Wavelet Spectra compared to Fourier Spectra

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Abstract

The relation between Fourier spectra and spectra obtained from wavelet analysis is established. Small scale asymptotic analysis shows that the wavelet spectrum is meaningful only when the analysing wavelet has enough vanishing moments. We relate these results to regularity theorems in Besov spaces. For the analysis of infinitly regular signals, a new wavelet, with infinit number of cancellations is proposed.

1 Introduction

Since its first definition by Grossmann and Morlet [1] the wavelet transform has been extensively studied mathematically (for reviews see [2] or [3]), and has been applied successfully to several topics in signal analysis and image processing. Its ability to decompose a signal into contributions localized both in space and scale, was found specially attractive to analyse inhomogeneous fields in fluid mechanics and turbulence ([4], [5], [6], [7] and [8]). However although the relationship between the decay of wavelet coefficients with scale and regularity properties of the signal has been clearly stated ([10], [11], [12], [2]), the relationship between spectral slopes obtained from wavelet spectra and spectral slopes obtained from classical Fourier spectra were still unclear. In fact the wavelet analysis depends both on the signal and on the analysing wavelet. Are the spectral slopes computed from a wavelet analysis strongly biaised by the wavelet? This is an important question for quantitative studies.

Indeed, wavelet analysis allows to define, locally in space, a power spectrum. When, this could not be achieved properly by Fourier techniques. With this local spectrum one is able to compare the spectral properties of the signal at different locations in physical space. As a Plancherel identity exists for the wavelet transform, this comparison is justified, wavelet spectral densities being additive contributions to the total energy of

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the signal. If we want to go further and compare the slopes of local spectra to the ones given by any analytical theory or to the slopes of global Fourier spectra, we need to check first if these slopes are consistent. This goes through the comparison of the mean spectrum obtained with wavelet analysis to the classical Fourier spectrum.

In section 2 we recall briefly definitions and properties of local and mean wavelet spectra and establish their connections with the Fourier spectrum.

In section 3 we demonstrate that wavelet spectra recover algebraically decreasing Fourier spectra only if the analysing wavelet has enough vanishing moments.

In section 4 we establish the fact that wavelets of finite order are not able to detect exponentially decreasing spectra.

Section 5 is devoted to the small scale behaviour of the local wavelet spectrum and its connection with regularity theorems.

Section 6 considers wavelets with infinite number of cancellations which overcome the problems shown in the preceeding sections.

For sake of simplicity all the derivations are done in the one dimensional case; however most of the results generalize easily in dimension 2 and more. Results for dimension larger than one are summarized in the appendix.

2 Wavelet and Fourier spectra, definitions and properties

Consider s a real signal of a real variable x. With suitable regularity properties, one may define its Fourier transform $\hat{s}(k)$ and its power spectrum $E(k)$:

$$\hat{s}(k) \;=\; \int_{-\infty}^{+\infty} s(x)\; e^{-ikx} dx \tag{1}$$

$$\boxed{E(k) \;=\; \frac{1}{2\pi}\; |\hat{s}(k)|^2 \;\; \text{for} \;\; k \geq 0} \;. \tag{2}$$

The total energy E of the signal is such that:

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} |s(x)|^2 \; dx = \frac{1}{4\pi}\int_{-\infty}^{+\infty} |\hat{s}(k)|^2 dk = \int_{0}^{+\infty} E(k) dk. \tag{3}$$

Now consider ψ a function of $L^1(IR) \cap L^2(IR)$: ψ is called an analysing wavelet, if it verifies, at least, the admissibility condition:

$$\int_{0}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega}\; d\omega \;<\; +\infty \;. \tag{4}$$

This imposes $\hat{\psi}(0) = 0$; the wavelet has a zero mean. A stronger condition is to impose cancellations up to some order p:

$$\int_{-\infty}^{+\infty} x^n \psi(x)\, dx = 0 \quad \text{for} \quad n = 0,\ 1 \ldots p-1, \quad \text{and} \quad \int_{-\infty}^{+\infty} x^p \psi(x)\, dx \neq 0\ . \tag{5}$$

Note that if ψ is integrable and square integrable, then the admissibility condition (4) is equivalent to a cancellation condition of order at least 0, and condition (5) is equivalent (up to a multiplicative constant) to the existence of a bounded, continuous function φ, with $\varphi(0) = 1$ and $\varphi(\infty) = 0$ such that:

$$\hat{\psi}(\omega) \;=\; \omega^p\ \varphi(\omega)\ . \tag{6}$$

Within suitable hypotheses, one may define the wavelet transform $\tilde{s}$ of the signal s:

$$\tilde{s}(a,b) \;=\; \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} s(x)\overline{\psi(\frac{x-b}{a})}\, dx, \tag{7}$$

where b is the position variable and $a > 0$ the scale. Alternatively, the wavelet transform can be computed from the Fourier transform of the signal:

$$\tilde{s}(a,b) \;=\; \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{+\infty} \hat{s}(\omega)\overline{\hat{\psi}(a\omega)}\ e^{i\omega b}\ d\omega. \tag{8}$$

We will suppose in the following that the wavelet ψ is real; however our results and formulas are still valid, up to a multiplicative constant, for complex progressive wavelets (see [1] for more details on progessive wavelets).
From the energy conservation property of the wavelet transform:

$$E \;=\; \frac{1}{2c_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |\tilde{s}(a,b)|^2\ \frac{da}{a^2}\ db, \tag{9}$$

where: $\quad c_\psi \;=\; \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega}\ d\omega$, we may define a local wavelet spectrum:

$$\boxed{\tilde{E}(k,x) \;=\; \frac{1}{2c_\psi\ k_0}\ |\tilde{s}(\frac{k_0}{k}, x)|^2 \ \text{ for } \ k \geq 0 \ \text{ and } \ x \in IR}\ , \tag{10}$$

where k_0 is the peak wavenumber of the analysing wavelet ψ.
The local spectrum measures the contribution to the total energy coming from the vicinity of point x and wavenumber k, this "vicinity" depending on the proper shape in physical and spectral space of the analysing wavelet ψ.

