 the mapping $J^k(M,p)$ is a submersion, thus $J^k(E) = J^k(M,p)^{-1}(j^k(Id_M))$ is a submanifold. (2) is clear. (3) and (4) are seen by looking at appropriate canonical charts. $\square$

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List of Symbols

$\alpha : J^r(M,N) \to M$ the source mapping of jets

$\beta : J^r(M,N) \to N$ the target mapping of jets
$C^\infty(E)$, also $C^\infty(E \to M)$ the space of smooth sections of a fiber bundle
$C^\infty(M, \mathbf{R})$ the space of smooth functions on M
d usually the exterior derivative
(E, p, M, S), also simply E usually a fiber bundle with total space E, base M, and standard fiber S
Fl^X_t, also $\mathrm{Fl}(t, X)$ the flow of a vector field X
$\mathbb{I}_k$, short for the $k \times k$-identity matrix $Id_{\mathbb{R}^k}$.
$\mathcal{L}_X$ Lie derivative
G usually a general Lie group with multiplication $\mu : G \times G \to G$, left translation λ, and right translation ρ
$J^r(E)$ the bundle of r-jets of sections of a fiber bundle $E \to M$
$J^r(M,N)$ the bundle of r-jets of smooth functions from M to N 12.2
$j^r f(x)$, also $j^r_x f$ the r-jet of a mapping or function f
$\ell : G \times S \to S$ usually a left action
M usually a (base) manifold
$\mathbb{N}$ natural numbers
$\mathbb{N}_0$ nonnegative integers
$\pi^r_l : J^r(M,N) \to J^l(M,N)$ projections of jets
$\mathbb{R}$ real numbers
$r : P \times P \to P$ usually a right action, in particular the principal right action of a principal bundle
TM the tangent bundle of a manifold M with projection $\pi_M : TM \to M$ 1.?
$\mathbb{Z}$ integers

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