From this local spectrum one may in turn define a mean wavelet spectrum $\tilde{E}(k)$:

$$\tilde{E}(k) \ = \ \int_{-\infty}^{+\infty} \ \tilde{E}(k,x) \ dx, \tag{11}$$

which is linked to the total energy by:

$$E \ = \ \int_{0}^{+\infty} \ \tilde{E}(k) \ dk. \tag{12}$$

From formulas (2, 8, 10, 11) one may derive easily the relationship between the Fourier spectrum $E(k)$ and the mean wavelet spectrum $\tilde{E}(k)$, namely:

$$\boxed{\tilde{E}(k) \ = \ \frac{1}{c_\psi \ k} \ \int_0^{+\infty} E(\omega) \ |\hat{\psi}(\frac{k_0 \omega}{k})|^2 \ d\omega} \ . \tag{13}$$

The wavelet spectrum appears then as an average of the Fourier spectrum wheighted by the square of the Fourier transform of the analysing wavelet shifted at wavenumber k. Note that the larger k is, the wider the averaging interval.

From this formula we may infer the results which are stated in the following sections: the behaviour of the wavelet spectrum at large wavenumbers depends strongly on the behaviour of the analysing wavelet at small wavenumbers.

In the following sections we will inspect whether wavelet spectra are able to detect algrebraically or exponentially decreasing power spectra.

3 Algebraically decreasing spectra

Consider a signal which Fourier spectrum behaves at small scales as $k^{-\alpha}$ $(\alpha > 1)$:

$$E(k) \ = \ k^{-\alpha} \qquad \text{for} \qquad k > k_a > 0 \ \ . \tag{14}$$

From formula (13) its wavelet spectrum is given by:

$$\tilde{E}(k) \ = \frac{1}{c_\psi \ k} \ \int_0^{k_a} E(\omega) \ \left|\hat{\psi}(\frac{k_0 \omega}{k})\right|^2 \ d\omega \ + \ \frac{1}{c_\psi \ k} \ \int_{k_a}^{+\infty} \ \omega^{-\alpha} \ \left|\hat{\psi}(\frac{k_0 \omega}{k})\right|^2 \ d\omega \tag{15}$$

Let us write $\hat{\psi}(\omega) = \omega^p \varphi(\omega)$ with φ continuous and $\varphi(0) = 1$, $p > 0$ as explained in (6). The wavelet spectrum then writes:

$$\begin{aligned} \tilde{E}(k) = k^{-(2p+1)} \frac{k_0^{2p}}{c_\psi} \int_0^{k_a} \omega^{2p} E(\omega) \left|\varphi(\frac{k_0 \omega}{k})\right|^2 d\omega \\ + k^{-(2p+1)} \frac{k_0^{2p}}{c_\psi} \int_{k_a}^{+\infty} \omega^{2p-\alpha} \left|\varphi(\frac{k_0 \omega}{k})\right|^2 d\omega \end{aligned} \tag{16}$$

As $\varphi(\frac{k_0\omega}{k}) \to 1$ when $k \to +\infty$, the first term behaves as $k^{-(2p+1)}$ when k goes to infinity (the total energy is supposed to be finite). The behaviour of the second term depends on the value of $2p - \alpha$. In consequence:

- If $\underline{\alpha < 2p+1}$ the second term of equation (16) rewrites:

$$k^{-\alpha} \; \frac{k_0^{\alpha-1}}{c_\psi} \; \int_{\frac{k_0 k_a}{k}}^{+\infty} \omega^{2p-\alpha} \, |\varphi(\omega)|^2 \, d\omega$$

  The integral has a finite non zero limit when k goes to infinity; this term behaves as $k^{-\alpha}$, dominating the first one. Thus the wavelet spectrum

$$\tilde{E}(k) \;\sim\; k^{-\alpha} \;\text{ for }\; k \to +\infty$$

  will exhibit at small scales the same slope [1] as the Fourier spectrum.

- If $\underline{\alpha > 2p+1}$, using Lebesgue theorem, the second integral in (16) has a finite non zero limit when k goes to infinity, and then the wavelet spectrum saturates at small scales to:

$$\tilde{E}(k) \;\sim\; k^{-(2p+1)} \;\text{ for }\; k \to +\infty$$

  and is independent of the signal.

- If $\underline{\alpha = 2p+1}$, similar analysis leads to a $k^{-\alpha} \ln k$ behaviour for the second term in (16), it dominates at small scales and

$$\tilde{E}(k) \;\sim\; k^{-\alpha} \ln k \;\text{ for } k \to +\infty \;.$$

The ability of the wavelet transform to detect algebraically decreasing spectrum at small scales depends on the behaviour of the analysing wavelet at large scales. Considering the inverse Fourier transform of the derivatives of $\hat{\psi}$, the behaviour of $\hat{\psi}$ in the vicinity of the origin is linked to the cancellation properties of the analysing wavelet in physical space.
A sufficient condition for the wavelet mean spectrum to exhibit the same behaviour as the $k^{-\alpha}$ Fourier spectrum is then:

$$\int_{-\infty}^{+\infty} x^n \;\psi(x)\, dx \;=\; 0 \qquad \text{for} \qquad 0 \leq \; n \; \leq \; \frac{\alpha - 1}{2} \tag{17}$$

These results are illustrated on figure 1. Given a Fourier sprectrum $E(k) \;=\; k^{-6}$, for $k > 1$, we computed, using (13) and a wavelet from the family of Gaussian derivatives

[1] As usual in turbulence studies when $E(k) \;\sim\; k^{\beta}$, β is called the slope of the spectrum, by reference to this spectrum drawn in log-log scale.

$\hat{\psi}(k) = k^p exp(-k^2/2)$, the mean wavelet spectrum $\tilde{E}(k)$. Spectra are displayed for various values of the cancellation order p. With $\alpha = 6$, the critical cancellation order is 2.5 . We find, as predicted above, that for $p = 2$ the mean wavelet spectrum saturates at k^{-5}, when for $p = 4$ and $p = 8$ the k^{-6} spectrum is recovered.

Spectra displayed on figure 2 come from the analysis of a vorticity field in a two-dimensional bi-periodic incompressible turbulent flow. The Fourier spectrum is steep at small scales, the mean wavelet spectrum, computed using isotropic Gaussian derivatives wavelets with p=2, 4, 8 and 16, is then far from it when the order of cancellation of the analysing wavelet is too low.

4 Exponentially decreasing spectra

Now consider a signal which Fourier spectrum decays at large wavenumber faster than any power of k. For example suppose:

$$E(k) \;=\; e^{-k^2} \tag{18}$$

(This particular case is an important example in 2D turbulence, the Gaussian vortex being a simple but powerfull model for coherent structures).

From (13) the mean wavelet spectrum writes:

$$\tilde{E}(k) \;=\; \frac{1}{c_\psi \; k} \int_0^{+\infty} \; e^{-\omega^2} \; \left|\hat{\psi}(\frac{k_0\omega}{k})\right|^2 \; d\omega \tag{19}$$

As in the preceeding sections, writting $\hat{\psi}(\omega) = \omega^p \varphi(\omega)$ with φ continuous, $\varphi(0) = 1$, $\varphi(\infty) = 0$ and $\varphi \in L^\infty(IR)$, we obtain:

$$\tilde{E}(k) \;=\; k^{-(2p+1)} \frac{k_0^{2p}}{c_\psi} \int_0^{+\infty} \; e^{-\omega^2} \omega^{2p} \left|\varphi(\frac{k_0\omega}{k})\right|^2 \; d\omega \tag{20}$$

Using Lebesgue theorem the integral converges to a finite, non zero limit for large wavenumbers k, so that:

$$\tilde{E}(k) \;\sim\; k^{-(2p+1)} \qquad \text{for} \;\; k \to +\infty$$

This result is clearly independent of the proper shape of the signal spectrum provided it decreases sufficiently rapidly at infinity. For Fourier spectra decreasing faster than algebraically, the wavelet spectrum at large wavenumbers is linked to the behaviour of the analysing wavelet at small wavenumbers, and not to the signal when the analysing wavelet has only a finite number of vanishing moments.

Numerical illustration is given on figure 3. The wavelet spectrum associated to $E(k) = e^{-k^2}$ was computed using Gaussian derivatives wavelets of order p=2, 8 and

16, in the wavenumber range $k > 1$. The discrepancy at small scales with the Fourier spectrum is clearly seen.

5 Connection with regularity theorems

In this section we recall some regularity theorems.

Let introduce the Besov spaces $B_p^{\alpha,\infty}$ with $1 \leq p \leq \infty$:

- For $0 < \alpha < 1$,

$$B_p^{\alpha,\infty}(IR) = \{f \in L^p(IR)\ ;\ \omega_p(f,h) = O(h^\alpha)\ ,\ \text{for } h \rightarrow 0\},$$

$$\text{where} \quad \omega_p(f,h) = \|f(x+h) - f(x)\|_{L^p}\ ,$$

- for $\alpha > 1$, α not an integer,

  $\alpha = [\alpha] + s,\quad 0 < s < 1$, ($[\alpha]$ is the integral part of α),

$$B_p^{\alpha,\infty}(IR) = \left\{f \in L^p(IR) \cap C^{[\alpha]}(IR)\ ;\ f^{([\alpha])} \in B_p^{s,\infty}\right\},$$

  where $C^m(IR)$ is the space of m-times continuously differentiable functions.

Note that for $p = \infty$, $B_\infty^{\alpha,\infty}$ is the classical Hölder space of order α.

Theorem 1: Consider f belonging to $B_1^{\alpha,\infty}(IR)$, then ([14]) the following estimation holds for its Fourier transform:

$$|\hat{f}(k)| = O\ (k^{-\alpha}) \ \text{ for } \quad k \rightarrow +\infty.$$

In terms of Fourier spectrum this writes : $E(k) = O(k^{-2\alpha})$

Theorem 2: Consider f belonging to $B_p^{\alpha,\infty}(IR)$, and an analysing wavelet ψ such that $\psi \in L^1(IR)$, $x^{p\alpha}\psi \in L^1(IR)$ ($x^\alpha \psi \in L^1(IR)$ if $p = \infty$) and ψ has $[\alpha]+1$ cancellations; then the wavelets coefficients of f verify:

$$\|\tilde{f}(a,.)\,\|_{L^p} = O\ (a^{\alpha+1/2}) \ \text{ for } \quad a \rightarrow 0.$$

Specifically one obtains,

- for $p = \infty$, then $|\tilde{f}(a,b)| = O\ (a^{\alpha+1/2})$,
- for $p = 1$, then $\int_{-\infty}^{+\infty} |\tilde{f}(a,x)|\ dx = O\ (a^{\alpha+1/2})$,

- for $p=2$, then $\tilde{E}(k) = O(k^{-(2\alpha+1)})$.

A statement similar to theorem 2 is valid for wavelet orthonormal basis ([2]); for the continuous transform, the case $p=\infty$ is well known and can be found for example in [10], for a direct demonstration of the general case see [15].

A main difference between Fourier analysis and wavelet analysis is that, when the reciprocal of theorem 1 is not true, the reciprocal of theorem 2 is true. That is the behaviour at small scales of the wavelet coefficients characterizes Besov spaces $B_p^{\alpha,\infty}(IR)$ ([2], [15]):

Theorem 3: Let $\alpha > 0$, and ψ an analysing wavelet such that ψ and all its derivatives up to order $[\alpha]+1$ are in $L^1(IR)$, if $\|\tilde{f}(a,.)\|_{L^p} \; = \; O(a^{\alpha+1/2})$ for $a \to 0$, then f belongs to $B_p^{\alpha,\infty}(IR)$.

Several comments can be done comparing these three theorems. First, L^∞ spectral estimations are not valid in the same functional spaces; Fourier transform works on L^1 functions, as wavelet transform works with L^p functions, and particularly with the L^∞ functions. However for regularities of identical orders the wavelet estimation is steeper than the Fourier one. The number of cancellations required in theorem 2 for the analysing wavelet is consistent with the results in section 3. Wavelet coefficients give a necessary and sufficient characterization of Besov spaces, when Fourier transform is only a necessary condition in the restricted case $p=1$; the difference comes from the fact that the Fourier coefficients are a mean over the whole space, when the wavelet coefficients, which are local in space and highly redundant, ensure a uniform bound in space.

These results clearly show the usefulness of the local wavelet spectrum from which one can deduce the local and global regularity properties of the signal, when the Fourier spectrum gives only necessary conditions. From this point of view let us recall a result on local wavelet behaviours ([10] and [11], [12] for more general results):

Theorem 4: Consider a bounded function f, locally integrable, Hölder continuous of order $\alpha > 0$ at point x_0 (i.e. $f \in C^{[\alpha]}(x_0)$ and $f^{([\alpha])}(x_0+h) - f^{([\alpha])}(x_0) = O(h^s)$ with $\alpha = [\alpha]+s$, $0 < s < 1$). Let ψ an analysing wavelet, such that $\psi \in L^1(IR)$, $x^\alpha \psi \in L^1(IR)$ and ψ has $[\alpha]+1$ cancellations, then:

$$|\tilde{f}(a,b)| \; = \; a^{\alpha+1/2} \; O(1 + \frac{|b-x_0|^\alpha}{a^\alpha}) \quad \text{when} \quad a \to 0, \tag{21}$$

Conversely, if (21) holds and if $f \in C^{\epsilon}(IR)$ for an $\epsilon > 0$, then $f^{([\alpha])}(x_0+h) - f^{([\alpha])}(x_0) = O(h^{\alpha}\ln(1/h))$.

6 Wavelets with infinite numbers of cancellations

From results exposed in the preceeding section, we need a wavelet with an infinite number of cancellations to be able to separate infinitly regular behaviours from finitly regular points. One way is to construct a wavelet with compact support in Fourier space: the Meyer's wavelet [2] for example. Indeed wavelets with compact support in Fourier space recover exactly algebraic spectral behaviours (see formula (13)). Unfortunately such a wavelet will have a bad -numerical- localization in physical space. Another wavelet with infinit vanishing moments was given by T. Paul in [16]; it expresses in Fourier space by the formula:

$$\hat{\psi}(k) = \exp(-\frac{\alpha}{2}\ln^2(|k|)) \; .$$

Figure 4 shows Paul's wavelet in spatial and Fourier spaces for $\alpha = 2,\ 4$ and 16.

Recently V. Perrier proposed the following family of wavelets, defined by their Fourier transforms:

$$\boxed{\hat{\pi}_n(k) = \alpha_n \;\exp\left(-\frac{1}{2}(k^2 + \frac{1}{k^{2n}})\right) \qquad n \geq 1} \; . \tag{22}$$

where α_n is chosen for normalisation.
π_n is a real function ($\hat{\pi}_n$ is even), with an infinit number of cancellations ($\hat{\pi}_n(k)$ decreases exponentially at the origin) and belongs to the Schwartz class $S(IR)$ (trivially $\hat{\pi}_n$ belongs to the Schwartz class at it is C^{∞} and decreases faster than any power of k at infinity; then π_n is also in the Schwartz class, as the Fourier transform is an isometry on $S(IR)$). Figure 5 displays functions π_n and their Fourier transforms $\hat{\pi}_n$ for $n = 2,\ 4$ and 16. Note that the peak wavenumber of π_n is $k_{n0} = n^{\frac{1}{2(n+1)}}$ and $1 \leq k_{n0} \leq 1.15$.

Now from results of Section 2, for algebraically decreasing Fourier spectra ($E(k) \sim k^{-\alpha}$) , the mean wavelet spectrum will, when using wavelet π_n, exhibit the same slope at small scales (see figure 6).

Further, the wavelet analysis of a signal which spectrum decays exponentially at small scales leads also to an exponentially decreasing mean wavelet spectrum. This is illustrated by the following calculation applied to a Gaussian spectrum $E(k) = e^{-k^2}$. From formula (13) its mean wavelet spectrum writes:

$$\tilde{E}(k) = \frac{{\alpha_n}^2}{c_{\pi_n}\ k_{n0}} \int_0^{+\infty} \exp -(\beta_n(k)\ \omega^2 + \frac{1}{\omega^{2n}})\ d\omega\ , \tag{23}$$

where:

$$\beta_n(k) = 1 + \left(\frac{k}{k_{n0}}\right)^2 \ .$$

Taking

$$\omega = [\beta_n(k)]^{\frac{-1}{2(n+1)}}\ \xi\ ,$$

we obtain:

$$\tilde{E}(k) = \frac{{\alpha_n}^2\,[\beta_n(k)]^{\frac{-1}{2(n+1)}}}{c_{\pi_n}\ k_{n0}} \int_0^{+\infty} \exp\left(-\,[\beta_n(k)]^{\frac{n}{n+1}}\,(\xi^2+\xi^{-2n})\right)\ d\xi\ . \tag{24}$$

Using Laplace's method[2] [17], the wavelet spectrum behaves at small scales as:

$$\tilde{E}(k)\ \sim\ \frac{\alpha_n^2}{c_{\pi_n}}\sqrt{\frac{\pi}{2(n+1)}}\ k^{-1}\ \exp\left(-c(n)\ \left(\frac{k}{k_{n0}}\right)^{2-\frac{2}{n+1}}\right), \tag{25}$$

with: $c(n) = n^{\frac{1}{n+1}}(1+\frac{1}{n})$.

This behaviour is not the same as the one of the Fourier spectrum, however it is worth to note that it is an exponentially decreasing behaviour when, with wavelets with finite number of cancellations, it was an algebraic behaviour. Further more for large n the wavelet spectrum at small scales is quite close to the Gaussian Fourier spectrum as:

$$\tilde{E}(k)\ \sim\ k^{-1}e^{-k^2} \qquad k \to +\infty\ \ \text{and large } n\ . \tag{26}$$

Thus, though unable to restore the correct exponential behaviour at small scales, π_n separates finitly regular functions from infinitly regular functions.

This property is illustrated on figure 7. The wavelet spectrum associated to $E(k) = e^{-k^2}$ was computed using π_n wavelets of order n=2, 4 and 8, in the wavenumber range $k > 1$. The small scales behaviour is to be compared to the one obtained with finite cancellation order wavelets (see figure 3).

7 Conclusion

We derived the analytical relation linking the mean wavelet spectrum to the classical Fourier spectrum. From this relation it turns out that, at small scales, if the Fourier

[2] Remember that Laplace's method gives: $\int_0^{+\infty} e^{x\phi(t)}dt \sim \sqrt{2\pi}\frac{e^{x\phi(t_0)}}{\sqrt{-x\phi''(t_0)}}$ *when* $x \to +\infty$, ϕ reaching its maximum at t_0.

spectrum is too steep, the wavelet spectrum is strongly biaised by the specific wavelet used in the analysis. We gave the inequality to be fulfilled between the spectrum slope and the cancellation order of the wavelet for the mean wavelet spectrum to be meaningfull. These derivations were made for asymptotically small scales, however we want to emphasis that relation (13) may be used on a practical basis to test, for a given wavelet and a given range of wavenumbers, what is the response of the wavelet spectrum to a given Fourier spectrum and whether its slope saturates or not. Further we presented and illustrated the properties of Perrier's wavelets which, built with an infinite number of cancellations, are able to detect any algebraical spectral decrease. These wavelets are not able to detect correctly exponential behaviours at small scales. However similar ideas can be applied to construct wavelets (with non compact support in Fourier space) able to recover exponential behaviour at small scales : the rate of decrease of the wavelet at the origin has to be increase.

Appendix : dimensions larger than one

Formulas established in section 2 generalize easily to dimension $n > 1$. Let us consider a real signal $s(\vec{x})$ of the n-dimensional variable $\vec{x}$; its power spectrum is given for a wavenumber k by:

$$E(k) \;=\; \frac{1}{2(2\pi)^n} k^{n-1} \int_{S^{n-1}} |\hat{s}(k\sigma)|^2 \; d\sigma \tag{27}$$

where the integration is done on S^{n-1}, the unit sphere of $I\!R^n$.

From the wavelet coefficients $\tilde{s}$ of s, defined for each scale $a > 0$, position $\vec{b} \in I\!R^n$, and rotation R in S^{n-1} by:

$$\tilde{s}(a, \vec{b}, R) \;=\; a^{-n/2} \int_{I\!R^n} s(\vec{x}) \; \psi \left(R^{-1}(\frac{\vec{x} - \vec{b}}{a}) \right) \; d\vec{x} \tag{28}$$

we may define the local wavelet spectrum:

$$\tilde{E}(k, \vec{x}) \;=\; \frac{1}{2 \; c_\psi} \frac{k^{n-1}}{k_0^n} \int_{S^{n-1}} \left| \tilde{s}(\frac{k_0}{k}, \vec{x}, \sigma) \right|^2 \; d\sigma \tag{29}$$

with $c_\psi \;=\; \int_{I\!R^n} \frac{|\hat{\psi}(\vec{\omega})|^2}{|\vec{\omega}|^n} \; d\vec{\omega}$.
Due to the properties of the wavelet transform, the mean wavelet spectrum, defined from the local wavelet spectrum by:

$$\tilde{E}(k) \;=\; \int_{I\!R^n} \tilde{E}(k, \vec{x}) \; d\vec{x}$$

is linked to the Fourier spectrum through the relation:

$$\tilde{E}(k) = \frac{1}{kc_\psi} \int_0^{+\infty} E(\omega) \left(\int_{S^{n-1}} \left| \hat{\psi}(\frac{k_0}{k}\sigma^{-1}\omega) \right|^2 d\sigma \right) d\omega \ . \tag{30}$$

Formula (30) is exactly the same as formula (13) derived in the one dimensional case, provided the wavelet has been isotropized (in Fourier space). Consequently, the results obtained in sections 3 and followings are still valid in dimension $n > 1$: the small scale behaviour of the mean wavelet spectrum depends on the rate of decrease of the wavelet at the origin in Fourier space. For isotropic (radial) wavelets this resumes to the number of cancellations of the one-dimensional generating function.

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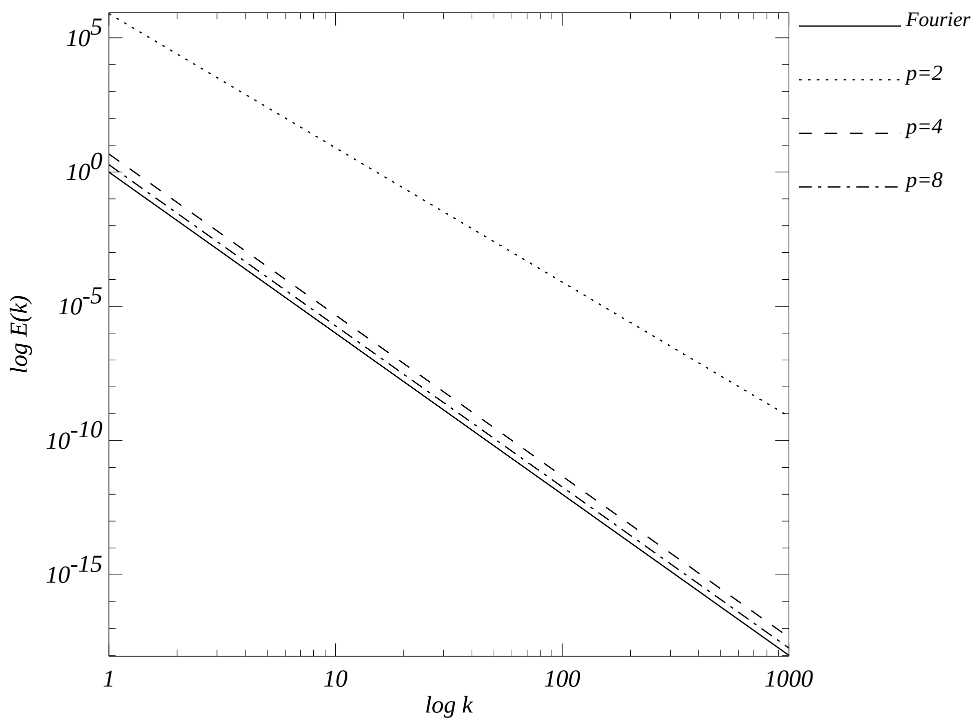

Figure 1: Fourier spectrum $E(k) = k^{-6}$ and associated wavelet spectra $\tilde{E}(k)$ for analysing wavelets with p cancellations, $\hat{\psi}(k) = k^p exp(-k^2/2)$, $p = 2, 4$ and 8 (log-log scale).

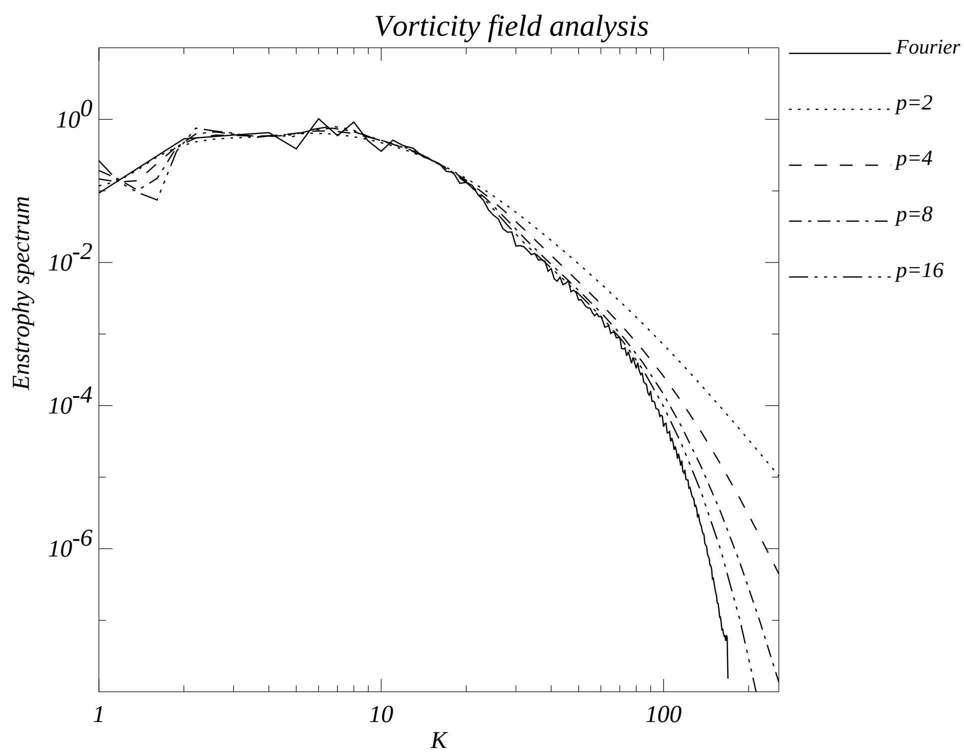

Figure 2: Fourier spectrum and wavelet spectra from the analysis of a vorticity field in a two-dimensional bi-periodic incompressible turbulent flow (enstrophy spectrum). Wavelets were isotropic Gaussian derivatives $\hat{\psi}(\vec{k}) = |\vec{k}|^p exp(-|\vec{k}|^2/2)$ for $p = 2,\ 4,\ 8,$ and 16 (log-log scale).

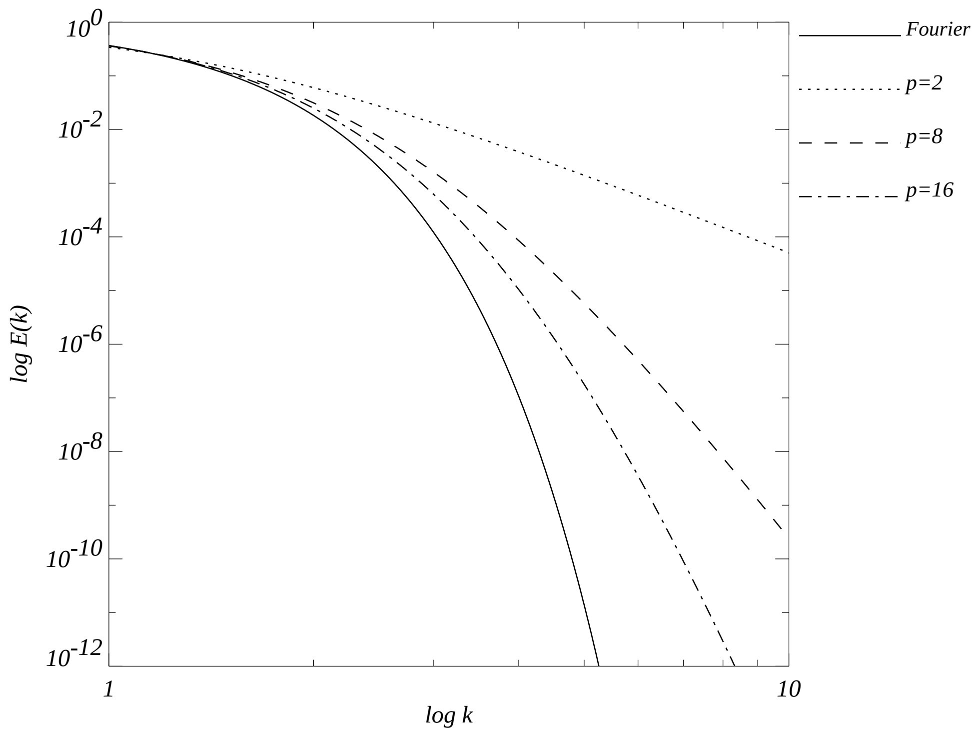

Figure 3: Fourier spectrum $E(k) = e^{-k^2}$ and associated wavelet spectra $\tilde{E}(k)$ for analysing wavelets with p cancellations, $\hat{\psi}(k) = k^p exp(-k^2/2)$, $p = 2$ 8 and 16 (log-log scale).

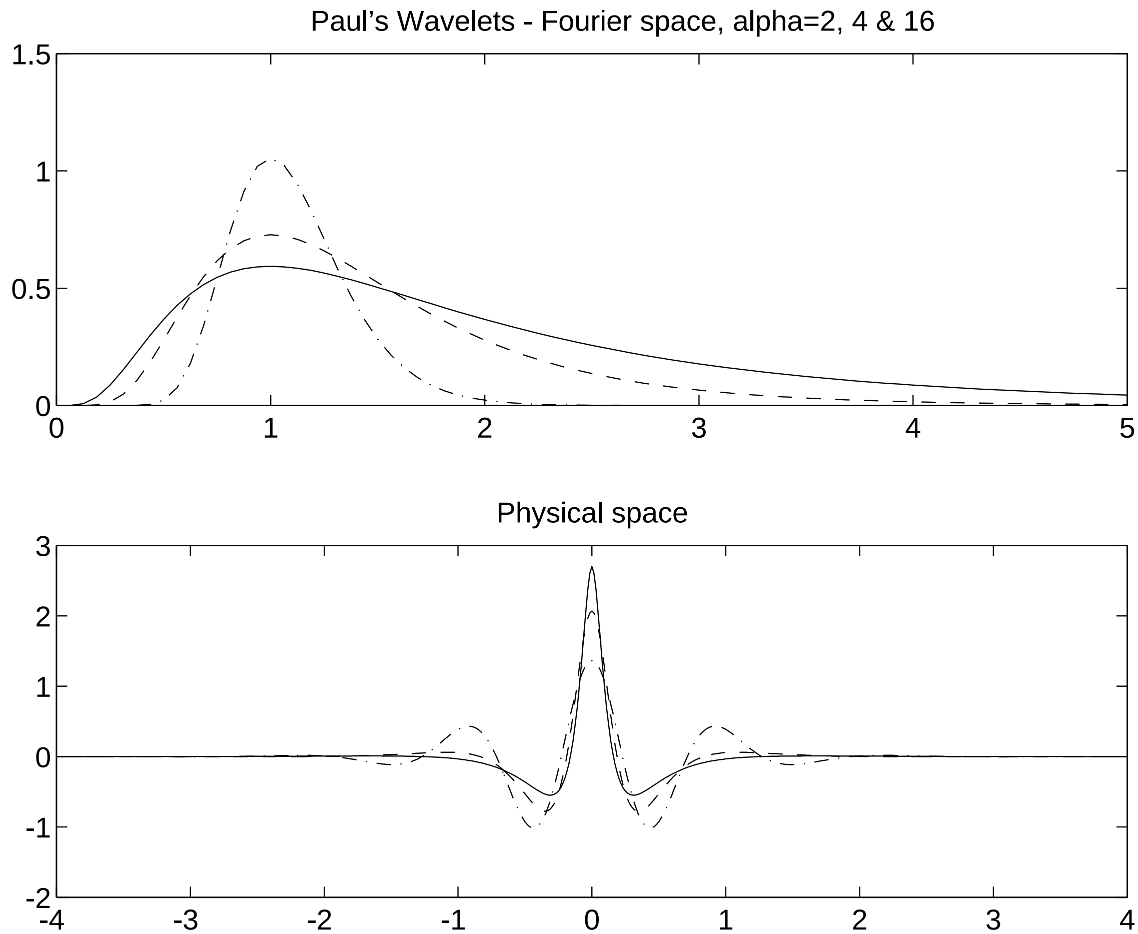

Figure 4: Paul's wavelet in physical and spectral space, for $\alpha = 2$ (continuous line), $\alpha = 4$ (dashed line) and $\alpha = 16$ (doted and dashed line).

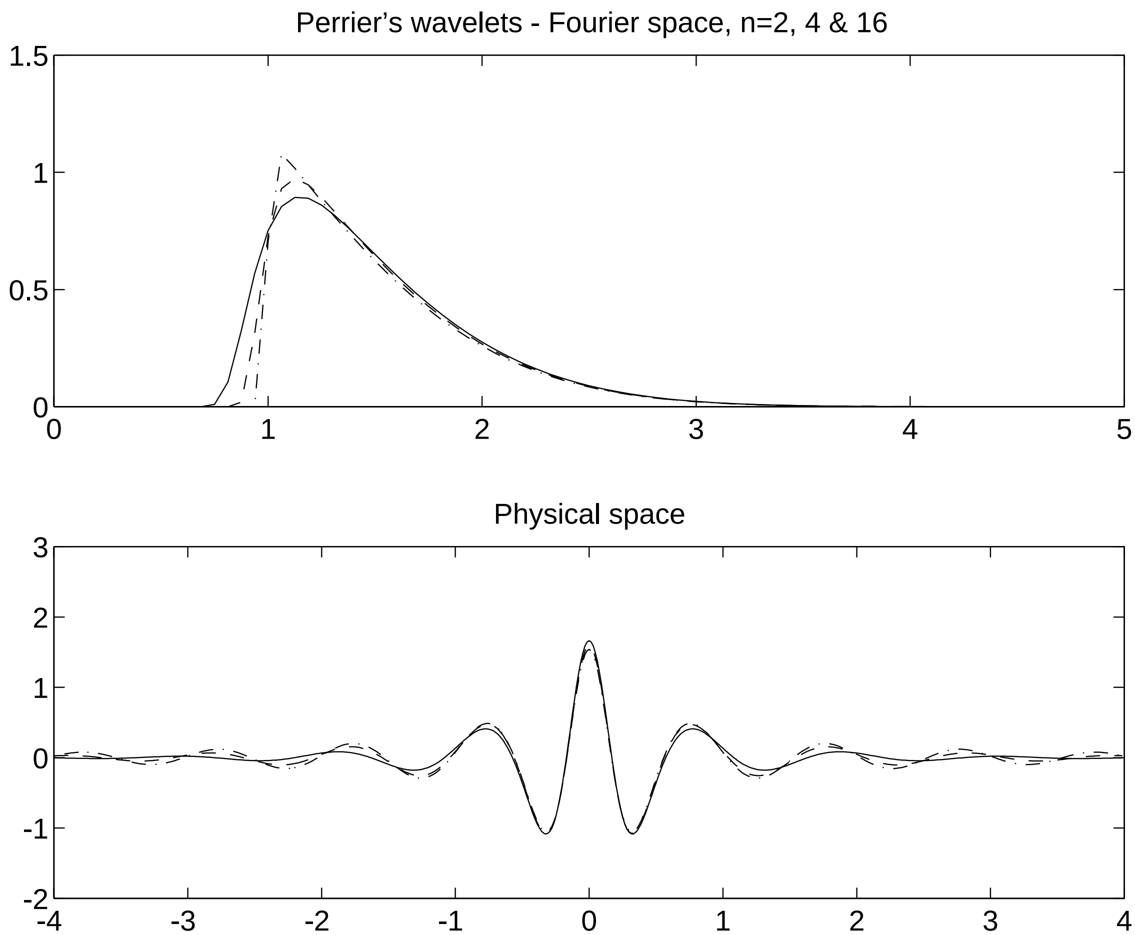

Figure 5: Perrier's wavelet in physical and spectral space,for $n = 2$ (continuous line), $n = 4$ (dashed line) and $n = 16$ (doted and dashed line).

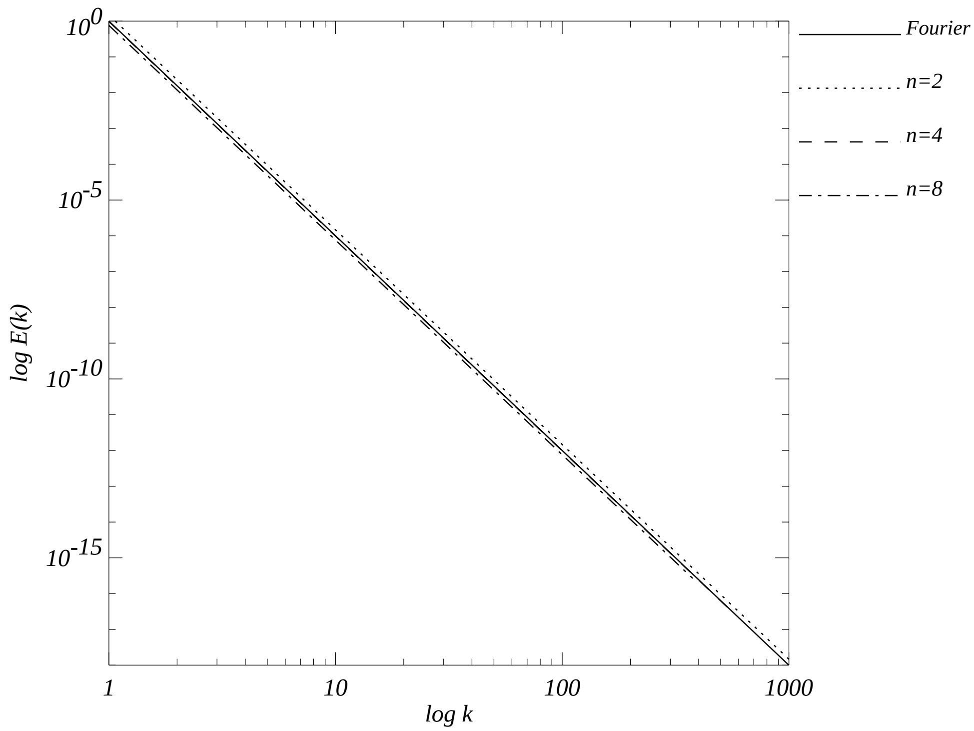

Figure 6: Fourier spectrum $E(k) = k^{-6}$ and associated wavelet spectra $\tilde{E}(k)$ computed with Perrier's wavelet π_n for $n = 2,\ 4$ and 8 (log-log scale).

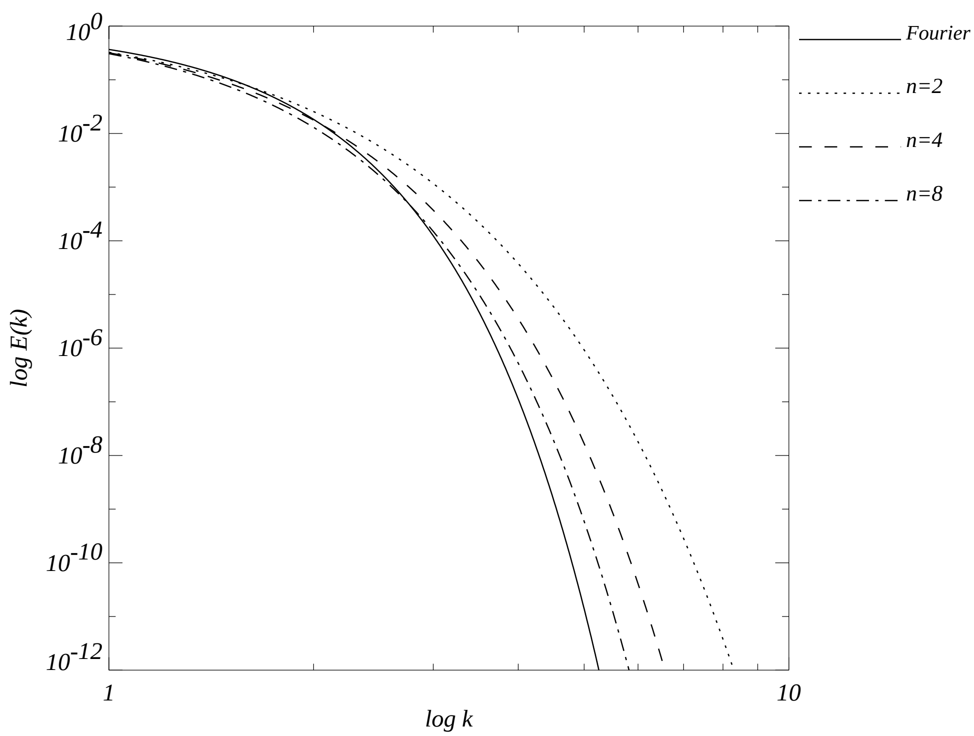

Figure 7: Fourier spectrum $E(k) = e^{-k^2}$ and associated wavelet spectra $\tilde{E}(k)$computed with Perrier's wavelet π_n with $n = 2,\ 4$ and 8 (log-log scale).